J. London Math. Soc. (2) 78 (2008) 172–182
doi:10.1112/jlms/jdn016

Stability of line bundle transforms on curves with respect to low codimensional subspaces

Ernesto C. Mistretta

Abstract

We show the stability of certain syzygies of line bundles on curves, which we call line bundle transforms. Furthermore, we prove the existence of reducible theta divisors for the transforms having integral slope. A line bundle transform is the kernel of the evaluation map on a subspace of the space of global sections.

1. *Introduction*

In the study of vector bundles on curves, it is a natural question to investigate the stability of kernels of evaluation maps of global sections. This was used in particular by Paranjape and Ramanan (cf. [**6**]) and Butler (cf. [**2**]) to prove normal generation of certain vector bundles, by Ein and Lazarsfeld (cf. [**3**]) to show the stability of the Picard bundle, and by Beauville (for example, in [**1**]) to study theta divisors.

Definition 1.1. Let C be a smooth projective curve over an algebraically closed field $\Bbbk$, and let E be a globally generated vector bundle over C. We call $M_{V,E} := \ker(V \otimes \mathcal{O}_C \twoheadrightarrow E)$ the *transform* of the vector bundle E with respect to the generating subspace $V \subset H^0(C, E)$, and $M_E := M_{H^0(E),E} = \ker(H^0(C,E) \otimes \mathcal{O}_C \twoheadrightarrow E)$ the *total transform* of E.

Starting from a result of Butler, who proved the stability of total transforms under certain hypotheses, we want to investigate the stability of transforms of line bundles by generic subspaces of certain codimensions.

Theorem 1.2 (Butler). *Let C be a smooth projective curve of genus $g \geqslant 1$ over an algebraically closed field $\Bbbk$, and let E be a semistable vector bundle over C with slope $\mu(E) \geqslant 2g$; then the vector bundle $M_E := \ker(H^0(C,E) \otimes \mathcal{O}_C \twoheadrightarrow E)$ is semistable. Furthermore, if E is stable and $\mu(E) \geqslant 2g$, then M_E is stable, unless $\mu(E) = 2g$, and either C is hyperelliptic or $\omega_C \hookrightarrow E$.*

It is natural to ask what happens when we take subspaces in the place of the vector space of global sections. Our results can be summarized in the following theorem.

Theorem 1.3. *Let $\mathcal{L}$ be a line bundle of degree d on a curve C of genus $g \geqslant 2$ such that $d \geqslant 2g + 2c$, with $1 \leqslant c \leqslant g$. Then $M_{V,\mathcal{L}}$ is semistable for a generic subspace $V \subset H^0(\mathcal{L})$ of codimension c. It is stable unless $d = 2g + 2c$ and the curve is hyperelliptic.*

Received 21 March 2007; revised 31 January 2008; published online 3 May 2008.

2000 *Mathematics Subject Classification* 14H60, 14F05, 14K25.

Similar results can be deduced by some constructions in Vincent Mercat's work [5] on Brill–Noether loci, but we think that in our case it is useful to give a more direct proof that applies to all line bundles of degree $d \geqslant 2g+2c$, and not only to generic line bundles.

Eventually, we observe the existence of theta divisors associated to the (semi)stable transforms having integer slope -2. These theta divisors are always non-integral, and in most cases reducible, and hence give further examples of stable vector bundles admitting a reducible theta divisor (cf. [1]).

Remark 1.4. A geometrical interpretation of this kind of results goes as follows: a generating subspace $V \subset H^0(C,\mathcal{L})$ gives rise to a base-point-free linear system $|V| \subset |\mathcal{L}|$ on the curve C, and determines a map $\varphi_V : C \to \mathbb{P}(V^*)$, which associates to a point $x \in C$, the hyperplane of global sections in V vanishing in x. The Euler sequence on $\mathbb{P}(V^*)$ is the dual of the tautological sequence

$$0 \longrightarrow \Omega_{\mathbb{P}(V^*)}(1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}(V^*)} \longrightarrow \mathcal{O}_{\mathbb{P}(V^*)}(1) \longrightarrow 0,$$

which, restricted to C, gives the evaluation sequence

$$0 \longrightarrow M_{V,\mathcal{L}} \longrightarrow V \otimes \mathcal{O}_C \longrightarrow \mathcal{L} \longrightarrow 0.$$

As the stability of a vector bundle is not affected by dualizing and tensorizing by a line bundle, we see that the stability of $M_{V,\mathcal{L}} = \Omega(1)_{|C}$ is equivalent to the stability of the restriction of the tangent bundle of the projective space $\mathbb{P}(V^*)$ to the curve C.

Therefore our theorem translates to the following theorem.

Theorem 1.5. *Let $C \subset \mathbb{P}^{d-g}$ be a genus $g \geqslant 2$ degree d non-degenerate smooth curve, where $d > 2g+2c$ and c is a constant such that $1 \leqslant c \leqslant g$. Then for the generic projection $\mathbb{P}^{d-g} \dashrightarrow \mathbb{P}^{d-g-c}$ the restriction $T_{\mathbb{P}^{d-g-c}|C}$ is stable.*

2. *Stability of transforms*

We essentially use the following two lemmas.

Lemma 2.1 [2, Lemma 1.10]. *Let C be a curve of genus $g \geqslant 2$, and let F be a vector bundle on C with no trivial summands and such that $h^1(F) \neq 0$. Suppose that $V \subset H^0(F)$ generates F. If $N = M_{V,F}$ is stable, then $\mu(N) \leqslant -2$. Furthermore, $\mu(N) = -2$ implies that either C is hyperelliptic, F is the hyperelliptic bundle, and N is its dual, or $F = \omega$ and $N = M_\omega$.*

The proof of this lemma is based on the result by Paranjape and Ramanan asserting the stability of M_ω (see [2, 6]).

Lemma 2.2. *Let $\mathcal{L}$ be a degree $d \geqslant 2g+2c$ line bundle on a curve C of genus $g \geqslant 2$ with $c \leqslant g$. Let $V \subset H^0(\mathcal{L})$ be a generating subspace of codimension c, and suppose that there exists a stable sub-bundle of maximal slope $N \hookrightarrow M_{V,\mathcal{L}}$ such that $0 \neq N \neq M_{V,\mathcal{L}}$ and $\mu(N) \geqslant \mu(M_{V,\mathcal{L}})$.*

Then there exist a line bundle F of degree $f \leqslant d-1$, a generating subspace $W \subset H^0(F)$, and an injection $F \hookrightarrow \mathcal{L}$ such that N fits into the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

That is, a destabilization of $M_{V,\mathcal{L}}$ must be the transform of a line bundle injecting into $\mathcal{L}$ such that the global sections we are transforming by are in V.

The importance of this lemma lies in the fact that we associate a line bundle F to a destabilizing N, and this allows us more easily to parameterize destabilizations and bound their dimension.

Proof. We remark that $\mu(M_{V,\mathcal{L}}) = -d/(d-g-c) \geqslant -2$ for $d \geqslant 2g+2c$. Consider a stable sub-bundle $N \hookrightarrow M_{V,\mathcal{L}}$ of maximal slope. Then it fits into the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & W\otimes\mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V\otimes\mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

Here $W \hookrightarrow V$ is defined by $W^* := \operatorname{Im}(V^* \to H^0(N^*))$, and hence W^* generates N^*; write $F^* := \ker(W^*\otimes\mathcal{O} \twoheadrightarrow N^*)$.

Then F is a vector bundle with no trivial summands. Moreover the morphism $F \to \mathcal{L}$ is not zero, as $W\otimes\mathcal{O}$ does not map to $M_{V,\mathcal{L}}$. We have to show that $\operatorname{rk} F = 1$ and $\deg F \leqslant d-1$. We distinguish the two cases: $h^1(F) = 0$ and $h^1(F) \neq 0$.

Case 1. Let us suppose that $h^1(F) = 0$. Then $h^0(F) = \chi(F) = \operatorname{rk} F(\mu(F)+1-g)$. On the other hand, $h^0(F) > \operatorname{rk} F$ as F is globally generated and not trivial.

Together, this yields

$$\mu(F) > g. \tag{2.1}$$

Furthermore

$$\mu(N) = -\frac{\deg F}{\dim W - \operatorname{rk} F} \leqslant -\frac{\mu(F)}{\mu(F)-g} = \mu(M_F), \tag{2.2}$$

as $\dim W \leqslant h^0(F) = \operatorname{rk} F(\mu(F)+1-g)$.

Consider the image $I = \operatorname{Im}(F \to \mathcal{L}) \subseteq \mathcal{L}$. The commutative diagram

$$\begin{array}{ccccc} W & \hookrightarrow & H^0(F) & \longrightarrow & H^0(I) \\ \downarrow & & & & \downarrow \\ V & & \hookrightarrow & & H^0(\mathcal{L}) \end{array}$$

shows that the map $W \to H^0(I)$ is injective and its image $W' \subset H^0(I)$ is contained in $V \subset H^0(\mathcal{L})$, and hence $N \hookrightarrow M_{W',I} \hookrightarrow M_{V,\mathcal{L}}$. As N is a sub-bundle of $M_{V,\mathcal{L}}$ of maximal slope, this yields $\mu(N) \geqslant \mu(M_{W',I})$; that is, $-\deg F/\operatorname{rk} N \geqslant -\deg I/\operatorname{rk} M_{W',I}$. Then

$$\deg F \leqslant \deg I\left(\frac{\operatorname{rk} N}{\operatorname{rk} M_{W',I}}\right) \leqslant \deg I \leqslant \deg\mathcal{L} = d.$$

If $\operatorname{rk} F \geqslant 2$, then $\mu(F) \leqslant \deg\mathcal{L}/2 = d/2$, and so

$$\mu(N) \leqslant \frac{-\mu(F)}{\mu(F)-g} \leqslant \frac{-d/2}{d/2-g} = \frac{-d}{d-2g} \leqslant \frac{-d}{d-g-c} = \mu(M_{V,\mathcal{L}}).$$

Here the first inequality is (2.2). For the second, we show that the function $-x/(x-g)$ is strictly increasing for $x > g$. Then use $\mu(F) > g$ due to (2.1). Equality holds only if $\operatorname{rk} F = 2$, $\deg F = d$, $W = H^0(F)$, and $g = c$. However, in this case we would find that

$$\dim W = h^0(F) = d+2-2g > d+1-g-c = \dim V,$$

which is impossible as by construction $W \hookrightarrow V$.

Hence $\operatorname{rk} F = 1$, and so $F = I$ is a globally generated and acyclic line bundle of degree $f \leqslant d$, and $\mu(N) = -f/(\dim W - 1)$.

It is easy to see that the case $f = d$ cannot hold, as in that case we cannot have $\mu(N) \geqslant \mu(M_{V,\mathcal{L}})$. Hence $f \leqslant d-1$.

Case 2. In the case $h^1(F) \neq 0$, by Lemma 2.1, $\mu(N) \leqslant -2$. Equality holds only if $F = \omega_C$ and $W = H^0(\omega)$, or if the curve C is hyperelliptic and F is the hyperelliptic bundle. In the latter case the only generating space of global sections is $H^0(F)$. In any case we have $f = \deg F < d-1$. □

Remark 2.3. The diagram in the statement of the lemma is a construction from Butler's proof of Theorem 1.2.

Remark 2.4. Looking carefully at the numerical invariants in the above proof, we can deduce some inequalities that will be useful in the following: let us consider again the diagram in the above lemma

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

and suppose that $h^1(F) = 0$. Let us write $f := \deg F$, $s := d - f$, and $b := \operatorname{codim}_{H^0(F)} W$. Then we can show that

$$0 < c - b < s \leqslant \frac{d}{g+c}(c-b). \tag{2.3}$$

In fact, as $W \hookrightarrow V$, and $W \neq V$,

$$d - s + 1 - g - b = h^0(F) - b = \dim W < \dim V = d + 1 - g - c,$$

and hence $c - b < s$. And as

$$-\frac{d-s}{d-s-g-b} = \mu(N) \geqslant \mu(M_{V,\mathcal{L}}) = -\frac{d}{d-g-c},$$

we see that $s(g+c) \leqslant d(c-b)$, and hence $c - b > 0$ and $s \leqslant (d/(g+c))(c-b)$.

2.1. *Line bundles of degree $d = 2g+2$*

A first consequence of these lemmas is the following proposition asserting semistability for hyperplane transforms of line bundles of degree $2g+2$.

Proposition 2.5. *Let $\mathcal{L}$ be a line bundle of degree $d = 2g+2$ on a curve C of genus $g \geqslant 2$. Then $M_{V,\mathcal{L}}$ is semistable for every generating hyperplane $V \subset H^0(\mathcal{L})$. It is strictly semistable if C is hyperelliptic.*

Proof. Let us prove the semistability of $M_{V,\mathcal{L}}$.

Consider a stable sub-bundle $N \hookrightarrow M_{V,\mathcal{L}}$ of maximal slope, and suppose that it destabilizes $M_{V,\mathcal{L}}$ in the strict sense; that is, $\mu(N) > -2 = \mu(M_{V,\mathcal{L}})$. By Lemma 2.2 we know that N fits into the following diagram, with F a line bundle.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

Moreover, $h^1(F) = 0$ since otherwise $\mu(N) \leqslant -2$ by Lemma 2.1. Hence $\deg F = 2g$ by (2.3), and $W = H^0(F)$ (we have $c = 1$, so $b = 0$ and $1 < s \leqslant 2$). Therefore

$$\operatorname{rk} N = h^0(F) - 1 = \deg F - g$$

and

$$\mu(N) = -\frac{\deg F}{\deg F - g} = -\frac{2g}{2g - g} = -2,$$

and this shows that it is not possible to find a strictly destabilizing N.

If the curve is hyperelliptic, then $M_{V,\mathcal{L}}$ is strictly semistable: we can show that there is a line bundle of degree -2 injecting in $M_{V,\mathcal{L}}$. In fact we can consider the line bundle A dual to the only g_2^1 of the curve, that is, the dual of the hyperelliptic bundle.

The hyperelliptic bundle A^* has $h^0(A^*) = 2$, and from the exact sequence

$$0 \longrightarrow M_{V,\mathcal{L}} \otimes A^* \longrightarrow V \otimes A^* \longrightarrow \mathcal{L} \otimes A^* \longrightarrow 0,$$

we see that there are destabilizations of $M_{V,\mathcal{L}}$ by the line bundle A if and only if

$$H^0(M_{V,\mathcal{L}} \otimes A^*) = \ker(\varphi : V \otimes H^0(A^*) \longrightarrow H^0(\mathcal{L} \otimes A^*)) \neq 0.$$

Counting dimensions, we see that the map φ cannot be injective:

$$\dim V \cdot \dim H^0(A^*) = (g+2)2 > g + 5 = \dim H^0(\mathcal{L} \otimes A^*). \qquad \square$$

In order to prove stability for non-hyperelliptic curves, though, we need to take a generic hyperplane, and not just a generating hyperplane. The following is a special case of a more general result, proved in Subsection 2.3.

THEOREM 2.6. *Let $\mathcal{L}$ be a line bundle of degree $d = 2g + 2$ on a curve C of genus $g \geqslant 2$. Then $M_{V,\mathcal{L}}$ is stable for a generic hyperplane $V \subset H^0(\mathcal{L})$ if and only if C is non-hyperelliptic.*

2.2. *Line bundles of degree $d > 2g + 2c$*

Here we show that for a generic subspace the transform of a line bundle of degree $d > 2g + 2c$ is stable. In contrast to Proposition 2.5, we have to consider generic hyperplanes, and not just generating hyperplanes.

THEOREM 2.7. *Let $\mathcal{L}$ be a line bundle of degree d on a curve C of genus $g \geqslant 2$ such that $d > 2g + 2c$, with $1 \leqslant c \leqslant g$. Then $M_{V,\mathcal{L}}$ is stable for a generic subspace $V \subset H^0(\mathcal{L})$ of codimension c.*

Proof. Let us proceed as in Proposition 2.5. We see that $-2 < \mu(M_{V,\mathcal{L}}) < -1$.

Consider a stable sub-bundle $N \hookrightarrow M_{V,\mathcal{L}}$ of maximal slope. By Lemma 2.2 we know that it fits into the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

We can conclude right away that $h^1(F) = 0$, as by Lemma 2.1 we would otherwise have $\mu(N) \leqslant -2$.

Therefore F is a globally generated line bundle with $h^1(F) = 0$, $\deg F =: d - s \leqslant d - 2$, and W is a b-codimensional subspace of $H^0(F)$. By Remark 2.4, we see that for every b with

$0 \leqslant b < c$ there is a finite number of values of s, giving rise to a possible destabilization of $M_{V,\mathcal{L}}$.

For any of those b and s we will construct a parameter space allowing F, W, and the subspace $V \subset H^0(\mathcal{L})$ to vary. For any such b and s we want to consider the parameter space $\mathcal{D}_{b,s}$, parameterizing subspaces $V \subset H^0(\mathcal{L})$ together with a destabilizing bundle of $M_{V,\mathcal{L}}$ of degree $s-d$ originating from a subspace W as in the construction above:

$$\begin{aligned}\mathcal{D}_{b,s} := \{(F, F \hookrightarrow \mathcal{L}, W \subset H^0(F)), V \subset H^0(\mathcal{L})) \mid F \in \mathrm{Pic}^{d-s}(C),\\ (\varphi : F \hookrightarrow \mathcal{L}) \in \mathbb{P}(H^0(F^* \otimes \mathcal{L})), W \in \mathbb{G}\mathrm{r}(b, H^0(F))\\ V \in \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L})), \varphi_{|W} : W \hookrightarrow V \subset H^0(\mathcal{L})\}.\end{aligned}$$

In order to estimate its dimension, we use the natural morphisms

$$\pi_{b,s} : \mathcal{D}_{b,s} \longrightarrow \mathrm{Pic}^{d-s}(C), \quad (F, F \hookrightarrow \mathcal{L}, W, V) \longmapsto F,$$

and $\rho_{b,s} : \mathcal{D}_{b,s} \to \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$, $(F, F \hookrightarrow \mathcal{L}, W, V) \mapsto V$.

The image of $\pi_{b,s}$ is formed by all the line bundles $F \in \mathrm{Pic}^{d-s}(C)$ such that $h^0(F^* \otimes \mathcal{L}) \neq 0$. In particular, $\dim \pi_{b,s}(\mathcal{D}_{b,s}) = \min(s, g)$, because the degree of $F^* \otimes \mathcal{L}$ is s. The fiber over $F \in \pi_{b,s}(\mathcal{D}_{b,s})$ has the same dimension as $\mathbb{P}(H^0(F^* \otimes \mathcal{L})) \times \mathbb{G}\mathrm{r}(b, (H^0(F))) \times \mathbb{G}\mathrm{r}(c, (H^0(\mathcal{L})/W))$.

By Clifford's theorem, $h^0(F^* \otimes \mathcal{L}) \leqslant s/2 + 1$ if $s \leqslant 2g$, and $h^0(F^* \otimes \mathcal{L}) = s + 1 - g$ otherwise. Therefore,

$$\begin{aligned}\dim \mathcal{D}_{b,s} &\leqslant \min(s, g) + \sup(s/2, s-g) + b(d-s-g+1-b) + c(s+b-c)\\ &\leqslant \tfrac{3}{2}s + b(d-s-g+1-b) + c(s+b-c).\end{aligned}$$

Claim: for g, d, c as in the hypothesis and s, b satisfying the inequalities of Remark 2.1, we have

$$\tfrac{3}{2}s + b(d-s-g+1-b) + c(s+b-c) < c(d+1-g) - c^2 = \dim \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L})).$$

Once we have proved the claim, it follows that for all s and b giving rise to possible destabilizations, the morphisms $\rho_{b,s} : \mathcal{D}_{b,s} \to \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$ have a locally closed image of dimension strictly smaller than $\mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$, and hence the generic subspace avoids all possible destabilizations of $M_{V,\mathcal{L}}$.

The claim is equivalent to

$$\frac{3s}{2(c-b)} + s + b < d + 1 - g.$$

Using inequalities (2.3) we get

$$\frac{3s}{2(c-b)} + s + b \leqslant \frac{3/2 + (c-b)}{g+c} d + b,$$

and hence we want to prove that

$$\frac{3/2 + (c-b)}{g+c} d + b < d + 1 - g,$$

which is equivalent to

$$\frac{b+g-1}{b+g-3/2} < \frac{d}{g+c},$$

and as $b \geqslant 0 \geqslant 2 - g$, we have

$$\frac{b+g-1}{b+g-3/2} \leqslant 2 < \frac{d}{g+c}. \qquad \square$$

2.3. *Line bundles of degree $d = 2g + 2c$*

We have shown in Subsection 2.1 that hyperplane transforms of a degree $2g+2$ line bundle are always semistable. We prove here that generic c-codimensional transforms of a degree $2g+2c$ line bundle are stable, except in the hyperelliptic case, where they are strictly semistable.

THEOREM 2.8. *Let $\mathcal{L}$ be a line bundle of degree $d = 2g+2c$ on a curve C of genus $g \geqslant 2$. Then $M_{V,\mathcal{L}}$ is semistable for a generic subspace $V \subset H^0(\mathcal{L})$ of codimension c. It is stable if and only if C is non-hyperelliptic.*

Proof. As in the proof of Theorem 2.7 we want to construct parameter spaces for destabilizations and verify by dimension count that the generic subspace avoids them. Let us consider a line bundle $\mathcal{L}$ of degree $d = 2g+2c$ on a curve C of genus $g \geqslant 2$, and the transform $M_{V,\mathcal{L}}$ for a subspace $V \subset H^0(\mathcal{L})$ of codimension c.

To show semistability, let us suppose that there is a destabilizing stable vector bundle $N \hookrightarrow M_{V,\mathcal{L}}$, with $\mu(N) > \mu(M_{V,\mathcal{L}}) = -2$. By Lemma 2.2 we know that it fits in the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

We can suppose that $h^1(F) = 0$ by Lemma 2.1. In this case we can follow the same computations as in Theorem 2.7: we have a parameter space for destabilizations

$$\begin{aligned} \mathcal{D}_{b,s} := \{(F, F \hookrightarrow \mathcal{L}, W \subset H^0(F)), V \subset H^0(\mathcal{L})) \mid F \in \operatorname{Pic}^{d-s}(C), \\ (\varphi : F \hookrightarrow \mathcal{L}) \in \mathbb{P}(H^0(F^* \otimes \mathcal{L})), W \in \mathbb{G}\mathrm{r}(b, H^0(F)) \\ V \in \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L})), \varphi_{|W} : W \hookrightarrow V \subset H^0(\mathcal{L})\}, \end{aligned}$$

with dimension bounded by

$$\dim \mathcal{D}_{b,s} \leqslant \tfrac{3}{2}s + b(d-s-g+1-b) + c(s+b-c),$$

with b and s satisfying $0 < c-b < s \leqslant (d/(g+c))(c-b)$. Except in the case $b=0$ and $g=2$, we can follow the very same proof as that of Theorem 2.7, and we see that this bound shows that the generic subspace avoids the destabilization locus.

In the case $b=0$ and $g=2$ as well, it can be easily shown that $\dim \mathcal{D}_{b,s} < \dim \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$, for all s giving rise to destabilizations.

To show that we have strict semistability in the hyperelliptic case, we can proceed as in Proposition 2.5, and show that the dual of the hyperelliptic bundle is a sub-bundle of $M_{V,\mathcal{L}}$ of slope -2.

To show that we have stability in the non-hyperelliptic case, we have to exclude slope -2 sub-bundles $N \hookrightarrow M_{V,\mathcal{L}}$. Again we can apply Lemma 2.2 and consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{V,\mathcal{L}} & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \end{array}$$

Here we can distinguish the two cases $H^1(F) = 0$ and $H^1(F) \neq 0$.

In the case $H^1(F) = 0$ we can again follow the same computations as in Theorem 2.7.

In the case $H^1(F) \neq 0$, Lemma 2.1 implies that $F = \omega$ and $N = M_\omega$, and hence the parameter space for destabilizations will be

$$\mathcal{D} := \{(\omega \hookrightarrow \mathcal{L}, V \subset H^0(\mathcal{L})) \mid H^0(\omega) \subset V\},$$

and it can be shown that $\dim \mathcal{D} < \dim \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$. ∎

3. *Theta divisors and transforms*

When a vector bundle has integer slope $\mu(E) = \mu \in \mathbb{Z}$, we can define the set

$$\Theta_E := \{P \in \operatorname{Pic}^\nu(C) \mid H^0(C, E \otimes P) \neq 0\},$$

where $\nu := g - 1 - \mu$.

As $\chi(E \otimes P) = 0$, either $\Theta_E = \operatorname{Pic}^\nu(C)$, or it carries the natural structure of an effective divisor in $\operatorname{Pic}^\nu(C)$. In the latter case we say that E *admits a theta divisor*. The class of this divisor in $H^2(\operatorname{Pic}^\nu(C), \mathbb{Z})$ is $\operatorname{rk} E \cdot \vartheta$, where ϑ is the class of the canonical theta divisor of $\operatorname{Pic}^\nu(C)$. Whenever a vector bundle admits a theta divisor, it is semistable. Moreover, strictly semistable vector bundles admitting a theta divisor have *non-integral* theta divisors.

However, there are examples of stable vector bundles with no theta divisor, or with a reducible theta divisor. Beauville shows in [**1**] that the total transform M_L of a degree $2g$ line bundle L on a genus g curve C always has a reducible theta divisor, and that if L is very ample, and C is not hyperelliptic, then M_L is stable.

The vector bundles considered above, that is, transforms of degree $d \geqslant 2g + 2c$ line bundles, with respect to c-codimensional subspaces of global sections, have slope μ such that $-2 \leqslant \mu < -1$. The case of integer slope $\mu = -2$ appears if and only if $d = 2g + 2c$.

Following the same argument as in [**1**], we prove that for the generic $V \subset H^0(C, \mathcal{L})$ those transforms always carry a non-integral theta divisor.

To prove that, for a generic $V \subset H^0(C, \mathcal{L})$ within the numerical conditions above, the transform $M_{V,\mathcal{L}}$ admits a theta divisor, we need the following lemma.

LEMMA 3.1. *Let P be a 2-dimensional vector space, let H be a vector space of dimension $n + c$, and let $K \subset P \otimes H$ be a subspace of dimension $2c$. If K contains no pure vectors, then the generic c-codimensional subspace $V \in \mathbb{G}\mathrm{r}(c, H)$ satisfies*

$$K \cap (P \otimes V) = 0.$$

Proof. We consider the map

$$\begin{aligned} f : \mathbb{G}\mathrm{r}(c, H) &\longrightarrow \mathbb{G}\mathrm{r}(2c, P \otimes H), \\ V &\longmapsto P \otimes V, \end{aligned}$$

and we claim that the image of f is not contained in the closed subscheme

$$Z := \{W \in \mathbb{G}\mathrm{r}(2c, P \otimes H) \mid \dim K \cap W \geqslant 1\}.$$

Let us observe at first that Z carries a filtration

$$Z = Z_1 \supseteq Z_2 \supseteq \ldots \supseteq Z_s := \{W \subset P \otimes H \mid \dim K \cap W \geqslant s\} \supseteq \ldots.$$

The tangent space of the Grassmannian $\mathbb{G}\mathrm{r}(2c, P \otimes H)$ at a point W is

$$T_W \, \mathbb{G}\mathrm{r}(2c, P \otimes H) = \operatorname{Hom}(W, P \otimes H/W).$$

The subscheme $Z_s \setminus Z_{s+1}$ is smooth and its tangent space at a point W is given by first-order deformations of $W \subset P \otimes H$ that deform $W \cap K$ into an s-dimensional subspace of K:

$$T_W(Z_s \setminus Z_{s+1}) = \{\varphi \in \operatorname{Hom}(W, P \otimes H/W) \mid \varphi(W \cap K) \subseteq K/(W \cap K)\}.$$

Also, the differential of the morphism f at the point $V \in \mathbb{G}\mathrm{r}(c, H)$ is the map

$$\begin{aligned} \mathrm{d}f_V : T_V \mathbb{G}\mathrm{r}(c, H) &\longrightarrow T_{P \otimes V} \mathbb{G}\mathrm{r}(2c, P \otimes H) \\ \varphi \in \operatorname{Hom}(V, H/V) &\longmapsto 1 \otimes \varphi \in \operatorname{Hom}(P \otimes V, P \otimes (H/V)). \end{aligned}$$

We can prove now that if $V \in \mathbb{G}\mathrm{r}(c, H)$ is a subspace such that $P \otimes V \in Z_s \setminus Z_{s+1}$, then

$$df_V(T_V \mathbb{G}\mathrm{r}(c, H)) \nsubseteq T_{P\otimes V}(Z_s \setminus Z_{s+1}) :$$

we claim that there exists a $\varphi \in \mathrm{Hom}(V, H/V)$ such that $1 \otimes \varphi \notin T_{P\otimes V}(Z_s \setminus Z_{s+1})$.

To see this, let us choose a basis (e_1, e_2) for P, and a vector $w = e_1 \otimes v_1 + e_2 \otimes v_2 \in K \cap (P \otimes V)$. By the hypothesis on K, $v_1 \nparallel v_2$. Let us consider now a vector $z = e_1 \otimes z_1 + e_2 \otimes z_2 \in (P \otimes (H/V))$ such that $z \notin (K/(P \otimes V \cap K))$.

Then if we choose a $\varphi \in \mathrm{Hom}(V, H/V)$ such that $\varphi(v_1) = z_1$ and $\varphi(v_2) = z_2$, we see that $(1 \otimes \varphi)(w) = z \notin K/(P \otimes V \cap K)$. Hence the image of a generic deformation of V avoids the subscheme $Z \subset \mathbb{G}\mathrm{r}(2n, P \otimes H)$. □

We can now prove the existence of theta divisors for generic transforms of slope -2.

Theorem 3.2. *Let $\mathcal{L}$ be a line bundle of degree $d = 2g + 2c$ on a genus g curve C, where $c \in \mathbb{N}$ is a positive integer and $g \geqslant 2$. Then, if $V \subset H^0(C, \mathcal{L})$ is a generic c-codimensional subspace, the transform $M_{V,\mathcal{L}}$ admits a non-integral theta divisor.*

Proof. We recall that $\mu(M_{V,\mathcal{L}}) = -2$. We have to show first that, for the generic $V \subset H^0(C, \mathcal{L})$, we have

$$\Theta_{M_{V,\mathcal{L}}} \neq \mathrm{Pic}^{g+1}(C),$$

that is, there is a $P \in \mathrm{Pic}^{g+1}(C)$ such that $H^0(M_{V,\mathcal{L}} \otimes P) = 0$. By the exact sequence

$$0 \longrightarrow M_{V,\mathcal{L}} \otimes P \longrightarrow V \otimes P \longrightarrow \mathcal{L} \otimes P \longrightarrow 0,$$

this is the same as a $P \in \mathrm{Pic}^{g+1}(C)$ such that the multiplication map

$$\mu : V \otimes H^0(P) \longrightarrow H^0(\mathcal{L} \otimes P)$$

is injective.

If P belongs to the divisor $D = (\omega_C) - C_{g-2} + C \subset \mathrm{Pic}^{g+1}(C)$ (that is, if P can be written in the form $P = \omega_C(x_1 - x_2 - \ldots - x_{g-1})$ for some points $x_1, x_2, \ldots, x_{g-1} \in C$), then either $h^0(P) > 2$, or $h^0(P) = 2$ and P has a base point. In either case this implies that μ is not injective for any V (cf. [**1**]).

Any P in $\mathrm{Pic}^{g+1}(C) \setminus D$ is base-point-free and has $h^0(P) = 2$. Let us fix such a P, and assume by generality that $h^1(L \otimes P^*) = 0$. We claim that for the generic $V \subset H^0(C, \mathcal{L})$ of codimension c the multiplication map $\mu : H^0(P) \otimes V \to H^0(P \otimes \mathcal{L})$ is injective. From the exact sequence

$$0 \longrightarrow P^* \longrightarrow H^0(P) \otimes \mathcal{O}_C \longrightarrow \; P \longrightarrow 0$$

we get

$$0 \longrightarrow H^0(P^* \otimes \mathcal{L}) \longrightarrow H^0(P) \otimes H^0(\mathcal{L}) \longrightarrow H^0(P \otimes \mathcal{L}) \longrightarrow 0,$$

and hence the map μ is injective if and only if the subspace $V \subset H^0(C, \mathcal{L})$ satisfies $H^0(P^* \otimes \mathcal{L}) \cap (H^0(P) \otimes V) = 0$. This is given by Lemma 3.1.

Hence we know that for the generic subspace V, the transform $M_{V,\mathcal{L}}$ admits a theta divisor. To observe that it is not integral, we notice that the set of points of $\Theta_{M_{V,\mathcal{L}}}$ contains the divisor D with cohomology class $(g-1)\vartheta$ (cf. [**4**]). As the cohomology class of $\Theta_{M_{V,\mathcal{L}}}$ is $(g+c)\vartheta$, it must be a non-integral divisor. □

As we have proved the existence of theta divisors for transforms with respect to subspaces of any codimension, this shows semistability in some cases not previously treated.

COROLLARY 3.3. *Let $\mathcal{L}$ be a line bundle of degree $d = 2g + 2c$ on a genus g curve C, where $c \in \mathbb{N}$ is any positive integer and $g \geqslant 2$. Then, if $V \subset H^0(C, \mathcal{L})$ is a generic c-codimensional subspace, the transform $M_{V,\mathcal{L}}$ is semistable.*

REMARK 3.4. If C is not hyperelliptic and $\mathcal{L}$ is a degree $d = 2g + 2c$ line bundle, where $d \notin 2(g-1)\mathbb{N}$, then the transform $M_{V,\mathcal{L}}$ of $\mathcal{L}$ with respect to a generic subspace $V \subset H^0(C, \mathcal{L})$ of codimension c admits a reducible theta divisor. In fact the set of points of $\Theta_{M_{V,\mathcal{L}}}$ contains the divisor

$$D = (\omega_C) - C_{g-2} + C \subset \operatorname{Pic}^{g+1}(C),$$

which is irreducible if C is not hyperelliptic, and with cohomology class $(g-1)\vartheta$. As the cohomology class of $\Theta_{M_{V,\mathcal{L}}}$ is $(g+c)\vartheta = (d/2)\vartheta$, it cannot be a multiple of $(g-1)\vartheta$, and so $\Theta_{M_{V,\mathcal{L}}}$ must be reducible.

Hence, if $c \leqslant g$ and $c \neq g-2$, then we have further examples of stable vector bundles (by Theorem 2.8) with reducible theta divisors.

4. *Conclusions*

We have proved the stability of transforms of line bundles with respect to subspaces of low codimension. On the other hand, it is rather easy to show the stability of transforms with respect to subspaces of low dimension: any stable vector bundle M^* of slope $\mu(M^*) > 2g - 1$ is globally generated. Hence we can choose any stable vector bundle M^* of determinant $\mathcal{L}$ and rank r, such that $r < d/(2g-1)$, where $\deg \mathcal{L} = d$. Choosing any generating subspace $V^* \subset H^0(M^*)$ of rank $r+1$, we get an exact sequence

$$0 \longrightarrow \mathcal{L}^* \longrightarrow V^* \otimes \mathcal{O} \longrightarrow M^* \longrightarrow 0.$$

Dualizing, we get an exact sequence

$$0 \longrightarrow M \longrightarrow V \otimes \mathcal{O} \longrightarrow \mathcal{L} \longrightarrow 0,$$

where M is a stable transform of $\mathcal{L}$. Hence, every stable bundle M of rank

$$\operatorname{rk}(M) = r < \frac{d}{2g-1}$$

and determinant $\mathcal{L}^*$ is a stable transform of $\mathcal{L}$. Thus the rational map

$$\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})^*) \dashrightarrow \mathcal{SU}(r, \mathcal{L}), \quad V \longmapsto (M_{V,\mathcal{L}})^*$$

is dominant.

By the same argument we see that there is only one globally generated vector bundle, among the vector bundles of determinant $\mathcal{L}$ and rank $d - g$ with no trivial summands, where $d = \deg \mathcal{L} \geqslant 2g$. Furthermore, this is semistable, and even stable if $d > 2g$. In fact, having such a globally generated bundle N, we can choose a vector space V of global sections of dimension $\operatorname{rk} N + 1$ generating N. This gives rise to the exact sequence

$$0 \longrightarrow \mathcal{L}^* \longrightarrow V \otimes \mathcal{O} \longrightarrow N \longrightarrow 0,$$

and dualizing,

$$0 \longrightarrow N^* \longrightarrow V^* \otimes \mathcal{O} \longrightarrow \mathcal{L} \longrightarrow 0.$$

However, as N is globally generated and has no trivial summands, $H^0(N^*) = 0$. And since V^* and $H^0(\mathcal{L})$ have the same dimension, $V^* \xrightarrow{\sim} H^0(\mathcal{L})$. Hence $N^* = M_{\mathcal{L}}$ is unique.

Table 1. *Transform stability according to rank.*

$\mathrm{rk}(M_{V,\mathcal{L}}) = r$	Stability	Map
$1 \leqslant r < \dfrac{d}{2g-1}$	Stable	$\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})^*) \dashrightarrow \mathcal{SU}(r,\mathcal{L})$ dominant
$\dfrac{d}{2g-1} \leqslant r < d-g-c$	??	??
$d-g-c \leqslant r < d-g$	Stable	$\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})^*) \dashrightarrow \mathcal{SU}(r,\mathcal{L})$
$r = d-g$	Stable	$\{*\} \hookrightarrow \mathcal{SU}(r,\mathcal{L})$

Therefore, when we consider the rational map

$$\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})^*) \dashrightarrow \mathcal{SU}(r,\mathcal{L}), V \longmapsto (M_{V,\mathcal{L}})^*,$$

we are saying that its image is made by globally generated bundles, and we can sum all this up in Table 1, where we suppose that $d > 2g + 2c$, with $1 \leqslant c \leqslant g$.

Here Theorem 2.7 corresponds to the existence of the rational map

$$\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})^*) \dashrightarrow \mathcal{SU}(r,\mathcal{L}).$$

Acknowledgements. I wish to thank my advisor Daniel Huybrechts for his help during my thesis, Arnaud Beauville for his interest and his comments, and my colleagues in Paris and Rome for their friendly advice.

References

1. A. Beauville, 'Some stable vector bundles with reducible theta divisor', *Manuscripta Math.* 110 (2003) 343–349.
2. D. C. Butler, 'Normal generation of vector bundles over a curve', *J. Differential Geom.* 39 (1994) 1–34.
3. L. Ein and R. Lazarsfeld, 'Stability and restrictions of Picard bundles, with an application to the normal bundles of elliptic curves', *Complex projective geometry*, Trieste, 1989, Bergen, 1989, London Mathematical Society Lecture Note Series 179 (Cambridge University Press, Cambridge, 1992) 149–156.
4. G. Farkas, M. Mustaţă and M. Popa, 'Divisors on $\mathcal{M}_{g,g+1}$ and the minimal resolution conjecture for points on canonical curves', *Ann. Sci. École Norm. Sup.* (4) 36 (2003) 553–581.
5. V. Mercat, 'Le problème de Brill–Noether pour des fibrés stables de petite pente', *J. reine angew. Math.* 506 (1999) 1–41.
6. K. Paranjape and S. Ramanan, 'On the canonical ring of a curve', *Algebraic geometry and commutative algebra*, vol. II (Kinokuniya, Tokyo, 1988) 503–516.

Ernesto C. Mistretta
Équipe de Topologie et Géométrie Algébriques
Institut de Mathématiques de Jussieu
175, rue de Chevaleret
75013 Paris
France

ernesto@math.jussieu.fr