Limiti (Esercizi) (05/11/2018)

lunedì 5 novembre 2018 13:47

Es: Calcolare

$$\lim_{x\to+\infty} 2^x \sin\frac{1}{2^x+1}$$

Sol:

$$\lim_{x\to+\infty} 2^x \sin\frac{1}{2^x+1} = \lim_{y\to0}\left(\frac{1}{y}-1\right)\sin y$$

$$y=\frac{1}{2^x+1}$$

$$2^x=\frac{1}{y}-1$$

$$=\lim_{y\to0}\left(\underbrace{\frac{\sin y}{y}}_{\xrightarrow[y\to0]{}1}-\underbrace{\sin y}_{\xrightarrow[y\to0]{}0}\right)=1$$

Esercizio importante

Dimostriamo che $\lim_{x\to0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

Osservo che

$$\lim_{x\to0}\frac{1-\cos x}{x^2}=\lim_{x\to0}\frac{1-\cos x}{x^2}\,\frac{1+\cos x}{1+\cos x}=$$

$(\cos x)^2+(\sin x)^2=1$

$\to\frac{1}{2}$

$$\lim_{x\to 0} \quad x^2 \quad \lim_{x\to 0} \quad x^2 \quad 1+\cos x$$

$(\cos x)^2 + (\operatorname{sen} x)^2 = 1$

$$= \lim_{x\to 0} \frac{1 - }{x^2(1+\cos x)} = \lim_{x\to 0} \underbrace{\frac{(\operatorname{sen} x)^2}{x^2}}_{\to 1} \cdot \overbrace{\frac{1}{1+\cos x}}^{\to \frac{1}{2}} = \frac{1}{2}$$

Esercizio importante

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

Ho $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\operatorname{sen} x}{\cos x} \cdot \frac{1}{x} = \lim_{x\to 0} \overbrace{\frac{\operatorname{sen} x}{x}}^{\to 1} \cdot \underbrace{\frac{1}{\cos x}}_{\to 1} = 1$

Esercizio importante

$$\lim_{x\to 0} \frac{\arctan x}{x} = 1$$

Ho 0 ∈ intervallo di invertibilità della tangente

e inoltre

$$\lim_{x\to 0} \frac{\arctan x}{x} = \lim_{y\to 0} \frac{y}{\tan y} = \lim_{y\to 0} \frac{1}{\frac{\tan y}{y}} = 1$$

$y = \arctan x$

$x = \tan \underbrace{\arctan x}_{y}$

$x = \tan y$

Ho utilizzato il fatto che

$$\lim_{x \to 0} \arctan x = 0$$

infatti $\forall\ \varepsilon > 0 \quad \exists\ \delta > 0 \mid |x| < \delta,\ x \neq 0 \Rightarrow |\arctan x| < \varepsilon$

infatti $|\arctan x| < \varepsilon$ è equivalente a dire

$$-\varepsilon < \arctan x < \varepsilon \qquad (1)$$

Non è restrittivo supporre $\varepsilon \in]0, \pi/2[$

Quando è che vale (1) ?

$$\tan(-\varepsilon) < \tan(\arctan x) < \tan \varepsilon$$

da cui

$$-\tan(\varepsilon) < x < \tan(\varepsilon)$$

Prendere $\delta = \tan(\varepsilon)$ e ottengo

$$|x| < \delta,\ x \neq 0 \Rightarrow |\arctan x| < \varepsilon$$

Limiti (Esercizi) 06/11/2018

martedì 6 novembre 2018 13:39

Esercizio importante

Sia $P(x)$ polinomio, $x_0 \in \mathbb{R}$.

$$\lim_{x \to x_0} P(x) = P(x_0)$$

Abbiamo

$$P(x) = a_0 + a_1 x + \dots + a_n x^n, \text{ dove } n \geq 0$$

è il grado di P

$\forall k \in \mathbb{N}$ $\quad \lim_{x \to x_0} x^k = x_0^k$

$$\lim_{x \to x_0} P(x) = \lim_{x \to x_0} \left(\underset{\downarrow\, a_0}{a_0} + \underset{\downarrow\, 0}{a_1 x} + \dots + \underset{\downarrow\, a}{a_n x^n} \right) = a_0 + a_1 x_0 + \dots + a_n x_0^n$$

$$= P(x_0)$$

Es: Calcolare

$$\lim_{x \to 0} \sqrt{|x|} \cos \frac{1}{x^2}$$

Sol:

$$\lim_{x \to 0} \sqrt{|x|} = 0, \quad \left| \cos\left(\frac{1}{x^2}\right) \right| \leq 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$$

$\to$ ma non ha limite per $x \to 0$

$$\Rightarrow \lim_{x \to 0} \sqrt{|x|} \cos\left(\frac{1}{x^2}\right) = 0$$

#

Es: $\lim_{x \to +\infty} (x + \operatorname{sen} x) = +\infty$

perché $x + \operatorname{sen} x \geq x - 1 \xrightarrow[x \to +\infty]{} +\infty$

Es: $\lim_{x \to +\infty} \ln x (\cos x - 2) = -\infty$

perché $(\ln x)\underbrace{(\cos(x) - 2)}_{\leq -1} \leq -1 \ln x \xrightarrow[\text{per } x \to +\infty]{} -\infty$

Es: $\lim_{x \to +\infty} (3 + \operatorname{sen} x)^x = +\infty$

perché $3 + \operatorname{sen} x \geq 2 \Rightarrow (3 + \operatorname{sen} x)^x \geq 2^x \to +\infty$

Es: $\lim_{x \to 0} (\cos x + 1)^{\ln x} = 0$

perché $0 \leq \cos x + 1 \leq 2 \Rightarrow 0 \leq (\cos x + 1)^{\ln x} \leq 2^{\ln x} \to 0$

MERCOLEDÌ 7/11

Esercizio

Calcolare $\lim_{x\to 0} (\cos x)^{1/x^2}$ $\quad (1^\infty)$

$$\lim_{x\to 0} (\cos x)^{1/x^2} = \lim_{x\to 0} e^{\frac{1}{x^2}\cdot \ln(\cos x)}$$

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x\to 0} \underbrace{\frac{1}{x^2}}_{\varphi(x)} \ln(\underbrace{\cos x}_{1+\varphi(x) \text{ con } \varphi(x)\to 0}) =$$

$$= \lim_{x\to 0} \frac{1}{x^2} \ln(1 + \underbrace{(\cos x - 1)}_{\varphi(x)}) =$$

$$= \lim_{x\to 0} \frac{1}{x^2}\cdot(\cos x - 1) = -\frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

Esercizio

Provare che $\boxed{\lim_{x\to 0} \frac{\arctan(\alpha x)}{x} = \alpha \quad \forall \alpha \in \mathbb{R}}$

Se $\alpha = 0$

$$\lim_{x \to 0} \frac{\arctan(\alpha x)}{x} = 0 \quad (\text{banale!})$$

Se $\alpha \neq 0$

$$\lim_{x \to 0} \frac{\arctan(\alpha x)}{x} = \lim_{y \to 0} \alpha \underbrace{\frac{\arctan y}{y}}_{\to 1} = \alpha$$

$\downarrow$

$y = \alpha x$

In modo analogo

$$\boxed{\lim_{x \to 0} \frac{\arcsin(\alpha x)}{x} = \alpha \quad \forall \alpha \in \mathbb{R}}$$

Esercizio

Calcolare $\displaystyle\lim_{x \to +\infty} \left(\frac{1 + |\sin x|}{x} \right)^x$

$\frac{1 + |\sin x|}{x} \to 0$ per $x \to +\infty$

$$0 \leq \frac{1 + |\sin x|}{x} \leq \frac{2}{x} \quad \text{per } x > 0$$

$0 \to 0$, $\frac{1+|\sin x|}{x} \to 0$, $\frac{2}{x} \to 0$

$\varphi, g : X \to \mathbb{R}$

$\varphi(x) \to 0$ per $x \to x_0$

$g(x) \to +\infty$ per $x \to x_0$

$\varphi(x) > 0$

$$\lim_{x \to x_0} \varphi(x)^{g(x)}$$

$$\lim_{x \to x_0} \varphi(x)^{g(x)} = \lim_{x \to x_0} e^{g(x) \ln \varphi(x)} \overset{y = g(x)\ln\varphi(x)}{=} \lim_{y \to -\infty} e^{y} = 0$$

$$\lim_{x \to x_0} \underset{\downarrow \atop +\infty}{g(x)} \underset{\to -\infty}{\ln \varphi(x)} = -\infty$$

$$\Rightarrow \lim_{x \to +\infty} \left(\frac{1 + |\operatorname{sen} x|}{x} \right)^{x} = 0$$

Esercizio

Calcolare $\displaystyle\lim_{x \to 0} \frac{\tan x - \operatorname{sen} x}{x^3} = \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\tan x}{x^3} - \frac{\operatorname{sen} x}{x^3} \qquad [+\infty - \infty] ??$$

$$\frac{\tan x}{x} \cdot \frac{1}{x^2} \to +\infty \qquad \frac{\operatorname{sen} x}{x} \cdot \frac{1}{x^2} \to +\infty$$

$$\lim_{x \to 0} \frac{\tan x - \operatorname{sen} x}{x^3} = \lim_{x \to 0} \frac{\frac{\operatorname{sen} x}{\cos x} - \operatorname{sen} x}{x^3} =$$

$$\lim_{x\to 0} \frac{\operatorname{sen} x \left(\frac{1}{\cos x} - 1\right)}{x^3} =$$

$$= \lim_{x\to 0} \frac{\operatorname{sen} x}{x} \cdot \frac{\frac{1}{\cos x} - 1}{x^2} =$$

$$= \lim_{x\to 0} \frac{\operatorname{sen} x}{x} \frac{\frac{1 - \cos x}{\cos x}}{x^2} =$$

$$= \lim_{x\to 0} \underbrace{\frac{\operatorname{sen} x}{x}}_{\downarrow \; 1} \cdot \underbrace{\frac{1}{\cos x}}_{\downarrow \; 1} \cdot \underbrace{\frac{1-\cos x}{x^2}}_{\downarrow \; 1/2} = \frac{1}{2}$$

Esercizio

Calcolare $\lim_{x\to 0} \left(\operatorname{sen} x^2\right)^{\frac{1}{\ln x^2}}$ $\quad (0^0)$

$$\lim_{x\to 0} \left(\operatorname{sen} x^2\right)^{\frac{1}{\ln x^2}} = \lim_{x\to 0} e^{\frac{\ln \operatorname{sen} x^2}{\ln x^2}}$$

$$\lim_{x\to 0} \frac{\ln(\operatorname{sen} x^2)}{\ln x^2} = \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\ln\left(\frac{\operatorname{sen} x^2}{x^2} \cdot x^2\right)}{\ln x^2} =$$

$$= \lim_{x \to 0} \frac{\ln x^2 + \ln \frac{\operatorname{sen} x^2}{x^2}}{\ln x^2} =$$

$$= \lim_{x \to 0} \left[1 + \underbrace{\frac{1}{\ln x^2} \cdot \ln\left(\frac{\operatorname{sen} x^2}{x^2}\right)}_{\downarrow \, 0} \right] = 1$$

$$\lim_{x \to 0} (\operatorname{sen} x^2)^{1/\ln x^2} = e$$

Esercizio

Calcolare

$$\lim_{x \to -\infty} (\overbrace{1 + 3\operatorname{sen} e^x}^{\to 1})^{\overbrace{\cos(e^{-x^2}) + \frac{1}{\operatorname{sen} e^x}}^{\to +\infty}}$$

$$\lim_{x \to -\infty} \operatorname{sen} e^x \underset{\substack{\downarrow \\ y = e^x}}{=} \lim_{y \to 0} \operatorname{sen} y = 0$$

$$\lim_{x \to -\infty} \cos(e^{-x^2}) \underset{\substack{\downarrow \\ y = e^{-x^2}}}{=} \lim_{y \to 0} \cos y = 1$$

$$\lim_{x \to -\infty} \left(1 + 3\,\mathrm{sen}\, e^{x}\right)^{\cos(e^{-x^2}) + \frac{1}{\mathrm{sen}\, e^{x}}} =$$

$$= \lim_{x \to -\infty} \left(1+3\,\mathrm{sen}\, e^{x}\right)^{\cos(e^{-x^2})} \cdot \left(1+3\,\mathrm{sen}\, e^{x}\right)^{\frac{1}{\mathrm{sen}\, e^{x}}} = e^3$$

$$\lim_{x \to -\infty} \left(1 + 3\,\mathrm{sen}\, e^{x}\right)^{\cos(e^{-x^2})} = 1$$

$$\lim_{x \to -\infty} \left(1 + 3\,\mathrm{sen}\, e^{x}\right)^{\frac{1}{\mathrm{sen}\, e^{x}}} = \lim_{y \to 0} (1+3y)^{1/y} = e^3$$

$$y = \mathrm{sen}\, e^{x} \to 0$$

<u>Esercizio</u>

Calcolare $\displaystyle \lim_{x \to +\infty} \left(1 + \frac{\arctan x}{x}\right)^{x}$

$$\lim_{x \to +\infty} \left(1 + \frac{\arctan x}{x}\right)^{x} =$$

$$y = \frac{x}{\arctan x} \to +\infty \qquad \varphi(x)$$

$$= \lim_{y \to +\infty} \left(1 + \frac{1}{y}\right)^{\varphi^{-1}(y)} = \qquad x = \varphi^{-1}(y)$$

$$= \lim_{y \to +\infty} \left[\left(1 + \frac{1}{y}\right)^{y}\right]^{\frac{\varphi^{-1}(y)}{y}} \qquad \frac{\varphi^{-1}(y)}{y} \to \frac{\pi}{2}$$

$$\lim_{x\to+\infty}\left(1+\frac{\operatorname{arctan}x}{x}\right)^{x}=$$

$$=\lim_{x\to+\infty}\left[\left(1+\frac{\operatorname{arctan}x}{x}\right)^{\frac{x}{\operatorname{arctan}x}}\right]^{\operatorname{arctan}x}=e^{\pi/2}$$

$$\downarrow$$

e (si pone $y=\frac{x}{\operatorname{arctan}x}$)

$$\lim_{x\to+\infty}\left(1+\frac{\operatorname{arctan}x}{x}\right)^{x}=\lim_{x\to+\infty}e^{x\ln\left(1+\frac{\operatorname{arctan}x}{x}\right)}$$

$$e^{\lim\limits_{x\to+\infty}\underbrace{x}_{f(x)}\ln\left(1+\underbrace{\frac{\operatorname{arctan}x}{x}}_{\varphi(x)\to0}\right)}=$$

$$=\lim_{x\to+\infty}x\cdot\frac{\operatorname{arctan}x}{x}=\frac{\pi}{2}$$

Esercizio Calcolare

$$\lim_{x\to+\infty}\left(\frac{x^4+2x^3+x}{x^4+x^2}\right)^{3x}\qquad(1^{\infty})$$

$$\lim_{x\to+\infty}\left(\frac{x^4+2x^3+x}{x^4+x^2}\right)^{3x} =$$

$$= \lim_{x\to+\infty} e^{3x\ln\left(\frac{x^4+2x^3+x}{x^4+x^2}\right)}$$

$$\lim_{x\to+\infty} 3x\ln\left(1+\left(\frac{x^4+2x^3+x}{x^4+x^2}-1\right)\right) =$$

$$= \lim_{x\to+\infty} 3x \ln\left(1+\frac{2x^3-x^2+x}{x^4+x^2}\right) =$$

$g(x)$

$\phi(x) \to 0$ per $x\to+\infty$

$$= \lim_{x\to+\infty} 3x\cdot\frac{2x^3-x^2+x}{x^4+x^2} =$$

$$= \lim_{x\to+\infty}\frac{6x^4-3x^3+3x^2}{x^4+x^2} = 6$$

$$\Rightarrow \lim_{x\to+\infty}\left(\frac{x^4+2x^3+x}{x^4+x^2}\right)^{3x} = e^6$$

Esercizio

Calcolare $\quad \displaystyle\lim_{x\to 0}\frac{e^x-1+\operatorname{sen} x}{(1+x)^2-1+\tan x}$

$$\lim_{x\to 0} \frac{e^x - 1 + \operatorname{sen} x}{(1+x)^2 - 1 + \tan x} = \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\dfrac{e^x - 1 + \operatorname{sen} x}{x}}{\dfrac{(1+x)^2 - 1 + \tan x}{x}} =$$

$$= \lim_{x\to 0} \frac{\dfrac{e^x - 1}{x} + \dfrac{\operatorname{sen} x}{x}}{\dfrac{(1+x)^2 - 1}{x} + \dfrac{\tan x}{x}} = \frac{2}{3}$$

Esercizio Sia $\alpha \in \mathbb{R}$. Calcolare

$$\lim_{x\to 0} \frac{e^{\alpha x} - \sqrt{1+x}}{\tan x} = \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{(e^{\alpha x} - 1) - (\sqrt{1+x} - 1)}{\tan x} =$$

$$= \lim_{x\to 0} \frac{\dfrac{e^{\alpha x} - 1}{x} - \dfrac{\sqrt{1+x} - 1}{x}}{\dfrac{\tan x}{x}} = \alpha - \frac{1}{2}$$

Per casa

- $\lim_{x \to +\infty} x\left(a^{1/x} - 1\right) = \ln a \quad , a > 0$

- $\lim_{x \to 0} \frac{a^x - b^x}{x} = \ln \frac{a}{b} \quad a, b > 0$