Successioni esercizi

Giovedì 15/11

Esercizio Calcolare

$$\lim_{n} \frac{(n-2)!\, n^{n+2} - (n+1)!\, n^{n-1}}{n^n \left[(n-2)! + 3^n\right]} =$$

$$= \lim_{n} \frac{n^{n-1} \cdot \left[(n-2)!\, n^3 - (n+1)!\right]}{n^n \left[(n-2)! + 3^n\right]} =$$

$$= \lim_{n} \frac{(n-2)!\, n^3 - (n+1)!}{n \left[(n-2)! + 3^n\right]} =$$

$(n+1)\cdot n \cdot (n-1) \cdot$
$\cdot (n-2)!$

$$= \lim_n \frac{(n-2)!\,[n^3 - \overbrace{(n+1)\cdot n(n-1)}^{n(n^2-1)}]}{n\,[(n-2)! + 3^n]} =$$

$$= \lim_n \frac{(n-2)!\, n}{n\,[(n-2)! + 3^n]} =$$

$$= \lim_n \frac{(n-2)!}{(n-2)! + 3^n} \overset{PdS}{=} 1$$

$$\frac{(n-2)!}{3^n} = \frac{(n-2)!}{3^{n-2}} \cdot \frac{1}{9} \qquad \frac{(n-2)!}{3^{n-2}} \to +\infty \qquad 3^n = o((n-2)!)$$

Esercizio Calcolare $\quad , \quad \alpha > 0$

$$\lim_n \frac{n^{n+1} + 3(n+1)^{n+1}}{n^n + n!} \operatorname{sen}\left(\frac{\pi}{n}\right)^\alpha$$

$$n^n + n! = n^n + o(n^n)$$

$$\lim_n 3 \frac{(n+1)^{n+1}}{n^{n+1}} = \lim_n 3 \left(1+\frac{1}{n}\right)^{n+1} = 3e$$

$$\frac{(n+1)^{n+1}}{n^{n+1}} = \left(\frac{n+1}{n}\right)^{n+1}$$

$$\Downarrow$$

$$3(n+1)^{n+1} =$$

$$= 3e\,n^{n+1} + o(n^{n+1})$$

$$\lim_n \frac{n^{n+1} + 3(n+1)^{n+1}}{n^n + n!} \operatorname{sen}\left(\frac{\pi}{n}\right)^\alpha =$$

$$= \lim_n \frac{n^{n+1} + 3e\, n^{n+1} + o(n^{n+1})}{n^n + o(n^n)} \operatorname{sen}\left(\frac{\pi}{n}\right)^\alpha =$$

Pd)
$$= \lim_n \frac{(3e+1)\, n^{n+1}}{n^n} \operatorname{sen}\left(\frac{\pi}{n}\right)^\alpha =$$

$$= \lim_n (3e+1)\, n \operatorname{sen}\left(\frac{\pi}{n}\right)^\alpha =$$

$\to 0$ per $n \to +\infty$

$$= \lim_n (3e+1)\, n \left[\left(\frac{\pi}{n}\right)^\alpha + o\left(\left(\frac{\pi}{n}\right)^\alpha\right)\right] =$$

$$= \lim_n (3e+1) \cdot n \cdot \frac{\pi^\alpha}{n^\alpha} =$$

$$= \begin{cases} +\infty & \text{se } \alpha < 1 \\ (3e+1)\pi & \text{se } \alpha = 1 \\ 0 & \text{se } \alpha > 1 \end{cases}$$

<u>Esercizio</u> Sia $\alpha > 0$. Calcolare

$$\lim_n \frac{\ln (n+5)! - \ln (n! + 5)}{\ln\left(2n^\alpha + \underbrace{\cos(n\pi)}_{=(-1)^n}\right)}$$

Numeratore

$$\ln (n+5)! - \ln (n!+5) =$$

$$= \ln \frac{(n+5)!}{n!+5} =$$

$$= \ln \Big[\underbrace{(n+5)(n+4)(n+3) \cdot (n+2) \cdot (n+1)\, n!}_{(n+5)!} \frac{1}{n!+5} \Big] =$$

$$= \ln \left[(n+5)(n+4) \cdots (n+1) \right] + \ln \left(\frac{n!}{n!+5} \right) =$$

$$= \ln \left[n^5 \cdot \frac{(n+5) \cdots (n+1)}{n^5} \right] + \ln \left(\frac{n!}{n!+5} \right) =$$

$$= \ln n^5 + \underbrace{\ln \left(\frac{(n+5) \cdot (n+4) \cdots (n+1)}{n^5} \right)}_{\downarrow 0} + \underbrace{\ln \left(\frac{n!}{n!+5} \right)}_{\downarrow 0}$$

$$= 5 \ln n + o(\ln n)$$

Denominatore

$$\ln \left(2n^\alpha + (-1)^n \right) = \ln \left(n^\alpha \left(2 + \frac{(-1)^n}{n^\alpha} \right) \right) =$$

$$= \ln n^\alpha + \underbrace{\ln \left(2 + \frac{(-1)^n}{n^\alpha} \right)}_{\downarrow \ln 2 \text{ per } n \to +\infty} = \alpha \ln n + o(\ln n)$$

$$\lim_{n} \frac{\ln (n+5)! - \ln (n!+5)}{\ln\left(2n^{\alpha} + \cos(n\pi)\right)} =$$

$$= \lim_{n} \frac{5 \ln n + o(\ln n)}{\alpha \ln n + o(\ln n)} \stackrel{PdS}{=} \frac{5}{\alpha}$$

Es: Calcolare

$$\lim_{n\to+\infty} \frac{(2n)! - n^n \cos n}{n^{2n} - (\operatorname{sen} n + 2)(3n)!}$$

Sol: $\boxed{\text{Numeratore}}$.

Proviamo a calcolare $\lim_{n\to+\infty}\frac{(2n)!}{n^n} = +\infty$

$$\frac{(2n)!}{n^n} = \frac{\overbrace{2n\cdot(2n-1)\cdot(2n-2)\cdots(n+1)}^{n \text{ fattori}}\cdot n!}{\underbrace{n\cdot n\cdots n}_{n\text{ fattori}}} = \underbrace{\frac{2n}{n}}_{\geq 1}\,\underbrace{\frac{(2n-1)}{n}}_{\geq 1}\cdots\underbrace{\frac{n}{n}}_{\geq 1}\cdot n! \geq n!$$

$\Rightarrow n^n\cos n = o((2n)!)$ perché $\lim_{n\to+\infty}\frac{n^n\cos n}{(2n)!} = \lim_n \frac{n^n}{(2n)!}\cos n = 0$

$\Rightarrow$ numeratore: $(2n)! + o((2n)!)$

$\boxed{\text{Denominatore}}$ Ho $n^{2n} - \underbrace{(\operatorname{sen} n+2)}_{\geq 1}(3n)!$. Mostro che

$\lim_n \frac{(3n)!}{n^{2n}} = +\infty$. Infatti

$$\frac{(3n)!}{n^{2n}} = \frac{\overbrace{3n\cdot(3n-1)\cdot(3n-2)\cdots(n+2)(n+1)}^{2n\text{ fattori}}\cdot n!}{\underbrace{n\cdot n\cdots n}_{2n\text{ fattori}}} = \overbrace{\underbrace{\frac{3n}{n}}_{\geq 1}\,\underbrace{\frac{(3n-1)}{n}}_{\geq 1}\cdots\underbrace{\frac{n}{n}}_{\geq 1}}^{2n\text{ fattori}}\; n!$$

$\Rightarrow n^{2n} = o((3n)!) = o((\operatorname{sen} n+2)(3n)!) \iff \lim_n \frac{n^{2n}}{(\operatorname{sen} n+2)(3n)!} = 0$

Il limite si risolve

$$\lim_{n} \frac{(2n)! + o((2n)!)}{-(3n)!(\ln n+2) + o((3n)!(\ln n+2))} \overset{PdS}{=} \lim_{n} -\frac{(2n)!}{(3n)!(\ln n+2)} =$$

$$= \lim_{n\to+\infty} - \frac{\cancel{2n!}}{3n\cdot(3n-1)\cdots(2n+1)\cdot\cancel{2n!}\,(\ln n+2)} =$$

$$= \lim_{n} - \underset{\substack{\vee\\ 1}}{\frac{1}{\ln n+2}} \cdot \underset{\substack{\downarrow\\ 0}}{\frac{1}{3n\cdot(3n-1)\cdots(2n+1)}} = 0 \qquad \#$$

<u>Formula di De Moivre-Stirling</u>:

$\forall n \geq 1\ \exists\, a_n \in\,]0,1[$ tale che

$$n! = n^n e^{-n}\sqrt{2\pi n}\; e^{a_n/12n}$$

$$\Rightarrow \lim_{n} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

$$\Rightarrow n! = n^n e^{-n}\sqrt{2\pi n} + o\left(n^n e^{-n}\sqrt{2\pi n}\right)$$

Es: Calcolare $\lim_n \frac{n!\, e^n \sqrt{n+1}}{(n+1)^{1/n}(n^{n+1} + \operatorname{sen} n)}$

Sol: $\boxed{\text{numeratore}}$ Utilizziamo la formula di De Moivre - Stirling

Otteniamo

$$n!\, e^n \sqrt{n+1} = \left(n^n e^{-n}\sqrt{2\pi n} + o\left(n^n e^{-n}\sqrt{2\pi n}\right)\right) e^n \sqrt{n+1}$$

$$= n^n \sqrt{2\pi n(n+1)} + \left(n^n \sqrt{2\pi n(n+1)}\right)$$

$\boxed{\text{denominatore}}$ Osservo che $\lim_{n\to+\infty} (n+1)^{1/n} = \lim_n e^{\frac{1}{n}\ln(n+1)} = 1$

Quindi

$$(n+1)^{1/n}(n^{n+1} + \operatorname{sen} n) = n^{n+1} + o(n^{n+1}) \quad \text{perché}$$

$$\lim_{n\to+\infty} \frac{\underbrace{(n+1)^{1/n}}_{\to 1}(n^{n+1} + \operatorname{sen} n)}{n^{n+1}} = \lim_n \underbrace{(n+1)^{1/n}}_{\downarrow 1} \cdot \frac{n^{n+1} + \operatorname{sen} n}{n^{n+1}} = 1$$

Il limite è ricondotto a

$$\lim_n \frac{n^n \sqrt{2\pi n(n+1)} + o\left(n^n \sqrt{2\pi n(n+1)}\right)}{n^{n+1} + o(n^{n+1})} \overset{\text{Pols}}{=} \lim_n \frac{n^n \sqrt{2\pi n(n+1)}}{n^{n+1}} =$$

$$= \lim_n \underbrace{\frac{\sqrt{2\pi n(n+1)}}{n}}_{\sqrt{2\pi \cdot \frac{n^2+n}{n^2}}} = \sqrt{2\pi}$$

$$\sqrt{2\pi \cdot \frac{n^2+n}{n^2}} \quad \frac{n^2+n}{n^2} \to 1$$