INTEGRALI
ESERCIZI

LUNEDÌ 7/1

Integrali immediati

$$\int \frac{x+1}{\sqrt{1-2x^2}}\,dx = \underbrace{-\frac{1}{4}\int \frac{\overset{D(1-2x^2)}{\boxed{-4x}}}{\sqrt{1-2x^2}}\,dx}_{=} + \int \frac{1}{\sqrt{1-2x^2}}\,dx$$

$$-\frac{1}{4}\,\frac{1}{1-\frac{1}{2}}\,(1-2x^2)^{1-\frac{1}{2}} + c = -\frac{1}{4}\int D(1-2x^2)\cdot(1-2x^2)^{-1/2}\,dx$$

$$\int f'(x)\,(f(x))^{\alpha}\,dx = \frac{1}{\alpha+1}\,(f(x))^{\alpha+1} + c$$

$$f(x) = 1-2x^2,\ \alpha = -\frac{1}{2}$$

$$\int \frac{x+1}{\sqrt{1-2x^2}}\,dx = -\frac{1}{2}\sqrt{1-2x^2} + \int \frac{1}{\sqrt{1-2x^2}}\,dx$$

$$D\,\text{arcsen}\,t = \frac{1}{\sqrt{1-t^2}}$$

$$\int \frac{1}{\sqrt{1-2x^2}}\,dx = \int \frac{1}{\sqrt{1-(\sqrt{2}x)^2}}\,dx =$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sqrt{2}}{\sqrt{1-(\sqrt{2}x)^2}}\,dx = \frac{1}{\sqrt{2}}\,\operatorname{arcsen}(\sqrt{2}x) + c$$

$$D\,\operatorname{arcsen}(\sqrt{2}x) = \frac{1}{\sqrt{1-2x^2}} \cdot \sqrt{2}$$

$$\int \frac{x+1}{\sqrt{1-2x^2}}\,dx = -\frac{1}{2}\sqrt{1-2x^2} + \frac{1}{\sqrt{2}}\,\operatorname{arcsen}(\sqrt{2}x) + c$$

- $\displaystyle\int \operatorname{sen}^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx =$

$\hookrightarrow \cos(2x) = \cos^2 x - \operatorname{sen}^2 x = 1 - 2\operatorname{sen}^2 x$

$$= \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos(2x)\,dx =$$

$$= \frac{1}{2}x - \frac{1}{2}\,\frac{1}{2}\int 2\cos(2x)\,dx =$$

$$D\,\operatorname{sen}(2x) = 2\cos(2x)$$

$$= \frac{1}{2}x - \frac{1}{4}\operatorname{sen}(2x) + c$$

- $\alpha \in \mathbb{R}, \quad \alpha \neq 0$

$$\int \frac{1}{1+\alpha^2x^2}\,dx = \int \frac{1}{1+(\alpha x)^2}\,dx =$$

$$D\,\operatorname{arctan}(\alpha x) = \alpha\,\frac{1}{1+(\alpha x)^2}$$

$$= \frac{1}{\alpha} \int \frac{\alpha}{1 + (\alpha x)^2}\, dx = \frac{1}{\alpha} \arctan(\alpha x) + c$$

- $$\int \frac{\cos x}{\sqrt{1 - 2\,\text{sen}\, x}}\, dx =$$

$$= -\frac{1}{2} \int (-2) \cos x \left(1 - 2\,\text{sen}\, x\right)^{-1/2} dx =$$

$\hookrightarrow D(1 - 2\,\text{sen}\, x) = -2\cos x$

$\hookrightarrow \int f'(x)\left(f(x)\right)^{\alpha} dx$

$\alpha = -\frac{1}{2}$

$$= -\frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} \left(1 - 2\,\text{sen}\, x\right)^{1 - \frac{1}{2}} + c =$$

$$= -\sqrt{1 - 2\,\text{sen}\, x} + c$$

- $$\int \frac{1}{1 + \text{sen}\, t}\, dt = \int \frac{1 - \text{sen}\, t}{(1 + \text{sen}\, t)(1 - \text{sen}\, t)}\, dt =$$

$$= \int \frac{1 - \text{sen}\, t}{1 - \text{sen}^2 t}\, dt = \int \frac{1 - \text{sen}\, t}{\cos^2 t}\, dt =$$

$$= \int \frac{1}{\cos^2 t}\, dt + \int \frac{(-\text{sen}\, t)}{\cos^2 t}\, dt =$$

" $\int f'(t)\left(f(t)\right)^{\alpha} dt$

$f(t) = \cos t$

$\alpha = -2$

$$= \tan t + \frac{1}{1 - 2} \cdot (\cos t)^{1-2} + c$$

$$\overset{!}{=} \tan t - \frac{1}{\cos t} + c$$

$$\cdot \int \frac{1}{\sqrt{x^2-9}}\, dx \quad , \quad \boxed{x > 3}$$

cerco primitive di $\frac{1}{\sqrt{x^2-9}}$ nell'intervallo $]3,+\infty[$

$$D \operatorname{settcosh} t = \frac{1}{\sqrt{t^2-1}}$$

$$\int \frac{1}{\sqrt{x^2-9}} = \int \frac{1}{3} \frac{1}{\sqrt{\left(\frac{x}{3}\right)^2-1}}\, dx =$$

$$D \operatorname{settcosh} \frac{x}{3} = \frac{1}{3} \frac{1}{\sqrt{\left(\frac{x}{3}\right)^2-1}}$$

$$= \operatorname{settcosh} \frac{x}{3} + c$$

Integrazione per parti

$$\cdot \int x^2 \ln x \, dx = \int \ln x \cdot D\left(\frac{1}{3} x^3\right) dx =$$

$$= \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^3 \cdot D(\ln x)\, dx =$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^3 \cdot \frac{1}{x}\, dx =$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + c$$

$$\cdot \int x^2 e^{2x}\,dx = \int x^2 \cdot D\left(\frac{1}{2}e^{2x}\right)dx =$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}\int e^{2x} D(x^2)\,dx =$$

$$= \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx =$$

$$= \frac{1}{2}x^2 e^{2x} - \int x\, D\left(\frac{1}{2}e^{2x}\right)dx =$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x}\,dx\right] =$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + c$$

$$\cdot \underbrace{\boxed{\int sen(2x)\, e^x\,dx}}_{\alpha} = \int sen(2x) \cdot D(e^x)\,dx =$$

$$= sen(2x)\, e^x - \int e^x D(sen(2x))\,dx =$$

$$= e^x sen(2x) - 2\int e^x \cos(2x)\,dx =$$

$$= e^x sen(2x) - 2\left[e^x \cos(2x) + 2\int e^x sen(2x)\,dx\right] =$$

$$= \boxed{e^x sen(2x) - 2e^x \cos(2x) - 4\int e^x sen(2x)\,dx}$$

$$\alpha = e^x sen(2x) - 2e^x \cos(2x) - 4\alpha$$

$$\Rightarrow \quad 5\int e^x \operatorname{sen}(2x)\,dx = e^x \operatorname{sen}(2x) - 2e^x\cos(2x)$$

$$\Rightarrow \int e^x \operatorname{sen}(2x)\,dx = \frac{1}{5}\, e^x\left(\operatorname{sen}(2x) - 2\cos(2x)\right) + c$$

fattore differenziale

$$\bullet \int x\,(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x^2 \cdot \ln x \cdot \frac{1}{x}\,dx =$$

$$= \frac{1}{2}x^2(\ln x)^2 - \int x \ln x\,dx =$$

$$= \frac{1}{2}x^2(\ln x)^2 - \left[\frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\cdot\frac{1}{x}\,dx\right] =$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + c$$

$$\bullet \int x \arctan(2x)\,dx =$$

$$= \frac{1}{2}x^2\arctan(2x) - \int x^2\,\frac{1}{1+4x^2}\,dx =$$

$$= \frac{1}{2}x^2\arctan(2x) - \frac{1}{4}\int \frac{4x^2+1-1}{1+4x^2}\,dx =$$

$$= \frac{1}{2}x^2\arctan(2x) - \frac{1}{4}\int\left(1 - \frac{1}{1+4x^2}\right)dx =$$

$$= \frac{1}{2}x^2\arctan(2x) - \frac{1}{4}x + \frac{1}{4}\,\frac{1}{2}\int \frac{2}{1+(2x)^2}\,dx =$$

$$= \frac{1}{2}x^2\arctan(2x) - \frac{1}{4}x + \frac{1}{8}\arctan(2x) + c$$

$$\cdot \int \operatorname{arcsen} x \, dx = \int \underset{\substack{\downarrow \\ \text{fattore differenziale}}}{1} \cdot \operatorname{arcsen} x \, dx =$$

$$= x \operatorname{arcsen} x + \frac{1}{2} \int \frac{\boxed{-2x}}{\sqrt{1-x^2}} \, dx = \qquad \boxed{-2x} \longrightarrow D(1-x^2)$$

$$= x \operatorname{arcsen} x + \frac{1}{2} \frac{1}{1-\frac{1}{2}} (1-x^2)^{1-1/2} + c =$$

$$= x \operatorname{arcsen} x + \sqrt{1-x^2} + c$$

Integrazione per sostituzione

$$\cdot \int_0^1 \frac{\sqrt{1-x^2}}{1+x} \, dx =$$

$x = \cos t$

$t = \arccos x \quad , \quad t \in [0, \frac{\pi}{2}]$

$$= \int_{\frac{\pi}{2}}^{0} \frac{\sqrt{1-\cos^2 t}}{1+\cos t} \, (\boxed{-} \operatorname{sen} t) \, dt =$$

$\boxed{-}$ → lo sfrutto per invertire gli estremi di integrazione

$$= \int_0^{\pi/2} \frac{\sqrt{1-\cos^2 t}}{1+\cos t} \cdot \operatorname{sen} t \, dt =$$

$$= \int_0^{\pi/2} \frac{\sqrt{\operatorname{sen}^2 t}}{1+\cos t} \operatorname{sen} t \, dt = \int_0^{\pi/2} \frac{\boxed{|\operatorname{sen} t|}}{1+\cos t} \operatorname{sen} t \, dt =$$

$= \operatorname{sen} t$ per $t \in [0, \frac{\pi}{2}]$

$$= \int_0^{\pi/2} \frac{\text{sen}^2 t}{1+\cos t}\, dt = \int_0^{\pi/2} \frac{1-\cos^2 t}{1+\cos t}\, dt =$$

$$= \int_0^{\pi/2} \frac{(1-\cos t)(1+\cos t)}{1+\cos t}\, dt =$$

$$= \int_0^{\pi/2} (1-\cos t)\, dt = (t-\text{sen}\, t)\Big|_0^{\pi/2} =$$

$$= \frac{\pi}{2} - 1$$

integriamo per parti

- $$\int_{-1}^{0} x \arcsin(x+1)\, dx = \underbrace{\frac{1}{2} x^2 \arcsin(x+1)\Big|_{-1}^{0}}_{\overset{\shortparallel}{0}} +$$

$$- \frac{1}{2} \int_{-1}^{0} x^2 \frac{1}{\sqrt{1-(x+1)^2}}\, dx = \downarrow$$

$x+1 = \cos t$

$t = \arccos(x+1)$

$0 \leq x+1 \leq 1 \quad , t \in \left[0, \frac{\pi}{2}\right]$

$$= -\frac{1}{2} \int_{\frac{\pi}{2}}^{0} \frac{(\cos t - 1)^2}{\boxed{\sqrt{1-\cos^2 t}}} \cdot (-\text{sen}\, t)\, dt =$$

$= |\text{sen}\, t| = \text{sen}\, t$ perché $t \in \left[0, \frac{\pi}{2}\right]$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{0} (\cos t - 1)^2\, dt = \frac{1}{2} \int_{\frac{\pi}{2}}^{0} \Big(\underset{\overset{\shortparallel}{\frac{1+\cos(2t)}{2}}}{\cos^2 t} - 2\cos t + 1\Big)\, dt =$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{0}\left(\frac{1}{2}\cos(2t) - 2\cos t + \frac{3}{2}\right)dt =$$

$= \dots$ per casa

- $$\int \frac{e^x}{e^{2x}+5e^x+6}\,dx =$$

$\downarrow$ $e^x = t$

$x = \ln t$

$$= \left[\int \frac{t}{t^2+5t+6}\cdot\frac{1}{t}\,dt\right] = \int \frac{1}{t^2+5t+6}\,dt =$$

$\downarrow$ $t = e^x$ Vogliamo una primitiva in x

$$= \int \frac{1}{(t+2)(t+3)}\,dt = \int\left[\frac{1}{t+2} - \frac{1}{t+3}\right]dt$$

$\downarrow$ $\frac{1}{(t+2)(t+3)} = \frac{A}{t+2} + \frac{B}{t+3}$, per qualche $A, B \in \mathbb{R}$ da calcolare

$(A = 1\ ,\ B = -1)$

$$= \int \frac{1}{t+2}\,dt - \int \frac{1}{t+3}\,dt = \left[\ln|t+2| - \ln|t+3| + c\right]$$

$\downarrow$ $t = e^x$

$$= \ln|e^x+2| - \ln(e^x+3) + c = \ln\left(\frac{e^x+2}{e^x+3}\right) + c.$$

- $$\int_0^{\pi/2} \frac{1}{2\cos x + \operatorname{sen} x + 3}\, dx$$

$$\operatorname{sen} x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} \quad , \quad \cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$$

$$\int_0^{\pi/2} \frac{1}{2\cos x + \operatorname{sen} x + 3}\, dx =$$

$$= \int_0^{\pi/2} \frac{1}{2\,\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} + \frac{2\tan(x/2)}{1+\tan^2(x/2)} + 3}\, dx =$$

$$= \int_0^{\pi/2} \frac{1+\tan^2(x/2)}{\tan^2(x/2)+2\tan(x/2)+5}\, dx =$$

$\downarrow$

$t = \tan\frac{x}{2}$

$x = 2\arctan t$

$$= \int_0^{1} \frac{1+t^2}{t^2+2t+5}\,\frac{2}{1+t^2}\, dt = 2\int_0^1 \frac{1}{\boxed{t^2+2t+5}}\, dt$$

$\checkmark$ polinomio di II° grado irriducibile, $\Delta < 0$

$$t^2+2t+5 = t^2+2t+1+4 = (t+1)^2+4$$

$$= 2\int_0^1 \frac{1}{4+(t+1)^2}\, dt = \frac{2}{4}\int_0^1 \frac{1}{1+\left(\frac{t+1}{2}\right)^2}\, dt =$$

$$= \int_0^1 \frac{1}{2} \frac{1}{1+\left(\frac{t+1}{2}\right)^2}\, dt = \arctan\left(\frac{t+1}{2}\right)\Big|_0^1 =$$

$$D\left(\frac{t+1}{2}\right) = \frac{1}{2}$$

$$= \frac{\pi}{4} - \arctan\frac{1}{2}$$

Integrazione di funzioni razionali fratte

Calcolare

$$\int \frac{P(x)}{Q(x)}\, dx \quad , \text{ con } P \text{ e } Q \text{ polinomi}$$

I° caso

$$\int \frac{1}{ax^2+bx+c}\, dx \quad \text{con } \Delta = b^2-4ac < 0$$

$\Rightarrow ax^2+bx+c$ è un polinomio irriducibile su $\mathbb{R}$ (cioè non è prodotto di polinomi di I° grado a coefficienti reali)

$$\int \frac{1}{\underset{4}{\boxed{4t^2+4t+10}}}\, dt = \int \frac{1}{(2t+1)^2+9}\, dt =$$

$$4t^2+4t+1+9 = (2t+1)^2+9$$

$$= \int \frac{1}{9} \frac{1}{1+\left(\frac{2t+1}{3}\right)^2} \, dt =$$

$$\downarrow$$

$$D \arctan\left(\frac{2t+1}{3}\right) =$$

$$= \frac{1}{9} \cdot \frac{3}{2} \arctan\left(\frac{2t+1}{3}\right) + c = \qquad = \frac{2}{3} \cdot \frac{1}{1+\left(\frac{2t+1}{3}\right)^2}$$

$$= \frac{1}{6} \arctan\left(\frac{2t+1}{3}\right) + c$$

- $$\int \frac{x+1}{x^2+4x+5} \, dx = \frac{1}{2} \int \frac{2x+2+2-2}{x^2+4x+5} \, dx =$$

$$\parallel$$

$$(x+2)^2+1$$

$$= \frac{1}{2} \int \left(\frac{2x+4}{x^2+4x+5} - \frac{2}{x^2+4x+5} \right) dx$$

$$\int \frac{2x+4}{x^2+4x+5} \, dx = \ln|x^2+4x+5) + c =$$

$$\downarrow \qquad = \ln(x^2+4x+5) + c$$

$$\ln|f(x)| + c = \int \frac{f'(x)}{f(x)} \, dx \qquad \text{con } f(x) = x^2+4x+5$$

$$\int \frac{x+1}{x^2+4x+5} \, dx = \frac{1}{2} \ln(x^2+4x+5) - \frac{1}{2} \int \frac{2}{x^2+4x+5} \, dx =$$

$$= \frac{1}{2} \ln(x^2+4x+5) - \int \frac{1}{(x+2)^2+1} \, dx =$$

$$= \frac{1}{2} \ln(x^2+4x+5) - \arctan(x+2) + c$$

II° caso

$$\int \frac{1}{ax^2+bx+c}\,dx \qquad \text{con } \Delta = b^2-4ac > 0$$

$\Rightarrow ax^2+bx+c$ è prodotto di due polinomi di I° grado a coefficienti reali

$$\cdot \int \frac{1}{x^2+7x+6}\,dx = \int \frac{1}{(x+6)(x+1)}\,dx$$

Trovo $A, B \in \mathbb{R} \mid \dfrac{1}{x^2+7x+6} = \dfrac{A}{x+6} + \dfrac{B}{x+1}$

$$\frac{1}{x^2+7x+6} = \frac{A}{x+6} + \frac{B}{x+1} = \frac{A(x+1)+B(x+6)}{(x+6)(x+1)} =$$

$$= \frac{(A+B)x + A + 6B}{x^2+7x+6}$$

$$\Leftrightarrow (A+B)x + A + 6B = 1$$

$$\begin{cases} A+B=0 \\ A+6B=1 \end{cases} \quad \begin{cases} B=-A \\ -5A=1 \end{cases} \quad \begin{cases} A=-\frac{1}{5} \\ B=\frac{1}{5} \end{cases}$$

$$\int \frac{1}{x^2+7x+6}\,dx = \frac{1}{5}\int\left(\frac{1}{x+1} - \frac{1}{x+6}\right)dx =$$

$$= \frac{1}{5}\left(\ln|x+1| - \ln|x+6|\right) + c$$

MARTEDI' 8/1

- $$\int \frac{3x+2}{x^2-5x+6}\,dx \qquad x^2-5x+6=(x-2)(x-3)$$

Cerchiamo $A, B \in \mathbb{R}$ tali che

$$\frac{3x+2}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}=$$

$$=\frac{\boxed{(A+B)x-3A-2B}}{x^2-5x+6} \quad =3x+2$$

$$\begin{cases} A+B=3 \\ -3A-2B=2 \end{cases} \qquad \begin{aligned} A&=-8 \\ B&=11 \end{aligned}$$

$$\int \frac{3x+2}{x^2-5x+6}\,dx=\int\left(\frac{11}{x-3}-\frac{8}{x-2}\right)dx=$$

$$=11\ln|x-3|-8\ln|x-2|+c$$

- $$\int \frac{x^2+1}{x^2-5x+6}\,dx$$

Ci si può sempre ricondurre al caso in cui il grado del numeratore è minore (strettamente) del grado del denominatore

$\frac{P(x)}{Q(x)}$, grado di P $\geq$ grado di Q

trovo due polinomi $D(x)$ e $R(x)$ tali che

grado R < grado di Q e

$$P(x) = D(x)\,Q(x) + R(x)$$

$$\Rightarrow \quad \frac{P(x)}{Q(x)} = \underbrace{D(x)}_{\text{polinomio}} + \underbrace{\frac{R(x)}{Q(x)}}_{\text{grado del numeratore è < grado del denominatore}}$$

$$x^2+1 = \underbrace{\boxed{1}}_{D(x)}\left(x^2-5x+6\right) \underbrace{+5x-5}_{R(x)}$$

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{x^2-5x+6}$$

$$\int \frac{x^2+1}{x^2-5x+6}\,dx = \int\left(1 + \frac{5x-5}{x^2-5x+6}\right)dx =$$

$$= x + \int \frac{5x-5}{x^2-5x+6}\,dx$$

→ opero come prima trovando A e B tali che

$$\frac{5x-5}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

- $$\int \frac{x^4+1}{x^3+8}\,dx$$

$$x^4+1 = \underset{D(x)}{x}\,(x^3+8)\quad -8x+1 \quad R(x)$$

$$\frac{x^4+1}{x^3+8} = x - \frac{8x-1}{x^3+8}$$

$$\int \frac{x^4+1}{x^3+8}\,dx = \int\left(x - \frac{8x-1}{x^3+8}\right)dx =$$

$$= \frac{1}{2}x^2 - \int \frac{8x-1}{x^3+8}\,dx \quad ??$$

$$x^3+8 = (x+2)(x^2-2x+4)$$

→ polinomio irriducibile, $\Delta < 0$

$$\frac{8x-1}{x^3+8} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4} =$$

$$= \frac{A(x^2-2x+4)+(Bx+C)(x+2)}{x^3+8} =$$

$$= \frac{(A+B)x^2+(2B+C-2A)x+4A+2C}{x^3+8}$$

$$\Rightarrow (A+B)x^2+(2B+C-2A)x+4A+2C = 8x-1$$

$$\begin{cases} A+B=0 \\ 2B+C-2A=8 \\ 4A+2C=-1 \end{cases}$$

$$A=-\frac{17}{12}$$

$$B=\frac{17}{12}$$

$$C=\frac{7}{3}$$

$$\int \frac{8x-1}{x^3+8}\,dx = -\frac{17}{12}\boxed{\int \frac{1}{x+2}\,dx} = \ln|x+2| + $$

$$+\int \frac{\frac{17}{12}x+\frac{7}{3}}{x^2-2x+4}\,dx$$

$$\int \frac{\frac{17}{12}x+\frac{7}{3}}{x^2-2x+4}\,dx = \frac{1}{12}\int \frac{17x+28}{x^2-2x+4}\,dx =$$

$$D(x^2-2x+4)=2x-2$$

$$= \frac{1}{12}\,\frac{17}{2}\int \frac{2x+56/17}{x^2-2x+4}\,dx =$$

$$\to \frac{17}{2}\cdot\left(2x+\frac{56}{17}\right)=17x+28$$

$$= \frac{17}{24}\int \frac{2x+56/17-2+2}{x^2-2x+4}\,dx =$$

$$\to \int \frac{2x-2}{x^2-2x+4}\,dx = \ln|x^2-2x+4| + +C$$

$$= \frac{17}{24}\int\left(\boxed{\frac{2x-2}{x^2-2x+4}} + \frac{56/17+2}{\boxed{x^2-2x+4}}\right)dx$$

$$\to (x-1)^2+3$$

$$\int \frac{56/17+2}{x^2-2x+4}\,dx = \frac{90}{17}\int \frac{1}{(x-1)^2+3}\,dx =$$

$$= \frac{30}{17}\int \frac{1}{1+\left(\frac{x-1}{\sqrt{3}}\right)^2}\,dx =$$

$$= \frac{30}{17}\sqrt{3}\,\operatorname{arctan}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

- $$\int \frac{1}{(x+1)(x-1)^2}\,dx$$

$$\frac{1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} =$$

$$= \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2} =$$

$$= \frac{\boxed{(A+B)x^2 + (C-2A)x + A-B+C} = 1}{(x+1)(x-1)^2}$$

$$\begin{cases} A+B=0 \\ C-2A=0 \\ A-B+C=1 \end{cases} \qquad \begin{aligned} A &= \frac{1}{4} \\ B &= -\frac{1}{4} \\ C &= \frac{1}{2} \end{aligned}$$

$$\int \frac{1}{(x+1)(x-1)^2}\,dx = \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{4}\int \frac{1}{x-1}\,dx +$$

$$+ \frac{1}{2}\int \frac{1}{(x-1)^2}\,dx =$$

$$= \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x-1| - \frac{1}{2}\,\frac{1}{x-1} + c$$

- $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1+x^2-x^2}{(1+x^2)^2}\,dx =$$

$$= \int \left(\frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2}\right) dx =$$

$$= \arctan x - \int x\,\frac{x}{(1+x^2)^2}\,dx$$

$$\int \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}\boxed{\int \frac{2x}{(1+x^2)^2}\,dx} =$$

$$= \int f'(x)(f(x))^{\alpha}\,dx$$

$f(x) = 1+x^2$

$\alpha = -2$

$$= \frac{1}{2}\cdot\frac{1}{1-2}\cdot(1+x^2)^{1-2} + c$$

$$= -\frac{1}{2}\,\frac{1}{1+x^2} + c$$

$$= \arctan x - \left[-\frac{1}{2}\,x\,\frac{1}{1+x^2} + \frac{1}{2}\int \frac{1}{1+x^2}\,dx\right] =$$

↓ integro per parti

$$= \frac{1}{2}\arctan x + \frac{1}{2}\,\frac{x}{1+x^2} + c$$

Altro metodo

$$\int \frac{1}{1+x^2}\,dx = \frac{x}{1+x^2} + \int \frac{2x^2}{(1+x^2)^2}\,dx$$

integro per parti prendendo 1 come fattore differenziale

$$\arctan x + c = \int \frac{1}{1+x^2}\,dx = \frac{x}{1+x^2} + 2\int \frac{x^2+1-1}{(1+x^2)^2}\,dx =$$

$$= \frac{x}{1+x^2} + 2\int \frac{1}{1+x^2}\,dx - 2\int \frac{1}{(1+x^2)^2}\,dx =$$

$$= \frac{x}{1+x^2} + 2\arctan x - 2\int \frac{1}{(1+x^2)^2}\,dx$$

→ e adesso lo ricavo!

- $$\int \frac{1}{(x^2+2x+5)^2}\,dx =$$

$x^2+2x+5 = (x+1)^2+4$

$$= \int \frac{\overset{x^2+2x+5}{\boxed{1+x^2+2x+4}} - x^2-2x-4}{(x^2+2x+5)^2}\,dx =$$

$$= \int \left(\frac{1}{x^2+2x+5} - \frac{x^2+2x+4}{(x^2+2x+5)^2} \right) dx$$

$$\int \frac{x^2+2x+4}{(x^2+2x+5)^2}\,dx = \boxed{\int \frac{4}{(x^2+2x+5)^2}\,dx} + \int \frac{x^2+2x}{(x^2+2x+5)^2}\,dx$$

$$\int \frac{1}{(x^2+2x+5)^2}\,dx = \int \frac{1}{x^2+2x+5}\,dx +$$

$$-4\int \frac{1}{(x^2+2x+5)^2}\,dx - \int \frac{x^2+2x}{(x^2+2x+5)^2}\,dx$$

$$5\int \frac{1}{(x^2+2x+5)^2}\,dx = \int \frac{1}{x^2+2x+5}\,dx +$$

$$-\int \frac{x^2+2x}{(x^2+2x+5)^2}\,dx$$

$$\int \frac{x^2+2x}{(x^2+2x+5)^2}\,dx = \int x\,\frac{x+2}{(x^2+2x+5)^2}\,dx =$$

$$= \frac{1}{2}\int x\,\frac{2x+4}{(x^2+2x+5)^2}\,dx =$$

$$= \frac{1}{2}\left[\int x\,\frac{2x+2}{(x^2+2x+5)^2}\,dx + \int \frac{2x}{(x^2+2x+5)^2}\,dx\right]$$

$$\int x\,\frac{2x+2}{(x^2+2x+5)^2}\,dx = -\frac{x}{x^2+2x+5} + \int \frac{1}{x^2+2x+5}\,dx$$

$$\int \frac{2x}{(x^2+2x+5)^2}\,dx = \int \frac{2x+2-2}{(x^2+2x+5)^2}\,dx =$$

$$= \int \frac{2x+2}{(x^2+2x+5)^2}\,dx - 2\int \frac{1}{(x^2+2x+5)^2}\,dx$$

$$5 \int \frac{1}{(x^2+2x+5)^2}\,dx = \int \frac{1}{x^2+2x+5}\,dx \; +$$

→ so calcolarlo

$$-\frac{1}{2}\left[-\frac{x}{x^2+2x+5} + \boxed{\int \frac{1}{x^2+2x+5}\,dx} + \right.$$

$$\left. + \boxed{\int \frac{2x+2}{x^2+2x+5}\,dx} - 2\int \frac{1}{(x^2+2x+5)^2}\,dx \right]$$

$$= \ln|x^2+2x+5| + c$$

$$\boxed{5 \int \frac{1}{(x^2+2x+5)^2}\,dx} = \frac{1}{2}\int \frac{1}{x^2+2x+5}\,dx + \frac{1}{2}\,\frac{x}{x^2+2x+5} +$$

$$-\frac{1}{2}\ln|x^2+2x+5| + \boxed{\int \frac{1}{(x^2+2x+5)^2}\,dx}$$

$$4 \int \frac{1}{(x^2+2x+5)^2}\,dx = \frac{1}{2}\int \frac{1}{x^2+2x+5}\,dx \; +$$

$$+\frac{1}{2}\,\frac{x}{x^2+2x+5} - \frac{1}{2}\ln|x^2+2x+5|$$

Lo stesso integrale si può calcolare a partire da

$$= \int \frac{1}{x^2+2x+5}\,dx$$ integrando per parti

$$= \frac{A}{x^2+2x+5} + \frac{Bx+C}{(x^2+2x+5)^2}$$

$$= \frac{x}{x^2+2x+5} - \int \boxed{x\,\frac{2x+2}{(x^2+2x+5)^2}}\,dx$$

[Integrali riconducibili a integrali di funzioni razionali fratte]

$$= \int_0^{\pi/2} \frac{1}{\operatorname{sen} x + \cos x}\,dx$$

$$\operatorname{sen} x = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$$

$$\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$$

$$= \int_0^{\pi/2} \frac{1+\tan^2(x/2)}{2\tan(x/2)+1-\tan^2(x/2)} =$$

$t = \tan(x/2)$

$$= \int_0^1 \frac{1+t^2}{2t+1-t^2} \cdot \boxed{\frac{2}{1+t^2}\,dt}$$

dx

$x = 2\arctan t$

$dx = D(2\arctan t)\,dt$

$= \dfrac{2}{1+t^2}\,dt$

$$= \int_0^1 \frac{2}{2t+1-t^2}\,dt = \ldots$$

utile esercizio per casa

- $$\int \frac{\tan x}{3\cos^2 x - \operatorname{sen}^2 x}\,dx$$

$$\operatorname{sen}^2 x = \frac{\tan^2 x}{1+\tan^2 x}$$

$$\cos^2 x = \frac{1}{1+\tan^2 x}$$

$$\int \frac{\tan x}{3\cos^2 x - \sin^2 x}\,dx = \int \frac{\tan x\,(1+\tan^2 x)}{3-\tan^2 x}\,dx$$

$$= \int \frac{t\,(1+t^2)}{3-t^2}\cdot\frac{1}{1+t^2}\,dt = \int \frac{t}{3-t^2}\,dt$$

$t = \tan x$, $x = \arctan t$

$$\cdot \int_0^{1/2} \frac{e^x}{e^{3x}-2e^x-4}\,dx = \int_1^{\sqrt{e}} \frac{t}{t^3-2t-4}\cdot\frac{1}{t}\,dt =$$

$t = e^x$, $x = \ln t$

$$= \int_1^{\sqrt{e}} \frac{1}{(t-2)(t^2+2t+2)}\,dt \qquad \frac{A}{t-2} + \frac{Bt+C}{t^2+2t+2}$$

$\hookrightarrow (t+1)^2+1$

$$= \frac{1}{10}\int_1^{\sqrt{e}} \left(\frac{1}{t-2} - \frac{t+4}{t^2+2t+2}\right) dt$$

$$\int \frac{1}{t-2}\,dt = \ln|t-2| + c$$

$$\int \frac{t+4}{t^2+2t+2}\,dt = \frac{1}{2}\int \frac{2t+2+6}{t^2+2t+2}\,dt =$$

$$= \frac{1}{2}\left[\int \frac{2t+2}{t^2+2t+2}\,dt + 6\int \frac{1}{(t+1)^2+1}\,dt\right] =$$

$$= \frac{1}{2}\left[\ln|t^2+2t+2| + 6\arctan(t+1)\right] + c$$

$$\bullet \int_1^2 \frac{\sqrt{1+x^2}}{x}\,dx =$$

$\cosh^2 t - \operatorname{senh}^2 t = 1$

$\cosh^2 t = 1 + \operatorname{senh}^2 t$

$\downarrow$

$x = \operatorname{senh} t \quad , \quad dx = (\cosh t)\,dt$

$$= \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} \frac{\sqrt{1+\operatorname{senh}^2 t}}{\operatorname{senh} t}\cosh t\,dt =$$

$$= \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} \frac{\sqrt{\cosh^2 t}}{\operatorname{senh} t}\cosh t\,dt = \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} \frac{\cosh^2 t}{\operatorname{senh} t}\,dt =$$

$$= \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} \frac{1+\operatorname{senh}^2 t}{\operatorname{senh} t}\,dt = \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} \left(\frac{1}{\operatorname{senh} t} + \operatorname{senh} t\right)dt =$$

$$= \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} \frac{1}{\operatorname{senh} t}\,dt + \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} \operatorname{senh} t\,dt$$

$$\cosh t \Big|_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} =$$

$$= \cosh(\operatorname{settsenh} 2) - \cosh(\operatorname{settsenh} 1)$$

$$\operatorname{senh} t = \frac{e^t - e^{-t}}{2}$$

$\downarrow$

$$= \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} 2} \frac{2}{e^t - e^{-t}}\,dt =$$

$\downarrow$

$y = e^t \quad , \quad t = \ln y$

$$= \int_{e^{\operatorname{settsenh} 1}}^{e^{\operatorname{settsenh} 2}} \frac{2}{y - \frac{1}{y}} \cdot \frac{1}{y}\,dy = \int_{e^{\operatorname{settsenh} 1}}^{e^{\operatorname{settsenh} 2}} \frac{2}{y^2 - 1}\,dy$$

$$\bullet \int_{-1/2}^{1} x\sqrt{\frac{2-x}{x+3}}\,dx =$$

$$t=\sqrt{\frac{2-x}{x+3}}$$

$$x=\frac{2-3t^2}{t^2+1}=-3+\frac{5}{t^2+1}$$

$$=-\int_{1}^{1/2}\left(-3+\frac{5}{1+t^2}\right)\frac{10t^2}{(t^2+1)^2}\,dt =$$

già calcolato una primitiva

?

$$=-30\boxed{\int_{1/2}^{1}\frac{t^2}{(t^2+1)^2}\,dt}+50\boxed{\int_{1/2}^{1}\frac{t^2}{(t^2+1)^3}\,dt}$$

$$\int_{1/2}^{1}\frac{t^2}{(t^2+1)^3}\,dt=\int_{1/2}^{1} t\cdot\frac{1}{2}\frac{2t}{(1+t^2)^3}\,dt \;=$$

→ per parti

$$= t\cdot\frac{1}{2}\cdot\frac{1}{1-3}\cdot(1+t^2)^{1-3}\Big|_{1/2}^{1}+$$

$$+\frac{1}{4}\boxed{\int_{1/2}^{1}\frac{1}{(1+t^2)^2}\,dt}$$

già calcolato una primitiva