INTEGRALI GENERALIZZATI ESERCIZI

LUNEDÌ 14/1

Esercizio

Studiare la convergenza di

$$\int_0^{+\infty} \frac{\arctan x}{x^{3/2}} \, dx$$

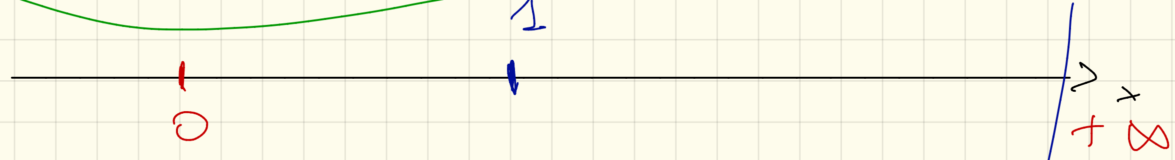
$= f(x)$

$f \in C^0(]0, +\infty[)$

- $\lim_{x \to 0} f(x) = +\infty$ (banale)
- Sto integrando in un intervallo illimitato

Per verificare se l'integrale converge bisogna studiare la convergenza di

$$\int_0^1 \frac{\arctan x}{x^{3/2}}\,dx \quad e \quad \int_1^{+\infty} \frac{\arctan x}{x^{3/2}}\,dx$$



→ integro una funzione continua ma illimitata in 0

integro una funzione continua e limitata su un intervallo illimitato

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• $$\int_0^1 \frac{\arctan x}{x^{3/2}}\,dx$$

$f(x)$, f è integrabile in ogni sottointervallo $[c, 1] \subseteq]0, 1]$

e b

$$\frac{\arctan x}{x^{3/2}} \sim \frac{x}{x^{3/2}} = \frac{1}{x^{1/2}} \quad \text{per } x \to 0$$

→ quando $\lim_{x \to 0}$ $$\frac{\frac{\arctan x}{x^{3/2}}}{1/x^{1/2}} = 1$$

$g(x)$

$$\int_0^1 \frac{1}{x^{1/2}}\,dx$$ converge o no?

Sì, converge!

$\Rightarrow$ per il criterio asintotico del confronto

$$\int_0^1 \frac{\arctan x}{x^{3/2}}\,dx \quad \text{converge}$$

- $$\int_1^{+\infty} \frac{\arctan x}{x^{3/2}}\,dx$$

$$\frac{\arctan x}{x^{3/2}} \sim \frac{1}{x^{3/2}} \quad \text{per } x \to +\infty$$

$$\lim_{x\to+\infty} \frac{\frac{\arctan x}{x^{3/2}}}{1/x^{3/2}} = \frac{\pi}{2} \qquad g(x) = \frac{1}{x^{3/2}}$$

$$\int_1^{+\infty} \frac{1}{x^{3/2}}\,dx \quad \text{converge o no?}$$

Sì, converge!

$\Rightarrow$ per il criterio asintotico del confronto

$$\int_1^{+\infty} \frac{\arctan x}{x^{3/2}}\,dx \quad \text{converge}$$

$$\Rightarrow \int_0^{+\infty} \frac{\arctan x}{x^{3/2}}\,dx \quad \text{converge}$$

Esercizio

Studiare la convergenza di

$$\int_1^{+\infty} \frac{\arctan x}{x^{3/2}(2+\operatorname{sen} x)}\,dx$$

$$2+\operatorname{sen} x \geq 1 \qquad \forall x \in \mathbb{R}$$

$$0 \leq \frac{\arctan x}{x^{3/2}(2+\operatorname{sen} x)} \leq \frac{\arctan x}{x^{3/2}}$$

$x \geq 1$

la integrale convergente in $[1,+\infty[$

$\Rightarrow$ per il criterio del confronto

$$\int_1^{+\infty} \frac{\arctan x}{x^{3/2}(2+\operatorname{sen} x)}\,dx \quad \text{converge}$$

Esercizio

Studiare la convergenza di

$$\int_0^1 \frac{\ln(1+\operatorname{sen}^2 x)}{\tan x^3}\,dx$$

$f(x)$, $f \in C^0(]0,1])$

Per studiare la convergenza guardo il comportamento della funzione per $x \to 0$

$$\frac{\ln(1+\overset{\to 0 \text{ per } x\to 0}{\sin^2 x})}{\tan x^3} \sim \frac{\sin^2 x}{x^3} \sim \frac{x^2}{x^3} = \frac{1}{x}$$

$$\int_0^1 \frac{1}{x}\,dx = +\infty \quad , \text{ diverge}$$

$$\Rightarrow \int_0^1 \frac{\ln(1+\sin^2 x)}{\tan x^3}\,dx \text{ diverge per}$$

il criterio asintotico del confronto

Esercizio

Studiare la convergenza di

$$\int_1^{+\infty} \frac{\sin^5(1/x)}{\ln(x^2+1)-2\ln x}\,dx \qquad f(x)$$

$$f(x) = \frac{\sin^5(1/x)}{\ln(x^2+1)-\ln x^2} = \qquad f \in C^0([1,+\infty[)$$

$$= \frac{\sin^5(1/x)}{\ln\left(\frac{x^2+1}{x^2}\right)} = \frac{\sin^5(1/x)}{\ln\left(1+\frac{1}{x^2}\right)}$$

Unico problema di integrabilità: sto integrando su un intervallo illimitato.

Studiamo il comportamento di f per $x \to +\infty$

$$\frac{\operatorname{sen}^5(1/x)}{\ln\left(1+\frac{1}{x^2}\right)} \sim \frac{1/x^5}{1/x^2} = \frac{1}{x^3}$$

$$\operatorname{sen}\frac{1}{x} = \frac{1}{x} + o(1/x) \quad \text{per } x \to +\infty$$

$$\ln\left(1+\frac{1}{x^2}\right) = \frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \quad \text{per } x \to +\infty$$

$$\int_1^{+\infty} \frac{1}{x^3}\,dx \quad \text{converge}$$

$$\Rightarrow \int_1^{+\infty} \frac{\operatorname{sen}^5(1/x)}{\ln(1+x^2)-2\ln x}\,dx \quad \text{converge}$$

per il criterio asintotico del confronto

Esercizio

Studiare la convergenza di

$$\int_{\sqrt{3}}^{+\infty} \frac{e^{x^4}}{1+e^{4x^4}}\,dx$$

$f(x)$, $f \in C^\infty(\mathbb{R})$

Studiamo il comportamento di f per $x \to +\infty$

$$\frac{e^{x^4}}{1+e^{4x^4}} \sim \frac{e^{x^4}}{e^{4x^4}} = \frac{1}{e^{3x^4}} = e^{-3x^4}$$

$$\int_{\sqrt{3}}^{+\infty} e^{-3x^4}\,dx \quad \text{converge o no?}$$

$$\int_{\sqrt{3}}^{+\infty} e^{-x}\,dx \quad \text{converge}$$

e $e^{-3x^4} = o(e^{-x})$ per $x \to +\infty$

cioè $\lim_{x\to+\infty} \frac{e^{-3x^4}}{e^{-x}} = 0$

$$\Rightarrow \int_{\sqrt{3}}^{+\infty} e^{-3x^4}\,dx \quad \text{converge}$$

$$\Rightarrow \int_{\sqrt{3}}^{+\infty} \frac{e^{x^4}}{1+e^{4x^4}}\,dx \quad \text{converge}$$

Osservazione importante

$$\boxed{\int_{1}^{+\infty} x^{\alpha} e^{-x}\,dx \quad \text{converge } \forall \alpha \in \mathbb{R}}$$

perché $\quad x^\alpha e^{-x} = o\left(e^{-x/2}\right)$

per $x \to +\infty$

$$\lim_{x\to+\infty} \frac{x^\alpha e^{-x}}{e^{-x/2}} = \lim_{x\to+\infty} x^\alpha e^{-x/2} = 0$$

e $\displaystyle\int_1^{+\infty} e^{-x/2}\,dx = 2e^{-1/2}$, converge

$$\boxed{\int_1^{+\infty} x^\alpha e^{-\beta x}\,dx \quad \text{converge } \forall \beta>0 \text{ e } \forall \alpha\in\mathbb{R}}$$

Esercizio

Studiare la convergenza di

$$\int_4^{+\infty} \frac{\ln(3+\operatorname{sen}x)}{\sqrt[4]{x^5-x^3+3}}\,dx$$

$f(x)$, $f\in C^\infty([4,+\infty[)$

$$x^5-x^3+3 = x^3(x^2-1)+3 \geq 3 \quad \forall x\geq 1$$

$$2 \leq 3+\operatorname{sen}x \leq 4$$

$$\ln 2 \leq \ln(3+\operatorname{sen}x) \leq \ln 4$$

$$\frac{\ln 2}{\sqrt[4]{x^5-x^3+3}} \le \frac{\ln(3+\operatorname{sen} x)}{\sqrt[4]{x^5-x^3+3}} \le \frac{\ln 4}{\sqrt[4]{x^5-x^3+3}}$$

$$\int_4^{+\infty} \frac{\ln 4}{\sqrt[4]{x^5-x^3+3}}\,dx \quad e \quad \int_4^{+\infty} \frac{\ln 2}{\sqrt[4]{x^5-x^3+3}}\,dx$$

convergono $\iff$ converge

$$\int_4^{+\infty} \frac{1}{\sqrt[4]{x^5-x^3+3}}\,dx$$

$$\Rightarrow \int_4^{+\infty} \frac{\ln(3+\operatorname{sen} x)}{\sqrt[4]{x^5-x^3+3}}\,dx \quad \text{converge}$$

$$\iff \int_4^{+\infty} \frac{1}{\sqrt[4]{x^5-x^3+3}}\,dx \quad \text{converge}$$

$$\frac{1}{\sqrt[4]{x^5-x^3+3}} \sim \frac{1}{\sqrt[4]{x^5}} = \frac{1}{x^{5/4}} \quad \text{per } x \to +\infty$$

$$e \int_4^{+\infty} \frac{1}{x^{5/4}}\,dx \quad \text{converge}$$

$$\Rightarrow \text{converge anche} \int_4^{+\infty} \frac{\ln(3+\operatorname{sen} x)}{\sqrt[4]{x^5-x^3+3}}\,dx$$

Esercizio

Trovare per quali $\alpha \in \mathbb{R}$ converge

$$\int_1^2 \frac{1}{(x-1)^\alpha \ln x}\,dx$$

$f(x)$, $f \in C^0(]1,2])$ $\forall \alpha$

$\Rightarrow$ f è integrabile in ogni sottointervallo $[c,2] \subseteq]1,2]$

Studiamo il comportamento di f per $x \to 1^+$

$$\frac{1}{(x-1)^\alpha \ln x} \sim \frac{1}{(x-1)^\alpha (x-1)} \quad \text{per } x \to 1^+$$

$\ln x = \ln(1+(x-1))$

$$\frac{1}{(x-1)^\alpha \ln x} \sim \frac{1}{(x-1)^{\alpha+1}} \quad \text{per } x \to 1^+$$

e $\displaystyle\int_1^2 \frac{1}{(x-1)^{\alpha+1}}\,dx$ converge $\Leftrightarrow$

$\alpha+1 < 1 \Leftrightarrow \alpha < 0$

$\Rightarrow \displaystyle\int_1^2 \frac{1}{(x-1)^\alpha \ln x}\,dx$ converge $\Leftrightarrow \alpha < 0$

[N.B.: $\dfrac{1}{(x-1)^\alpha \ln x} \geq 0 \quad \forall x > 1$]

Esercizio

Trovare per quali $\alpha \in \mathbb{R}$ converge

$$\int_0^1 \frac{\operatorname{sen} t \, |\ln t|}{t^\alpha \, |t-1|^{-\alpha}} \, dt$$

$f(t)$

$f \in C^0(]0,1[)$

Ma $\lim_{t \to 0^+} f(t)$ e $\lim_{t \to 1^-} f(t)$

potrebbero essere infiniti

$f(t) \geq 0 \quad \forall t \in]0,1[$

La convergenza dell'integrale equivale alla convergenza di entrambi gli integrali

$$\int_0^{1/2} \frac{\operatorname{sen} t \, |\ln t|}{t^\alpha \, |t-1|^{-\alpha}} \, dt \quad e \quad \int_{1/2}^1 \frac{\operatorname{sen} t \, |\ln t|}{t^\alpha \, |1-t|^{-\alpha}} \, dt$$

- $$\int_0^{1/2} \frac{\operatorname{sen} t \, |\ln t|}{t^\alpha \, |t-1|^{-\alpha}} \, dt$$

Studio il comportamento di f per $t \to 0^+$

$$\frac{\operatorname{sen} t\,|\ln t|}{t^\alpha\,|t-1|^{-\alpha}} \sim \frac{t\cdot|\ln t|}{t^\alpha} = \frac{1}{t^{\alpha-1}}|\ln t|$$
per $t\to 0^+$

$\operatorname{sen} t = t + o(t)$ per $t\to 0$

$$\lim_{t\to 1} |t-1|^{-\alpha} = 1$$

$$\int_0^{1/2} \frac{|\ln t|}{t^{\alpha-1}}\,dt$$ converge $\iff \alpha-1<1$
$\iff \alpha<2$

$$\int_0^{1/2} \frac{\operatorname{sen} t\,|\ln t|}{t^\alpha\,|t-1|^{-\alpha}}\,dt$$ converge $\iff \boxed{\alpha<2}$
per il criterio asintotico del confronto

$$\int_{1/2}^{1} \frac{\operatorname{sen} t\,|\ln t|}{t^\alpha\,|t-1|^{-\alpha}}\,dt$$

Studio il comportamento di f
per $t\to 1^-$

$$\frac{\operatorname{sen} t\,|\ln t|}{t^\alpha\,|t-1|^{-\alpha}} \sim \frac{|t-1|}{|t-1|^{-\alpha}} = \frac{1}{|t-1|^{-\alpha-1}}$$
per $t\to 1^-$

e $$\int_{1/2}^{1} \frac{1}{|t-1|^{-\alpha-1}}\,dt$$ converge $\iff$
$-\alpha-1<1 \iff \alpha>-2$

$$\int_{1/2}^{1} \frac{\operatorname{sen} t \, |\ln t|}{t^{\alpha} \, |t-1|^{-\alpha}} \, dt \quad \text{converge} \iff \boxed{\alpha > -2}$$

per il criterio asintotico del confronto

$$\Rightarrow \int_{0}^{1} \frac{\operatorname{sen} t \, |\ln t|}{t^{\alpha} \, |t-1|^{-\alpha}} \, dt \quad \text{converge} \iff -2 < \alpha < 2$$

Esercizio

Studiare la convergenza di

$$\int_{0}^{+\infty} \frac{\operatorname{senh} x^2}{x^{\alpha} \, e^{\alpha(x^2+x)}} \, dx$$

$f(x)$, $f \in C^{\infty}(]0,+\infty[)$
$f(x) \geq 0$
$\forall x > 0$

L'integrale converge $\iff$ convergono entrambi

$$\int_{0}^{1} \frac{\operatorname{senh} x^2}{x^{\alpha} \, e^{\alpha(x^2+x)}} \, dx \quad e \quad \int_{1}^{+\infty} \frac{\operatorname{senh} x^2}{x^{\alpha} \, e^{\alpha(x^2+x)}} \, dx$$

- $\displaystyle\int_{0}^{1} \frac{\operatorname{senh} x^2}{x^{\alpha} \, e^{\alpha(x^2+x)}} \, dx$

Studiamo il comportamento di f per $x \to 0^+$

$$\frac{\operatorname{senh} x^2}{x^{\alpha} \, e^{\alpha(x^2+x)}} \sim \frac{x^2}{x^{\alpha}} = \frac{1}{x^{\alpha-2}} \quad \text{per } x \to 0^+$$

e $\displaystyle\int_0^1 \frac{1}{x^{\alpha-2}}\,dx$ converge $\iff \alpha-2<1$
$\iff \alpha<3$

$$\int_0^1 \frac{\operatorname{senh} x^2}{x^\alpha e^{\alpha(x^2+x)}}\,dx \quad \text{converge} \iff \boxed{\alpha<3}$$

per il criterio asintotico del confronto

- $\displaystyle\int_1^{+\infty} \frac{\operatorname{senh} x^2}{x^\alpha e^{\alpha(x^2+x)}}\,dx$

Studiamo il comportamento di f per $x\to+\infty$

$$\frac{\operatorname{senh} x^2}{x^\alpha e^{\alpha(x^2+x)}} \sim \frac{e^{x^2}}{x^\alpha e^{\alpha(x^2+x)}} \quad \text{per } x\to+\infty$$

$$\lim_{t\to+\infty} \frac{\operatorname{senh} t}{e^t} = \frac{1}{2}$$

$$\lim_{x\to+\infty} \frac{e^{\alpha(x^2+x)}}{e^{\alpha x^2}} = \lim_{x\to+\infty} e^{\alpha x} = \begin{cases} +\infty & \text{se } \alpha>0 \\ 1 & \text{se } \alpha=0 \\ 0 & \text{se } \alpha<0 \end{cases}$$

$f \sim g$ per $x\to x_0$ se $\displaystyle\lim_{x\to x_0}\frac{f(x)}{g(x)} = l$,
$l\in\mathbb{R}$, $l\neq 0$

$$\frac{\operatorname{senh} x^2}{x^\alpha e^{\alpha(x^2+x)}} \sim \frac{e^{(1-\alpha)x^2-\alpha x}}{x^\alpha} \quad \text{per } x \to +\infty$$

- Se $1-\alpha > 0$ $(\alpha < 1)$

$$\lim_{x\to+\infty} \frac{e^{(1-\alpha)x^2-\alpha x}}{x^\alpha} = +\infty$$

$\Rightarrow$ l'integrale diverge

- Se $1-\alpha = 0$ $(\alpha = 1)$

$\frac{\operatorname{senh} x^2}{x\, e^{x^2+x}} \sim \frac{e^{-x}}{x}$ per $x \to +\infty$ e

$\int_1^{+\infty} \frac{e^{-x}}{x}\,dx$ converge

$\Rightarrow \int_1^{+\infty} \frac{\operatorname{senh} x^2}{x^\alpha e^{\alpha(x^2+x)}}\,dx$ converge per $\alpha = 1$

- $1-\alpha < 0$ $(\alpha > 1)$

$$\frac{e^{(1-\alpha)x^2-\alpha x}}{x^\alpha} = \frac{e^{(1-\alpha)x^2}}{x^\alpha} \cdot e^{-\alpha x} \le e^{-\alpha x}$$

definitivamente per $x \to +\infty$

$\downarrow$
0 per $x \to +\infty$

e $\int_1^{+\infty} e^{-\alpha x}\,dx$ converge per $\alpha > 1$

oppure, $e^{(1-\alpha)x^2-\alpha x} \le 1 \Rightarrow \frac{e^{(1-\alpha)x^2-\alpha x}}{x^\alpha} \le \frac{1}{x^\alpha}$

e $\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ converge per $\alpha > 1$

$$\int_1^{+\infty} \frac{\operatorname{senh} x^2}{x^\alpha e^{\alpha(x^2+x)}}\,dx \quad \text{converge} \iff \boxed{\alpha \geq 1}$$

$$\int_0^{+\infty} \frac{\operatorname{senh} x^2}{x^\alpha e^{\alpha(x^2+x)}}\,dx \quad \text{converge} \iff$$

$$1 \leq \alpha < 3$$

Esercizio

Trovare per quali $\alpha, \beta \in \mathbb{R}$ converge

$$\int_0^{+\infty} \frac{\arctan^\alpha x}{x^\beta}\,dx$$

$$\int_0^{+\infty} \frac{\arctan^\alpha x}{x^\beta}\,dx \quad \text{converge} \iff$$

convergono entrambi

$$\int_0^2 \frac{\arctan^\alpha x}{x^\beta}\,dx \quad e \quad \int_2^{+\infty} \frac{\arctan^\alpha x}{x^\beta}\,dx$$

- $$\int_0^2 \frac{\arctan^\alpha x}{x^\beta}\,dx$$

Per $x \to 0^+$

$$\frac{(\arctan x)^\alpha}{x^\beta} \sim \frac{x^\alpha}{x^\beta} = \frac{1}{x^{\beta-\alpha}}$$

$$\int_0^2 \frac{1}{x^{\beta-\alpha}}\,dx \quad \text{converge} \iff \beta-\alpha<1$$

$$\Rightarrow \int_0^2 \frac{(\arctan x)^\alpha}{x^\beta}\,dx \quad \text{converge} \iff \boxed{\beta-\alpha<1}$$

per il criterio asintotico del confronto

- $$\int_2^{+\infty} \frac{(\arctan x)^\alpha}{x^\beta}\,dx$$

Per $x \to +\infty$

$$\frac{(\arctan x)^\alpha}{x^\beta} \sim \frac{1}{x^\beta} \quad \text{e} \int_2^{+\infty} \frac{1}{x^\beta}\,dx \text{ converge}$$

$$\iff \beta > 1$$

$$\int_2^{+\infty} \frac{(\arctan x)^\alpha}{x^\beta}\,dx \quad \text{converge} \iff \boxed{\beta > 1}$$

per il criterio asintotico del confronto

$$\Rightarrow \int_0^{+\infty} \frac{(\arctan x)^\alpha}{x^\beta}\,dx \quad \text{converge} \iff$$

$$\beta > 1 \quad \text{e} \quad \beta-\alpha<1$$

$$\iff \quad 1 < \beta < \alpha+1$$