LIMITI DI SUCCESSIONI

Gli esercizi denotati con un asterisco sono un po' più difficili

$$\lim_{n\to+\infty} \frac{5n^3+7n-2}{2n^3} = \frac{5}{2}$$

$$\lim_{n\to+\infty} \frac{5n^3+7n-2}{2(n+4)^3} = \frac{5}{2}$$

$$\lim_{n\to+\infty} \frac{n \operatorname{sen} n}{n^2+1} = 0$$

$$\lim_{n} \sqrt[n]{\frac{n^2+3n+2}{n^2}} = 1$$

$$\lim \left(\sqrt{n^2+2n-1} - \sqrt{n^2+2}\right) = 1$$

$$\lim n\left(\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}}\right) = 1$$

$$\lim \frac{2^{n+1}+3^{n+1}}{2^n+3^n} = 3$$

$$\lim \left(n - \sqrt{n^2+1}\right) = 0$$

$$\lim \sqrt[n]{3^n + 7^n} = 7$$

$$\lim \left(\sqrt{n^2+n} - \sqrt{n^2+7}\right) = \frac{1}{2}$$

$$\lim \left(1 + \frac{1}{n^2}\right)^{n} = 1$$

$$\lim \left(1 + \frac{1}{n^2}\right)^{n^3} = +\infty$$

$$\lim \left(1 + \frac{1}{n^n}\right)^{n!} = 1$$

$$\lim \left(1 + \frac{1}{n}\right)^{\frac{1}{\operatorname{sen} \frac{1}{n}}} = e$$

$$\lim \left(1 + \operatorname{sen} \frac{1}{n}\right)^{n} = e$$

$$\lim \frac{1}{\log n^4 \ \operatorname{sen}\left(-\frac{1}{n}\right)} = -\infty$$

$$\lim \left(n^4 + 1\right)\left(1 - \cos \frac{3}{n^2}\right) = \frac{9}{2}$$

$$\lim \left(\operatorname{sen} \frac{1}{n}\right)^{1 - \cos \frac{1}{n}} = 1$$

$$\lim \frac{\left(1+\frac{1}{n}\right)^{\frac{1}{6}}-1}{\frac{1}{n}} = \frac{1}{6}$$

$$\lim \left(\sqrt{n^2+2n} - \sqrt[3]{n^3+2n^2}\right) = \frac{1}{3} \quad \circledast$$

$$\lim_{n\to+\infty} \frac{\operatorname{tg}\frac{1}{n}}{\frac{1}{n}} = 1$$

$$\lim_{n\to+\infty} \frac{\operatorname{tg}\frac{a}{n}}{\frac{1}{n}} = a \qquad a\in\mathbb{R}$$

$$\lim \frac{\log\left(1+\operatorname{sen}\left(\operatorname{arctg}\frac{1}{n}\right)\right)}{\frac{1}{n}} = 1$$

$$\lim \frac{\log\left(\cos\frac{1}{n}\right)}{\frac{1}{n^2}} = -\frac{1}{2}$$

$$\lim \left(\cos\frac{1}{n}\right)^{n^2} = \frac{1}{\sqrt{e}}$$

$$\lim \left(\frac{n+\cos\frac{1}{n}}{n}\right)^{n} = e$$

$$\lim \left(\operatorname{sen} \left(\operatorname{sen}^4 n \right) \right)^{\frac{n}{4}} = 0 \qquad \circledast$$

$$\lim \left(\cos \left(1 + \frac{\cos n}{2} \right) \right)^{n} = 0 \qquad \circledast$$

$$\lim \left(\frac{n}{3} + \frac{1}{n} \right) \operatorname{sen} \frac{5}{n} = \frac{5}{3}$$

$$\lim \frac{e^n - 2^{n \log n}}{n^n} = 0$$

$$\lim_{n \to \infty} \frac{n}{2^{\log n}} = +\infty$$

$$\lim_{n} \frac{n}{3^{\log n}} = 0$$

$$\lim \frac{n^{\pi} + 5n^2 - 7 \log n}{\pi^{\log n} + n^3} = +\infty$$

$$\lim \frac{n^{\pi} + 5n^2 - 7 \log n}{\pi^{\log n} + n^4} = 0$$

$$\lim \left(\sqrt[7]{\cos \frac{1}{n}} - 1 \right) n^2 = -\frac{1}{14}$$

$$\lim \frac{\operatorname{sen}\left(-\frac{1}{n^2}\right)}{1 - \cos \frac{1}{n}} = -2$$

$$\lim_{n \to +\infty} \frac{\operatorname{arctg}\left((-1)^n \frac{1}{n}\right)}{(-1)^n \frac{1}{n}} = 1$$

$$\lim \frac{\operatorname{arctg}\left(\frac{1}{n} - \frac{1}{n^2}\right)}{\frac{1}{n}} = 1$$

$$\lim_{n \to +\infty} \left[\left(1 + \frac{1}{n^2}\right)^n - 1\right] n = 1 \qquad (\circ)$$

Svolgimento di $(\circ)$:

Vedi pagina seguente, dopo aver provato a svolgerlo

$$\lim \frac{n^2}{\underset{n}{-}\log n} = +\infty$$

$$\lim \frac{n}{\underset{v}{-}\log n} = 0$$

$$\lim \frac{n^{\alpha n}}{n!} = \begin{cases} +\infty & \text{se } \alpha \geq 1 \\ 0 & \text{se } \alpha < 1 \end{cases}$$

$$\lim \frac{\sqrt[3]{1 + \frac{1}{n^\alpha}} - 1}{\log\left(1 + \frac{1}{n^\beta}\right)} = \begin{cases} +\infty & \text{se } \beta > \alpha \\ \frac{1}{3} & \text{se } \beta = \alpha \\ 0 & \text{se } \beta < \alpha \end{cases}$$

$\alpha > 0$
$\beta > 0$

$$\lim \left(1 + \log\left(1 + \frac{1}{n}\right)\right)^n = e$$

$$\lim_{n \to +\infty} \left(\sqrt{1 + \frac{1}{n}}\right)^n = \sqrt{e}$$

$$\lim \frac{\log(1 + e^n)}{\sqrt{1 + n^2}} = 1$$

Svolgimento di (•)

utilizzando $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

posso scrivere

$$\left(1+\frac{1}{n^2}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n^2}\right)^k =$$

$$= 1 + n\frac{1}{n^2} + \frac{n(n-1)}{2}\frac{1}{n^4} +$$

$$+ \frac{n(n-1)(n-2)}{6}\frac{1}{n^6} + \dots$$

il valore massimo di $\binom{n}{k}$ è per $k = \frac{n}{2}$ (se n è pari)

e si ha

$$\binom{n}{\frac{n}{2}} = \frac{n!}{\frac{n}{2}!\left(n-\frac{n}{2}\right)!} =$$

$$= \frac{n(n-1)\cdot(n-2)\cdot \dots \cdot\left(\frac{n}{2}+1\right)}{\left(n-\frac{n}{2}\right)!} =$$

$$= \frac{n(n-1)(n-2)\cdot \dots \left(\frac{n}{2}+1\right)}{\frac{n}{2}\left(\frac{n}{2}-1\right)\cdot\left(\frac{n}{2}-2\right)\cdot \dots \cdot 1} =$$

$$= \frac{n(n-1)\cdot(n-2)\cdot \ \dots \ \cdot \left(\frac{n}{2}+1\right)}{\frac{n}{2}\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)\cdot \ \dots \ \left(\frac{n-(n-2)}{2}\right)} =$$

$$= \frac{n(n-1)(n-2)\cdot \ \dots \ \cdot\left(\frac{n}{2}+1\right)}{n(n-2)(n-4)\cdot \ \dots \ (n-(n-2))} \quad 2^{n/2}$$

(se n dispari abbiamo una cosa analoga)

Allora:

$$\binom{n}{\frac{n}{2}}\left(\frac{1}{n^2}\right)^{n/2} = \frac{n(n-1)\cdot(n-2)\ \dots \ \left(\frac{n}{2}+1\right)}{n(n-2)(n-4)\ \dots \ 2}\left(\frac{\sqrt{2}}{n}\right)^{n}$$

In conclusione:

$$\left(1+\frac{1}{n^2}\right)^n = 1+\frac{1}{n}+\frac{n(n-1)}{2}\frac{1}{n^4} + \text{termini di ordine inferiore}$$

per cui $\left[\left(1+\frac{1}{n^2}\right)^n - 1\right] n =$

$$= 1+\frac{n(n-1)}{2}\frac{1}{n^3} + \text{altri termini che hanno limite } 0 \text{ per } n\to+\infty$$

Altro svolgimento di $(\bullet)$

$$\left(1+\frac{1}{n^2}\right)^n = \left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{1/n} =$$

$$= e^{\frac{1}{n}\log\left(1+\frac{1}{n^2}\right)^{n^2}}$$

per cui

$$\left[\left(1+\frac{1}{n^2}\right)^n - 1\right] n =$$

$$= \frac{e^{\frac{\log\left(1+\frac{1}{n^2}\right)^{n^2}}{n}} - 1}{\frac{1}{n}} =$$

$$= \underbrace{\frac{e^{\frac{\log\left(1+\frac{1}{n^2}\right)^{n^2}}{n}} - 1}{\frac{\log\left(1+\frac{1}{n^2}\right)^{n^2}}{n}}}_{\xrightarrow[n\to\infty]{} 1} \quad \underbrace{\log\left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]}_{\xrightarrow[n\to\infty]{} 1}$$

Attenzione: si potrebbe essere tentati di usare il limite notevole

$$\lim_{n \to +\infty} \frac{e^{1/n} - 1}{1/n} = 1$$

visto che $\left(1 + \frac{1}{n^2}\right)^n = \left[\left(1 + \frac{1}{n^2}\right)^{n^2}\right]^{1/n}$

e $\lim \left(1 + \frac{1}{n^2}\right)^{n^2} = e$

ma, in generale, bisogna prestare molta attenzione a queste sostituzioni.

Anche se in questo caso $\lim \frac{e^{1/n} - 1}{1/n}$ e $\lim \frac{\left(1 + \frac{1}{n^2}\right)^n - 1}{1/n}$ sono uguali

bisogna evitare di passare al limite ~~solo~~ in "tempi diversi".

Ad esempio: $\left(1 + \frac{1}{n}\right) \xrightarrow[n \to \infty]{} 1$

$$1^n = 1$$

ma $\left(1 + \frac{1}{n}\right)^n$ non tende a 1

$$\lim_{n \to +\infty} \frac{\operatorname{sen}\left(+\frac{1}{n^{\alpha}}\right)}{1 - \cos\left(\frac{1}{n}\right)^{\beta}} =$$

$$\cancel{\lim_{n \to +\infty} \operatorname{arctg} (-1)^n \frac{1}{n}}$$

$$\lim_{n \to +\infty} \left[\log\left(2 - \cos e^{-n}\right)\right] \left(1 + \frac{2}{n}\right)^{n^2}$$

$$\lim_{n \to +\infty} \left(\frac{n + \cos \frac{1}{n}}{n}\right)^{n}$$