Degree course: Mathematics
Programme of the first part of
Introduction to Partial Differential Equations – a.y. 2023/2024
Teachers: Claudio Marchi and Fabio Paronetto

2.10.2023 Introduction to the course: general informations and brief presentation of the course contents.

Some notations, definition of a partial differential equation (PDE), classification in linear, semi-linear, quasi-linear and fully non-linear equations. Typical problems one can deal with when studying a PDE.
Some examples (transport equation, Laplace equation, heat equation, wave equation).
Classification of linear second order partial differential equations in $\mathbf{R}^2$.

3.10.2023 Classification of linear second order partial differential equations in $\mathbf{R}^n$. Examples.
Some recalls: the space $C^k(\Omega)$, the space $C^k(\bar{\Omega})$, sets of regularity C^k or Lipschitz. Examples. The divergence theorem.
The continuity equation: starting to compute $\dfrac{d}{dt}\displaystyle\int_{E(t)} \rho(x,t)\,dx$ ...

9.10.2023 The continuity equation or equations of conservation of mass and some particular cases: the transport equation, the heat equation, the Laplace equation.

A brief introduction to the theory of distributions: definition, examples, δ (Dirac's delta), some sequences approximating δ, convolution of a distribution with a function, convolution of δ with a function ($\delta * \psi = \psi$).

10.10.2023 The Laplace equations: some physical examples, some possible (linear) generalisations of $-\Delta$, typical boundary conditions (Dirichlet, Neumann, Robin), definition of harmonic function.
Variational nature of harmonic functions: Dirichlet's principle (also in dimension 1).

16.10.2023 Energy methods for uniqueness: Dirichlet, Neumann, Robin problems.
A compatibility conditions for the Neumann problem. Holomorphic functions and harmonic functions.
The Laplacian is rotations invariant. Looking for radial harmonic functions. Definition of fundamental solutions for a general differential operator. The fundamental solutions for the Laplacian.

17.10.2023 Proof of the following fact: $-\Delta E = \delta$ in dimension 2.
Some properties of harmonic functions: definitions of subharmonic and superharmonic functions, the mean value property fir C^2 functions, the maximum principle for subharmonic and superharmonic functions. (Strong and weak maximum principles without proof). Some simple consequences of the maximum principles. Proof of the maximum principle for subharmonic functions $u \in C^2$ satisfying $-\Delta u \leqslant 0$.
Comparison for the Dirichlet problems.
Green's identities and some simple consequences.

23.10.2023 Uniqueness via maximum principle for the Dirichlet problems.
A representation for C^2 functions: the Stokes identity. Some consequences, in particular: there are no harmonic functions $u \not\equiv 0$ compactly supported, a harmonic function $u \in C^2$ is in fact C^∞. A function u harmonic in Ω is locally analytic in Ω (without proof).
Another representation formula for C^2 functions following by the Stokes identity. Consequence: a function $u \in C^2$ is sub-harmonic (superharmonic) if and only if $-\Delta u \leqslant 0$ ($-\Delta u \geqslant 0$).

Towards the solution of the Dirichlet problem: Green function, Poisson kernel and possible representation formula for the solution of the Poisson equation with Dirichlet boundary datum.
Simmetry of the Green function (without proof).

24.10.2023 The Green function for a ball. Poisson formula in a ball and solution of the Dirichlet problem in a ball. In general $-\Delta u = f$ with $f \in C^0(\Omega)$ has no C^2 solutions (without proof).
Some consequences of the Poisson formula: u continuous and satisfying the mean value property is C^2 and harmonic.
The Harnack inequality (1 and 1-bis).

30.10.2023 The Harnack inequality (2). Comments.
Results about sequences of harmonic functions (Harnack principle).
Theorem of Liouville. Comments.

The Dirichlet problem in a bounded domain (the Perron's method): definition of some classes of sub-harmonic and super-harmonic functions.
Definition of harmonic lifting.

31.10.2023 The harmonic lifting $h_{v;x_o,\rho}$ of a sub-harmonic function v in a ball $B_\rho(x_o)$.
The function $v_{x_o,\rho} \geqslant v$, where $v_{x_o,\rho} = v$ in $\Omega \setminus B_\rho(x_o)$ and $v_{x_o,\rho} = h_{v;x_o,\rho}$ in $\Omega \setminus B_\rho(x_o)$.
The function $v_{x_o,\rho}$ is sub-harmonic if v is sub-harmonic.
The exterior ball condition. Comments and examples.

In $n = 2$: an open set Ω of class C^2 satisfies the exterior ball condition, an open set Ω of class C^1 but not C^2 may not satisfy the exterior ball condition. The classes $\sigma(\Omega;\varphi)$ and $\Sigma(\Omega;\varphi)$. The function u_φ is harmonic in Ω, $u_\varphi(x) = \sup\{v(x)\,|\,v \in \sigma(\Omega;\varphi)\}$.
The same holds for U_φ, $U_\varphi(x) = \inf\{w(x)\,|\,w \in \Sigma(\Omega;\varphi)\}$.

6.11.2023 Given $v \in \sigma(\Omega;\varphi)$ and $w \in \Sigma(\Omega;\varphi)$ one has that $w - v \geqslant 0$ and, in particular, $u_\varphi \leqslant U_\varphi$.
The function $u_\varphi \in C^0(\bar{\Omega})$, $u_\varphi = \varphi$ in $\partial\Omega$ and $u_\varphi = U_\varphi$.
Theorem (Perron): if Ω, open and bounded, satisfies the exterior ball condition then the Dirichlet problem admits a unique solution.
Barriers and regular points. Theorem: the Dirichlet problem admits a unique solution if and only if all the points of the boundary are regular. The exterior cone condition. Examples of regular sets: convex sets, C^2 sets and in general the sets satisfying the exterior ball condition, but also the exterior cone condition.

7.11.2023 Some exercises.