Degree course: Mathematics and Mathematical engineering
Programme of
Introduction to Partial Differential Equations – a.y. 2024/2025
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Common part

30.9.2024 Introduction to the course: general informations and brief presentation of the course contents.

Some notations, definition of a partial differential equation (PDE), classification in linear, semi-linear, quasi-linear and fully non-linear equations.
Typical problems one can deal with when studying a PDE.
Some examples (transport equation, Laplace equation, heat equation, wave equation).
Classification of linear (and quasi-linear) second order partial differential equations in $\mathbf{R}^n$.
Examples.

2.10.2024 Some recalls: the space $C^k(\Omega)$, the space $C^k(\bar{\Omega})$, sets of regularity C^k or Lipschitz. Examples. The divergence theorem.
The continuity equation or equations of conservation of mass and some particular cases: the transport equation, the heat equation, the Laplace equation.

Recalls on the convolutions between two functions f, g.

7.10.2024 The lesson was not held due to the absence of many students.

9.10.2024 Definition of a family of mollifiers (or approximate identity) $(\rho_n)_n$. Some recalls on convergence of $\rho_n * f$ for $f \in C^0$ and $f \in L^p$, $p \in [1, +\infty)$.
A brief introduction to the theory of distributions: definition, examples, δ (Dirac's delta), some sequences approximating δ, examples and exercises.
Convolution of a distribution with a function.

14.10.2024 Convolution of δ with a function ($\delta * \psi = \psi$).

The Laplace equations: some physical examples, typical boundary conditions (Dirichlet, Neumann, Robin), definition of harmonic function.
Variational nature of harmonic functions: Dirichlet's principle (also in dimension 1). Comments and a mention to the example of Hadamard.

16.10.2024 Two exercises: minimisation of the Dirichlet functional in dimension1 and an example in dimension 2.
Energy methods for uniqueness: Dirichlet, Neumann, Robin problems.
A compatibility conditions for the Neumann problem. Holomorphic functions and harmonic functions.
The Laplacian is rotations invariant.

21.10.2024 Looking for radial functions satisfying $-\Delta u = \delta$ in the distributional sense. Definition of fundamental solutions for a general differential operator. The fundamental solutions E for the Laplacian. Proof of the following fact: $-\Delta E = \delta$ in dimension 2.

Some properties of harmonic functions: definitions of subharmonic and superharmonic functions. Comments.
In dimension 1 the only functions that are both subharmonic and superharmonic are $u(x) = ax + b$, $a, b \in \mathbf{R}$.

23.10.2024 The mean value property for C^2 functions; the maximum principle for subharmonic and superharmonic functions in a bounded and connected domain.
A convex function on a convex domain is subharmonic.
(Strong and weak maximum principles without proof). Some simple consequences of the maximum principles. EX: direct proof of the maximum principle for subharmonic functions $u \in C^2$ satisfying $-\Delta u \leqslant 0$.
Uniqueness via maximum principle for the Dirichlet problems.
Comparison for the Dirichlet problems.
Stability and continuous dependence on the boundary data for the Dirichlet problem.
Green's identities and some simple consequences.

28.10.2024 A representation for C^2 functions: the Stokes identity. Some consequences, in particular: there are no harmonic functions $u \not\equiv 0$ compactly supported, a harmonic function $u \in C^2$ is in fact C^∞. A function u harmonic in Ω is locally analytic in Ω (without proof).
Another representation formula for C^2 functions following by the Stokes identity. Consequence: a function $u \in C^2$ is sub-harmonic (superharmonic) if and only if $-\Delta u \leqslant 0$ ($-\Delta u \geqslant 0$).

Towards the solution of the Dirichlet problem: Green's function, Poisson kernel and possible representation formula for the solution of the Poisson equation with Dirichlet boundary datum.
Simmetry of the Green's function (without proof): $G(x, y) = G(y, x)$.
The Poisson kernel is harmonic.

Comment: once called G^x the function $\Omega \ni y \mapsto G(x, y)$ one has that $-\Delta G^x = \delta_x$ in $\mathcal{D}'(\Omega)$ and $G^x = 0$ in $\partial\Omega$.

30.10.2024 The Green function for a ball. Poisson formula in a ball and solution of the Dirichlet problem in a ball. In general $-\Delta u = f$ with $f \in C^0(\Omega)$ has no C^2 solutions (without proof).
Some consequences of the Poisson formula: u continuous and satisfying the mean value property is C^2 and harmonic.
The Harnack inequality (1 and 1-bis).

4.11.2024 The Harnack inequality (2). Comments.
Results about sequences of harmonic functions (Harnack principle).
Theorem of Liouville. Comments.

The Dirichlet problem in a bounded domain (the Perron's method): definition of some classes of sub-harmonic and super-harmonic functions: the classes $\sigma(\Omega; \varphi)$ and $\Sigma(\Omega; \varphi)$ with Ω bounded open set and $\varphi \in C^0(\partial\Omega)$.
Definition of harmonic lifting. The harmonic lifting $h_{v;x_o,\rho}$ of a sub-harmonic function v in a ball $B_\rho(x_o)$. The function $v_{x_o,\rho} \geqslant v$, where $v_{x_o,\rho} = v$ in $\Omega \setminus B_\rho(x_o)$ and $v_{x_o,\rho} = h_{v;x_o,\rho}$ in $\Omega \setminus B_\rho(x_o)$.
The function $v_{x_o,\rho}$ is sub-harmonic if v is sub-harmonic.
The exterior ball condition. Comments and examples.
In $n = 2$: an open set Ω of class C^2 satisfies the exterior ball condition.

6.11.2024 An open set Ω of class C^1 but not C^2 may not satisfy the exterior ball condition.
The function u_φ is harmonic in Ω, $u_\varphi(x) = \sup\{v(x) \,|\, v \in \sigma(\Omega; \varphi)\}$.
The same holds for U_φ, $U_\varphi(x) = \inf\{w(x) \,|\, w \in \Sigma(\Omega; \varphi)\}$.
Given $v \in \sigma(\Omega; \varphi)$ and $w \in \Sigma(\Omega; \varphi)$ one has that $w - v \geqslant 0$ and, in particular, $u_\varphi \leqslant U_\varphi$.
The function $u_\varphi \in C^0(\bar{\Omega})$, $u_\varphi = \varphi$ in $\partial\Omega$ and $u_\varphi = U_\varphi$.
Theorem (Perron): if Ω, open and bounded, satisfies the exterior ball condition then the Dirichlet problem admits a unique solution.
Barriers and regular points.

11.11.2024 Theorem: the Dirichlet problem admits a unique solution if and only if all the points of the boundary are regular. The exterior cone condition.
Examples of regular sets: convex sets, C^2 sets and in general the sets satisfying the exterior ball condition, but also the exterior cone condition.
A harmonic bounded function defined in $\Omega \setminus \{x_o\}$ admits a unique harmonic extension to $\{x_o\}$.

A brief mention to existence in external domains, capacity of a compact set and Wiener criterion.

Only for mathematical engineering

In the following the symbol $(*)$ denotes the result has been proved.

1.10.2024 Definition of a topological space. Comments and examples.
Definition of a σ-algebra. Comments and examples. Measurable sets. σ-algebra of Borel. The Lebesgue measure as a measure on $\mathbf{R}^n$ defined on the σ-algebra of Borel when $\mathbf{R}^n$ is endowed with the standard topology. There are sets in $\mathbf{R}^n$ that are not measurable.
Measurable functions in $\mathbf{R}^n$. Examples of measurable functions: continuous functions and the Dirichlet function. The set $\mathbf{Q}$ of rational numbers is measurable $(*)$ and has measure zero $(*)$.

Definition of a *norm* and of a *scalar product* on a vectorial space. Examples. The Cauchy-Schwarz inequality $(*)$.

8.10.2024 Normed spaces. A scalar product on a vectorial space induces a norm. Definition of a complete space. Banach spaces and Hilbert spaces. The function $f_p : \mathbf{R}^2 \to [0, +\infty)$, $f_p(x, y) = (|x|^p + |y|^p)^{1/p}$ for $p \in (0, +\infty)$; f_p defines a norm on $\mathbf{R}^2$ for $p \geqslant 1$ and cannot be a norm for $p \in (0, 1)$. Meaning of *almost everywhere* and of *for almost every*.
The space L^p with $p \in [1, \infty]$. Hölder's inequality and Minkowski's inequality. L^p is a Banach space, L^2 is a Hilbert space.
Theorem: given a sequence $(u_n)_n$ converging in L^p, $p \in [1, +\infty]$ there exists a subsequence pointwise converging almost everywhere.

15.10.2024 The dual space of a Banach space. Norm in the dual space. A linear map from a Banach space X to a Banach space Y is continuous if and only if is bounded. An example of an unbounded linear map valued in $\mathbf{R}$. The Riesz theorem (in L^p). The weak topology. A characterisation of convergence of sequences with respect to the weak topology.

22.10.2024 The set $S = \{x \in X \mid \|x\|_X = 1\}$ is not closed in the weak topology if X has infinite dimension.
Theorem: a bounded sequence $\{f_n\}_n$ in L^p admits a subsequence $\{f_{n_k}\}_k$ such that $\int f_{n_k} g \, dx \to \int f g \, dx$ for every $g \in L^{p'}$ if $p \in [1, +\infty)$ and for every $g \in L^1$ if $p = +\infty$. Definition of separable space. Examples L^p with $p \in [1, +\infty)$.

Sobolev spaces in dimension 1: definition of weak derivative, $W^{1,p}(I)$ and $W^{m,p}(I)$, I interval.
Density results: $C^m(I) \cap L^p(I)$ and $C^m(\bar{I})$ are dense in $W^{m,p}(I)$.
Definition of $W_0^{1,p}(I)$.

Theorem: $W^{1,p}(I) \subset C^0(\bar{I})$ and the injection $W^{1,p}(I) \hookrightarrow C^0(\bar{I})$ is continuous for every $p \in [1, +\infty]$, is compact for $p \in (1, +\infty]$, and the injection $W^{1,p}(I) \hookrightarrow L^q(I)$ is compact for every $q \in [1, +\infty)$.

29.10.2024 Examples and comments on the last theorem of the previous lesson. Lipschitz continuous and Hölder continuous functions. A function $u \in W^{1,p}$ satisfies $u(x) - u(y) = \int_x^y u'(t)\, dt$ and is $1/p'$-Hölder continuous for $p \in (1, +\infty)$, is Lipschitz continuous for $p = +\infty$.
The Poincaré inequality in a bounded interval. Comments and examples.
The dual space of $W_0^{1,p}(I)$, I interval, for $p \in [1, +\infty)$.

5.11.2024 Representation of a linear form in $W_0^{1,p}(I)$.

Sobolev spaces in dimension greater than 1: definition of $W^{1,p}(\Omega)$ and $W^{m,p}(\Omega)$, $\Omega \subset \mathbf{R}^n$. Examples of functions in $W^{1,p}(\Omega)$ in $\mathbf{R}^2$: a non-differentiable function, an unbounded function, a bounded discontinuous function that cannot be extended to a continuous function.
Density results.
Sobolev inequalities (embeddings of $W^{m,p}(\Omega)$ into other spaces).
An example of a function in L^q for every $q \in [2, +\infty)$, but not bounded.

12.11.2024 Examples of functions in $L^p(\mathbf{R})$ that have no limit at infinity.
Motivation of the exponent p^* in the Sobolev inequalities.
Compactness: Rellich-Kondrachov theorem. Counterexample in $\mathbf{R}$. The space $W_0^{1,p}(\Omega)$. The Poincaré inequality in a bounded set Ω.
The trace operator.

19.11.2024 Other comments on the trace. A function $u \in W_0^{1,p}$ iff $Tu = 0$ in $\partial\Omega$. The dual space of $W_0^{1,p}(\Omega)$. The triplet ($p = 2$) $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$ also for $p \geqslant \frac{2n}{n+2}$.

If $\{u_n\}_n$ weakly converges to u in a Hilbert space H and $\|u_n\|_H$ converges to $\|u\|_H$ then $\{u_n\}_n$ stronglyly converges to u in H (proved).

Some examples of solutions of a second order equation in dimension 1.

26.11.2024 Recalls on the orthogonal projection on a closed subspace of a Hilbert space and the orthogonal subspace.
The Lax-Milgram theorem (with the proof).
Application to linear partial differential equations, in particular for the existence and uniqueness of the solution to the problem

$$\begin{cases} -\mathrm{div}(a\, Du) + b\, Du + c\, u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

with a matrix, b vector, c scalar and $a_{ij}, b_i, c \in L^\infty(\Omega)$ and suitable assumptions on a, b, c.

3.12.2024 In Ω bounded one can solve also the problem the problem

$$\begin{cases} -\mathrm{div}(a\,Du) = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

How to solve the problem the problem

$$\begin{cases} -\mathrm{div}(a\,Du) + b\,Du + c\,u = f & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases}$$

with $\varphi \in L^2(\partial\Omega)$ (knowing that $\varphi = T\phi$ for some $\phi \in H^1(\Omega)$).
A priori estimates for this last problem.
Euler-Lagrange equation for the functional $Fu = 2^{-1}[\int_\Omega (aDu, Du)dx + \int_\Omega cu^2 dx] - \langle f, u\rangle$. Comments on the minimisation problem.
Difference between Dirichlet and Neumann conditions for the weak formulation of $-\Delta u = f$.
Regularity results. Weak maximum principle. Harnack inequality. Comments.

10.12.2024 For function $f : [0, T] \to B$, B Banach space, definition of a measurable function, Bochner integral, space $L^p(0, T, B)$. Evolution triple $V \subset H \subset V'$ ($H = L^2(\Omega)$ and V such that $H^1_0(\Omega) \subset V \subset H^1(\Omega)$) and the spaces $\mathcal{V}, \mathcal{H}, \mathcal{V}', \mathcal{W}$. The space $\mathcal{W}$ continuously embeds in $C^0([0, T]; H)$.
Abstract position of a parabolic problem. A priori estimates (dependence of the solution on the data) (with proof).

17.12.2024 Existence and uniqueness of the solution of a linear parabolic problem (with proofs).
Some regularity results. Weak maximum principle. Harnack inequality. Examples.