C BOUDARY REGULARITY

EXERCISE C. 1 Prove that the two definitions of set of class C^1 given at class are equivalent.

E - CONVOLUTION

EXERCISE E.1 Consider $f, g : \mathbb{R} \to \mathbb{R}$, $f, g \in C^1(\mathbb{R})$. Prove that

$$\frac{d}{dx}\left(f * g\right)(x) = f' * g(x) = f * g'(x).$$

Analogously if $f, g : \mathbb{R}^n \to \mathbb{R}$

$$\frac{\partial}{\partial x_i}(f * g)(x) = \frac{\partial f}{\partial x_i} * g(x) = f * \frac{\partial g}{\partial x_i}(x)$$

EXERCISE E.2 Consider a function $g : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}$ and a function $u_0 : \mathbb{R}^n \to \mathbb{R}$.

Denote by "x" the variable in $\mathbb{R}^n$ and by "t" the variable in $(0, +\infty)$ and for convenience we write

$$g_t(x) \quad \text{for} \quad g(x,t)$$

(so $g_t : \mathbb{R}^n \to \mathbb{R}$ is a family of functions defined in $\mathbb{R}^n$)

Suppose $g, \frac{\partial g}{\partial t} \in C^0(\mathbb{R}^n)$.

Suppose to know that $g_t * u_0$ is the solution of the following problem

$$\circledast \quad \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(0) = u_0 & \text{in } \mathbb{R}^n \end{cases}.$$

(The convolution is made with respect only to the spatial variable

$$g_t * u_0 (x) = \int_{\mathbb{R}^n} g_t(y)\, u_0(x-y)\, dy \quad , \quad \text{but}$$

the solution will depend both on x and t).

Prove that if u_0 is harmonic (i.e. $\Delta u_0 = 0$)

$g_t * u_0$ is independent of t.

[Hint: use the previous exercise]

EXERCISE E.3 A function $g : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}$, or a family $\{g_t\}_{t>0}$, such that $g_t * u_0$ is the solution of $\circledast$ above does exist.

If $n = 1$ this family is the following

$$g_t(x) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}$$

Verify that $g_t * g_s = g_{t+s}$ $\circledast$

[In fact ⊛ is true for every $\alpha > 0$ and g_t^α where

$$g_t^\alpha(x) = \frac{1}{\sqrt{\alpha \pi t}} e^{-\frac{x^2}{\alpha t}}$$
]

What can you deduce from that ?

(Rethink to this exercise when you will see the heat equation)

COMPUTATIONS (only as a check after you tried by yourself)

for simplicity we consider $\alpha = 1$ in ⊛

$$g_t * g_s(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{\pi t}} e^{-\frac{y^2}{t}} \frac{1}{\sqrt{\pi s}} e^{-\frac{(x-y)^2}{s}} dy$$

$$e^{-\frac{y^2}{t}} e^{-\frac{(x-y)^2}{s}} = e^{-\frac{sy^2 + t(x-y)^2}{ts}}$$

$$s{}^2 + tx^2 + ty^2 - 2txy = \left(\sqrt{t+s}\, y - \frac{xt}{\sqrt{t+s}}\right)^2 + tx^2 - \frac{x^2t^2}{t+s}$$

$$g_t * g_s(x) = \frac{1}{\pi\sqrt{st}} \int_{\mathbb{R}} e^{-\frac{1}{ts}\left(\sqrt{t+s}\, y - \frac{xt}{\sqrt{t+s}}\right)^2 - \frac{x^2}{ts}\left(t - \frac{t^2}{t+s}\right)} dy$$

$$y = \left(\sqrt{ts}\, \xi + \frac{xt}{\sqrt{t+s}}\right)\frac{1}{\sqrt{t+s}}$$

$$\left(\sqrt{t+s}\, y - \frac{xt}{\sqrt{t+s}}\right)\frac{1}{\sqrt{ts}} = \xi$$

$$g_t * g_s(x) =$$

$$= \frac{1}{\pi \sqrt{ts}} \int e^{-\xi^2} e^{-\frac{x^2}{ts}\left(\frac{\cancel{t^2} + st - \cancel{t^2}}{t+s}\right)} \frac{\sqrt{ts}}{\sqrt{t+s}} \, d\xi$$

$$= \frac{1}{\pi \sqrt{t+s}} e^{-\frac{x^2}{t+s}} \int e^{-\xi^2} d\xi =$$

$$= \frac{1}{\sqrt{\pi (t+s)}} e^{-\frac{x^2}{t+s}}$$

E - DISTRIBUTIONS

EXERCISE E.4

Compute the derivative D_x of the distributions

$$\tilde{H}(x,y) := \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$H_f(x,y) := \begin{cases} f(y) & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\tilde{H}_f(x,y) := \begin{cases} f(x,y) & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad f \in C^1([0,+\infty) \times \mathbb{R})$$

EXERCISE E.5

Given a distribution T

show that there exists $D_v T$ $(v \in \mathbb{R}^n,\ |v| = 1)$

where $D_v \phi$ denotes the directional derivative of ϕ

in the direction v and

$$\langle D_v T, \phi \rangle = - \langle T, D_v \phi \rangle$$

EXERCISE E.6

Compute the derivative D_v of the distributions

$$H_v(x,y) := \begin{cases} 1 & \text{if } (x,y) \in A \\ 0 & \text{if } (x,y) \notin A \end{cases}$$

where $v = (v_1, v_2)$, $|v| = 1$, and

$$A = \{ (x,y) \in \mathbb{R}^2 \mid \langle (x,y), (v_1, v_2) \rangle \geq 0 \}$$

EXERCISE E.7

Find a function defined in $\mathbb{R}^2$ whose some derivative (of some order) is δ ($\delta = \delta_{(0,0)}$)

EXERCISE E.8

Suppose to know that, given $T \in \mathcal{D}'(\mathbb{R}^n)$ and $h \in \mathcal{D}(\mathbb{R}^n)$, $T * h$ is a function in $L^1_{loc}(\mathbb{R}^n)$.

Suppose moreover to know that there exists a sequence of $L^1_{loc}(\mathbb{R}^n)$ functions $\{f_k\}_k$ such that

$f_k \to T$ in $\mathcal{D}'$.

Show that $T * \phi$ is C^∞ for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

6 - VARIATIONAL FORMULATION OF AN ELLIPTIC EQ.

EX 6.1 Given a symmetric matrix $a = (a_{ij}(x))_{i,j=1}^{n}$, $a_{ij} \in C^1(\Omega)$ with $\Omega \overset{\circ}{\subseteq} \mathbb{R}^n$, satisfying

$$\lambda |\xi|^2 \le (a(x) \cdot \xi, \xi) \le \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n$$

with $\Lambda \ge \lambda > 0$ given constants,

find F such that the equation

$$- \operatorname{div} (a(x) \cdot Du) = f \quad \text{in } \Omega$$

is the Euler-Lagrange equation corresponding to F.

EX 6.2 (*) Find a functional F such that

$$- \Delta u + D\phi \cdot Du = f \quad \text{in } \Omega$$

is the Euler-Lagrange equation corresponding to F.

[Hint: find a suitable function ω by which multiply $|Du|^2$]

EXERCISE 6.3

Find the solution of the minimum problem ($n = 1$)

$$\mathcal{F}u = \frac{1}{2}\int_0^1 |u'(x)|^2 dx \longrightarrow \min$$

$$u(0) = a \quad , \quad u(1) = b$$

EX 6.4 Apply the energy method to study uniqueness for the Neumann and Robin problems (N) and (R)

$$(N) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g & \text{in } \partial\Omega \end{cases}$$

$$(R) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ \alpha \dfrac{\partial u}{\partial \nu} + \beta u = g & \text{in } \partial\Omega \end{cases} \qquad \alpha, \beta > 0$$

? Have they a unique solution ?

H - FUNDAMENTAL SOLUTION

EX H.1 Say for which $p \in [1, +\infty)$ one has $\mathcal{E} \in L^p(B_1)$

EX H.2 Verify that $\mathcal{E}$ and $\nabla \mathcal{E} \in L^1_{loc}(\mathbb{R}^n)$ both for $n = 2$ and for $n \geq 3$.

EX H.3 Write the laplacian of u, $\Delta u = u_{xx} + u_{yy}$ in $\mathbb{R}^2$, in polar coordinates.

I - PROPERTIES OF HARMONIC FUNCTIONS

EX I.1 Prove that if $u \in C^2(\Omega)$ is harmonic in Ω then u^2 is subharmonic in Ω.

EX I.2 Consider $n = 1$.

Compute for which $\alpha \in \mathbb{R}$ the function $u(x) = |x|^\alpha$ is subharmonic (u defined in its natural domain).

EX I.3 If $u : \Omega \to \mathbb{R}$ is convex (concave) then u is subharmonic (superharmonic).

EX I.4 Consider $u : \mathbb{R}^n \to \mathbb{R}$ harmonic and suppose $u \in L^p(\mathbb{R}^n)$ for some $p \in [1, +\infty)$.

Prove that $u \equiv 0$.

[Hint - Show that $|u|^p$ is subharmonic using the Jensen's inequality]

<u>Jensen's inequality</u> Consider a measure space

(X, Σ, μ), X set
Σ σ-algebra in X
μ positive measure
(i.e. $\mu(A) \geq 0$ for every $A \in \Sigma$)

and suppose $\mu(X) = 1$.

Consider a function $f : X \to \mathbb{R}$, $f \in L^1(X, \mu)$, and $\varphi : \mathbb{R} \to \mathbb{R}$ convex. Then

$$\varphi\left(\int_X f \, d\mu \right) \leq \int_X \varphi \circ f \, d\mu$$

Clearly it is not necessary to have $\mu(X) = 1$, but $\mu(X) < +\infty$. In this case one needs to replace the measure μ by the measure ν defined by $\nu(A) := \dfrac{\mu(A)}{\mu(X)}$ for $A \in \Sigma$

EX I.5 Show that if $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and $Du \in L^2(\mathbb{R}^n)$ then u is constant.

EX I.6 Prove that if $u \in C^2(\mathbb{R}^n)$ is harmonic (in $\mathbb{R}^n$) and $\int_{\mathbb{R}^n} u^2(x) \, dx < +\infty$ then $u \equiv 0$.

[Hint: use I.1 and Hölder inequality]

EX I.7 Consider $u : \Omega \to \mathbb{R}$ harmonic and <u>positive</u>. Say for which values of β the function $v = u^\beta$ is harmonic and subharmonic.

EX I.8 Consider u as in Ex I.3.
Verify that $v(x) = \log u(x)$ is superharmonic.
What can we say about $v(x) = f(u(x))$
with $u : \Omega \to \mathbb{R}$ harmonic and
$f : \mathbb{R} \to \mathbb{R}$ convex?

[hint: suppose first $f \in C^2(\mathbb{R})$.
In the general case approximate f with a sequence $\{f_n\}_n$, f_n convex and C^2, $f_n \to f$ locally uniformly.]

EX I.9 Let Ω be $\{x \in \mathbb{R}^n \mid |x| > 1\}$.
Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ harmonic in Ω and satisfying

$$\lim_{|x| \to +\infty} u(x) = 0.$$

Prove that

$$\max_{\Omega} |u| = \max_{\partial\Omega} |u|$$

EX I.10 Let $u \in C^2(\mathbb{R}^2)$ subharmonic

For $r > 0$ denote $M(r)$ the quantity $\max_{x \in \partial B_r(0)} u(x)$

Prove that for each $r_1, r_2 > 0$,

$r_1 < r_2$, one has

$$M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} M(r_2)$$

for every $r \in (r_1, r_2)$.

H, L, N - DIRICHLET PROBLEM

Ex L/N.1 Write the solutions of the following problems

$$\begin{cases} -u'' = 0 & [r, R] \subseteq \mathbb{R} \\ u(r) = 1 \\ u(R) = 0 \end{cases}$$

$$\begin{cases} -\Delta u = 0 & \text{in } B_R(0) \setminus \overline{B_r(0)} \\ u = 1 & \text{in } \partial B_r(0) \\ u = 0 & \text{in } \partial B_R(0) \end{cases} \qquad 0 < r < R, \quad n \geq 2$$

Ex L.2 Compute Δv with $v(x) = \log|x|$ in $\mathbb{R}^n \setminus \{0\}$.

Ex L.3 Consider $\varphi \in C^0(\partial B_1(0))$ and u the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } B_1(0) \subseteq \mathbb{R}^n \\ u = \varphi & \text{in } \partial B_1(0) \end{cases}$$

Can we guess the value $u(0)$?

[hint: clearly in dependence on the informations, i.e. φ]

EX N.1 If $\partial\Omega$ is of class C^2 then Ω satisfies the exterior sphere condition.
(prove it in $n = 2$)

EX N.2 Show that if $\partial\Omega$ is of class C^1 (but not C^2) Ω could not satisfy the ex. sphere cond.
(in $n = 2$)