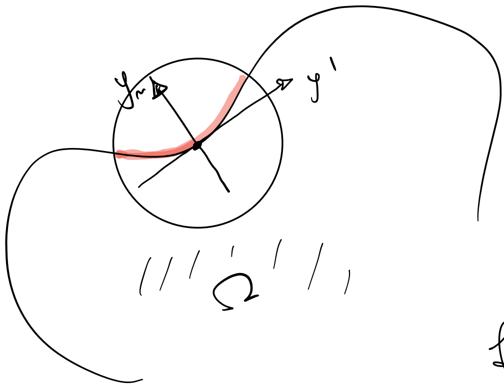
LOOK AT THE SOLUTIONS <u>ONLY</u>

<u>AFTER</u> TRYING TO SOLVE THE

EXERCISES

C.1 | Suppose first we have a system of coordinates (y', y_n) around $x_0 \in \partial\Omega$ with the origin $(0',0)$ corresponding to x_0 and a function $u \in C^1$ for which the two conditions 1. and 2. given in the definition are verified.



Then it is sufficient to consider a change of variables $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\phi(x) = y \qquad \left(\phi(x_0) = (0',0) \right)$$

(ϕ is a rototranslation, so it is bijective, C^1 and ϕ^{-1} is C^1)

Then it is sufficient to consider

$$f : U \to \mathbb{R}^n, \qquad U = \phi^{-1}(V \times u(V))$$

$$f(x) := (\phi(x))_n - u((\phi(x))')$$

where $\phi(x) = ((\phi(x))', \phi(x))$, $(\phi(x))' \in \mathbb{R}^{n-1}$

By the choice of u we have that $f(x) = 0$ for $x \in \partial\Omega$ and $(i \in \{1, \dots n\})$

$$\frac{\partial f}{\partial x_i}(x) = \sum_{j=1}^{n-1} \frac{\partial u}{\partial y_j}((\phi(x))') \frac{\partial \phi_j}{\partial x_i}(x) + \frac{\partial \phi_n}{\partial x_i}(x)$$

or $\nabla f(x) = \nabla U(\phi(x)) \cdot D\phi(x)$ where

$U(y) = y_n - u(y')$

If $\phi(x) = R \circ T(x)$

R rotation

T translation $\quad (T(x) = x - x_0)$

$D\phi(x) = R$ $\quad$ R constant matrix with determinant equal to 1 (R is orthogonal)

Then, since $\nabla U(y) = \left(\frac{\partial u}{\partial y_1}, \dots \frac{\partial u}{\partial y_{n-1}}, 1\right)$ clearly satisfies $|\nabla U| \neq 0$, so ∇f does.

Now suppose $\exists f$ satisfying the second definition, i.e.

$f : U \to \mathbb{R}$, U neighbourhood of $x_0 \in \partial\Omega$,

with $f \in C^1(U)$, $|\nabla f(x_0)| \neq 0$,

$$\partial\Omega \cap U = \{x \in U \mid f(x) = 0\}.$$

Just for sake of simplicity we can suppose $x_0 = (0, \dots 0) \in \mathbb{R}^n$ (otherwise we translate as before) and, since $|\nabla f(0)| \neq 0$, suppose

$$\frac{\partial f}{\partial x_n}(0) \neq 0 .$$

Then, by the implicit functions theorem (or Dini's theorem) we have that the set

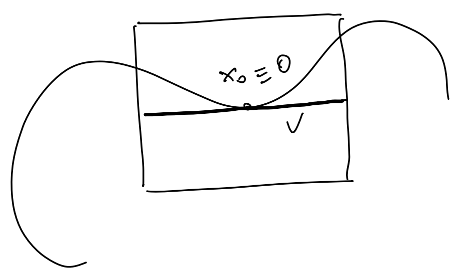
$Z = \{ x \in U \mid f(x) = 0 \}$ is a graph, at least in a neighbourhood of 0, of the first $n-1$ variables, i.e.

$$f(x_1, \dots x_{n-1}, u(x_1, \dots x_{n-1})) = 0$$

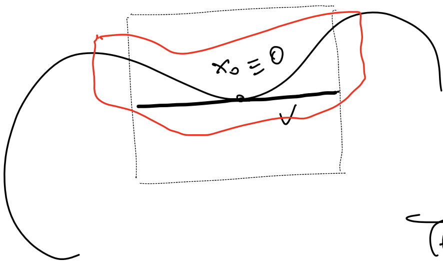
for some $u : V \to \mathbb{R}$, where V neighbourhood of $(0, \dots 0) \in \mathbb{R}^{n-1}$

and u is $C^1(V)$.

Then, by Dini's theorem, we can suppose that the neighbourhood U, taking a smaller set if necessary, is $V \times I$ with I interval, V compact and $I = [a, b]$ $(a < 0, b > 0)$, $U \cap \partial\Omega$ is the graph of $u : V \to \mathbb{R}$.



By the continuity of f, since $f = 0$ in $U \cap \partial\Omega$ and $\frac{\partial f}{\partial x_n}(0) \neq 0$, we can find a set S where
$f < 0$ if $x \in S \cap \Omega$
$f > 0$ if $x \in S \cap \Omega^c$



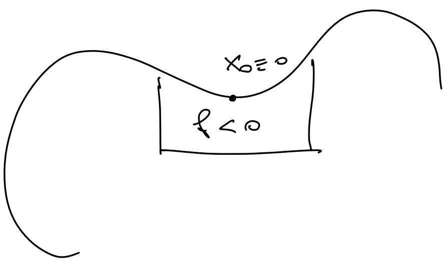
or vice verse (depending on the sign of $\frac{\partial f}{\partial x_n}$)

Then, taking a smaller U if necessary $(U \cap S)$,

we have found that

$f < 0$ if $x \in U \cap \Omega$
$f > 0$ if $x \in U \cap \Omega^c$ (or vice verse)

This in particular means that $f(x) < 0$ if $x_n < u(x_1, \dots x_n)$ $(x \in U \cap \Omega)$



and we are done. //

E.3 We do the computations for $n = 1$.

$$g_t * g_s(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{\pi t}} e^{-\frac{y^2}{t}} \frac{1}{\sqrt{\pi s}} e^{-\frac{(x-y)^2}{s}} dy$$

$$e^{-\frac{y^2}{t}} e^{-\frac{(x-y)^2}{s}} = e^{-\frac{sy^2 + t(x-y)^2}{ts}}$$

$$s^2 + tx^2 + ty^2 - 2txy = \left(\sqrt{t+s}\, y - \frac{xt}{\sqrt{t+s}}\right)^2 + tx^2 - \frac{x^2t^2}{t+s}$$

$$g_t * g_s(x) = \frac{1}{\pi\sqrt{st}} \int_{\mathbb{R}} e^{-\frac{1}{ts}\left(\sqrt{t+s}\, y - \frac{xt}{\sqrt{t+s}}\right)^2 - \frac{x^2}{ts}\left(t - \frac{t^2}{t+s}\right)} dy$$

$$y = \left(\sqrt{ts}\,\xi + \frac{xt}{\sqrt{t+s}}\right)\frac{1}{\sqrt{t+s}}$$

$$\left(\sqrt{t+s}\, y - \frac{xt}{\sqrt{t+s}}\right)\frac{1}{\sqrt{ts}} = \xi$$

$$g_t * g_s(x) =$$

$$= \frac{1}{\pi\sqrt{ts}} \int e^{-\xi^2} e^{-\frac{x^2}{ts}\left(\frac{\cancel{t^2} + st - \cancel{t^2}}{t+s}\right)} \frac{\sqrt{ts}}{\sqrt{t+s}}\, d\xi$$

$$= \frac{1}{\pi\sqrt{t+s}} e^{-\frac{x^2}{t+s}} \int e^{-\xi^2} d\xi =$$

$$= \frac{1}{\sqrt{\pi(t+s)}} e^{-\frac{x^2}{t+s}}$$

When you will see the heat equation
relook at this exercise : what can you derive
from $g_t * g_s = g_{t+s}$?

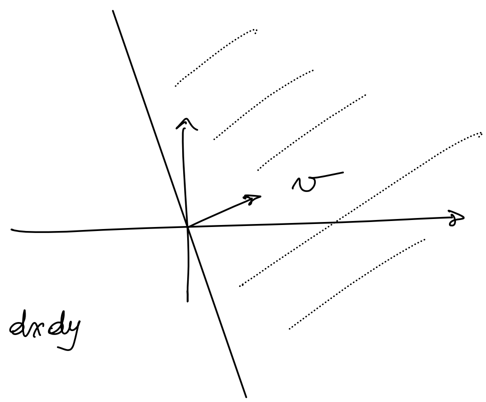
E.4 We have to compute $D_x T_{\tilde{H}}$, that is

$$\langle D_x T_{\tilde{H}}, \phi \rangle = \int_{\mathbb{R}} \phi(0,y)\, dy$$

$$\langle D_x T_{H_f}, \phi \rangle = \int_{\mathbb{R}} f(y)\, \phi(0,y)\, dy$$

$$\langle D_x T_{\tilde{H}_f}, \phi \rangle = \int_{\mathbb{R}} dy \int_0^{+\infty} \frac{\partial f}{\partial x}(x,y)\, \phi(x,y)\, dx + \int_{\mathbb{R}} f(0,y)\, \phi(0,y)\, dy$$

E.6 The set A is the dotted one



$$\langle D_v T_{H_v}, \phi \rangle = - \iint_A \langle \nabla \phi, v \rangle\, dx\, dy$$

(first show E.5)

E.7 For instance, the derivative D^α with $\alpha = (1,1)$ (i.e. ∂_{xy}) of the function

$$f(x,y) = \begin{cases} 1 & \text{if } x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

E.8 We start showing that $T * \phi$ is a continuous function.

Consider $\{f_k\}_{k\in\mathbb{N}} \subseteq L^1_{loc}$ approximating T, i.e. $T_{f_k} \to T$ in $\mathcal{D}'$ where $\langle T_{f_k}, \psi\rangle_{\mathcal{D}'\times\mathcal{D}} := \int f_k \psi$

Then $T * \phi(x) = \lim_{k\to+\infty} \langle T_{f_k}, \tau_x \check{\phi}\rangle$

$$\langle T_{f_k}, \tau_x \check{\phi}\rangle = \int_{\mathbb{R}^n} f_k(y)\, \phi(x-y)\, dy$$

$$\langle T_{f_k}, \tau_{x_0} \phi\rangle = \int_{\mathbb{R}^n} f_k(y)\, \phi(x_0-y)\, dy$$

$$\left| \langle T_{f_k}, \tau_x \check{\phi}\rangle - \langle T_{f_k}, \tau_{x_0} \phi\rangle \right| =$$

$$= \left| \int_K f_k(y) \big(\phi(x-y) - \phi(x_0-y)\big)\, dy \right| \leq$$

$$\leq \| f_k \|_{L^1(K)} \| \tau_x \check{\phi} - \tau_{x_0} \check{\phi} \|_{L^1(K)}$$

$$\leq \| \tau_x \check{\phi} - \tau_{x_0} \check{\phi} \|_{L^1(K)}$$

where $K = \operatorname{supp} \tau_x \check{\phi} \cup \operatorname{supp} \tau_{x_0} \check{\phi}$

Then we conclude that, since ϕ is uniformly continuous, that for every $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

if $|x - x_0| < \delta$ then $\| \tau_x \check{\phi} - \tau_{x_0} \check{\phi} \|_{L^1(K)} < \varepsilon$

and in particular

$$\left| T_{f_k} * \phi(x) - T_{f_k} * \phi(x_0) \right| < \varepsilon \quad \text{uniformly in } k \in \mathbb{N}$$

In particular

$$\lim_{x \to x_0} \lim_{k \to +\infty} \langle T_{f_k}, \tau_x \check{\phi} \rangle =$$

$$= \lim_{k \to +\infty} \lim_{x \to x_0} \langle T_{f_k}, \tau_x \check{\phi} \rangle = T * \phi(x_0)$$

Since with the same argument one gets

$$\lim_{x \to x_0} T * D^\alpha \phi(x) = T * D^\alpha \phi(x_0)$$

one gets that, given $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$T * \phi \in C^\infty(\mathbb{R}^n).$$

G.1

$$\mathcal{F}u = \frac{1}{2}\int_\Omega (a(x)\cdot Du, Du)\,dx - \int_\Omega fu\,dx$$

product matrix – vector

$(\cdot,\cdot)$ scalar product in $\mathbb{R}^n$

G.2

$$\mathcal{F}u = \frac{1}{2}\int_\Omega e^{-\phi(x)}|Du|^2\,dx - \int_\Omega fu\,dx$$

G.3

The solution is : $u(x) = (b-a)x + a$

Indeed consider

$$\mathcal{F}u = \frac{1}{2}\int_0^1 |u'(x)|^2\,dx \longrightarrow \min \quad \text{with } u \in C^1([0,1])$$

$$u(0) = a, \quad u(1) = b$$

and choose a function $\varphi \in C^1_c((0,1))$ so that

$$u + t\varphi \in C^1([0,1]) \quad \text{and}$$

$$(u+t\varphi)(0) = a, \quad (u+t\varphi)(1) = b.$$

Differentiate with respect to t

$$\frac{d}{dt} \mathcal{F}(u+t\varphi) = \frac{1}{2} \int_0^1 \frac{d}{dt} \left(u'(x) + t\varphi'(x)\right)^2 dx =$$

$$= \frac{1}{2} \int_0^1 2\left(u'(x) + t\varphi'(x)\right) \varphi'(x)\, dx$$

and impose $\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(u+t\varphi) = 0$, that is

$$\int_0^1 u'(x)\, \varphi'(x)\, dx = 0 .$$

This holds for every $\varphi \in C^1_c((0,1))$

i.e. (integrating by parts) $\displaystyle -\int_0^1 u''(x)\, \varphi(x)\, dx = 0 \quad \forall \varphi \in C^1_c((0,1))$.

This implies that $u''(x) = 0$, i.e.

$$u(x) = \alpha x + \beta \qquad \text{for some } \alpha, \beta \in \mathbb{R} .$$

Taking into account $u(0) = a$, $u(1) = b$

one finds the minimum.

6.4

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g & \text{in } \partial\Omega \end{cases}$$

As done for the Dirichlet problem, suppose to have u_1 and u_2 two solutions and define $u := u_1 - u_2$ which satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega \end{cases}$$

One has

$$\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u \, d\mathcal{H}^{n-1} = 0$$

and since $\dfrac{\partial u}{\partial \nu} = 0$ in $\partial\Omega$ $\quad \displaystyle\int_\Omega |\nabla u|^2 dx = 0$

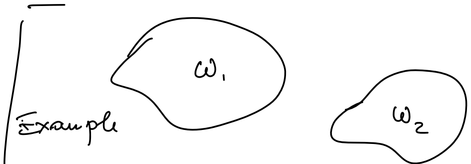
i.e. u constant in each connected component of Ω. Then we can <u>only</u> conclude that

$u = c_i$ in ω_i, $c_i \in \mathbb{R}$, $\quad \Omega = \omega_1 \cup \dots \cup \omega_k$

ω_i open and connected

That is

$u_1 = u_2 + c_i$ in ω_i



$\Omega = \omega_1 \cup \omega_2$

Then the solution of $\begin{cases} -\Delta u = f \\ \frac{\partial u}{\partial \nu} = g \end{cases}$

is unique modulo an additive constant (or there are infinite solutions, $u + c$, $c \in \mathbb{R}$ and u a solution).

For the Robin boundary conditions $(\alpha, \beta > 0)$

$$\begin{cases} -\Delta u_i = f & \text{in } \Omega \\ \alpha \frac{\partial u_i}{\partial \nu} + \beta u_i = g & \text{in } \partial\Omega \end{cases} \qquad i = 1, 2$$

the solution is unique, since we can proceed as above writing ($u = u_1 - u_2$, $\alpha \frac{\partial u}{\partial \nu} + \beta u = 0$)

$$\frac{\partial u}{\partial \nu} = -\frac{\beta u}{\alpha}.$$

H.2 We verify $E \in L^1_{loc}(\mathbb{R}^n)$ for $n \geq 3$, being the computation for ∇E $(n \geq 3)$ similar and the case $n = 2$ easier.

Consider the "spherical" coordinates in $\mathbb{R}^n$, $n \geq 3$,

$F(\vartheta_1, \dots \vartheta_{n-1}, \rho)$ with $\rho = |x|$, $x \in \mathbb{R}^n$,

$\left(\vartheta_i \in (0, \pi) \quad \vartheta_{n-1} \in (0, 2\pi) \right)$ $\vartheta_1, \dots \vartheta_{n-1}$ suitable angolar coordinates
$i = 1, \dots n-2$

The Jacobian of F (that we do not calculate) satisfies

$$(\bullet) \quad |DF| \leq \rho^{n-1}$$

Clearly the only thing to verify is that E (and its gradient) is integrable on sets containing the origin, being $E \in C^0(\mathbb{R}^n \setminus \{0\})$. Take, for sake of simplicity, $B_1(0)$. Then

$$\int_{B_1(0)} \frac{1}{|x|^{n-2}} \, dx = \int_0^{\pi} d\vartheta_1 \dots \int_0^{\pi} d\vartheta_{n-2} \int_0^{2\pi} d\vartheta_{n-1} \int_0^1 \frac{d\rho}{\rho^{n-2}} |DF(\vartheta_1, \dots \vartheta_{n-1}, \rho)|$$

$$\underset{\text{by } (\bullet)}{\leq} \text{const} \int_0^1 \frac{1}{\rho^{n-2}} \rho^{n-1} \, d\rho < +\infty$$

We can write

$$\int_{B_1(0)} \frac{1}{|x|^{n-2}}\, dx \lesssim \int_0^1 \frac{1}{\rho^{n-2}} \rho^{n-1}\, d\rho$$

where by $\ldots \lesssim \ldots$ we mean $\ldots \leq$ constant $\ldots$

Compute the gradient ∇E and do an analogous estimate for $n \geq 3$.

For $n=2$ use the polar coordinates.

H.3 LAPLACIAN IN POLAR COORDINATES

$$F(\rho,\theta) = (\rho\cos\theta,\ \rho\,\mathrm{sen}\,\theta)$$

$$F^{-1}(x,y) = \left(\sqrt{x^2+y^2},\ g(x,y)\right)$$ where

$$g(x,y) = \begin{cases} \operatorname{atan}\frac{y}{x} & \text{if } x>0,\ y\geq 0 \\ \pi/2 & \text{if } x=0,\ y>0 \\ \operatorname{atan}\frac{y}{x}+\pi & \text{if } x<0 \\ \frac{3}{2}\pi & \text{if } x=0,\ y<0 \\ \operatorname{atan}\frac{y}{x}+2\pi & \text{if } x>0,\ y<0 \end{cases}$$

In the following, given $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$, we denote

$$\tilde{u}(\rho,\theta) := u\circ F(\rho,\theta) = u(\rho\cos\theta,\ \rho\,\mathrm{sen}\,\theta)$$

and

$$u(x,y) = \tilde{u}\circ F^{-1}(x,y) = \tilde{u}\left(F^{-1}(x,y)\right). \qquad \circledast$$

Now everything is depending on x and y using $\circledast$.

$$u_x = \tilde{u}_\rho \frac{x}{\sqrt{x^2+y^2}} - \tilde{u}_\theta \frac{y}{x^2+y^2}$$

$$u_{xx} = \tilde{u}_{\rho\rho} \frac{x^2}{x^2+y^2} - \tilde{u}_{\rho\theta} \frac{y}{x^2+y^2} \frac{x}{\sqrt{x^2+y^2}} +$$

$$+ \tilde{u}_\rho \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} \right) +$$

$$- \tilde{u}_{\theta\theta} \frac{xy}{(x^2+y^2)^{3/2}} + \tilde{u}_{\theta\theta} \frac{y^2}{(x^2+y^2)^2} + \tilde{u}_\theta \frac{2xy}{(x^2+y^2)^2}$$

$$u_y = \tilde{u}_\rho \frac{y}{\sqrt{x^2+y^2}} + \tilde{u}_\theta \frac{x}{x^2+y^2}$$

$$u_{yy} = \tilde{u}_{\rho\rho} \frac{y^2}{x^2+y^2} + \tilde{u}_{\rho\theta} \frac{x}{x^2+y^2} \frac{y}{\sqrt{x^2+y^2}} +$$

$$+ \tilde{u}_\rho \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{(x^2+y^2)^{3/2}} \right) +$$

$$+ \tilde{u}_{\rho\theta} \frac{xy}{(x^2+y^2)^{3/2}} + \tilde{u}_{\theta\theta} \frac{x^2}{(x^2+y^2)^2} - \tilde{u}_\theta \frac{2xy}{(x^2+y^2)^2}$$

Summing the expressions for u_{xx} and u_{yy} you obtain the laplacian depending on the derivatives with respect to ρ and θ of $\tilde{u}$.

These terms are to be evaluated in $F^{-1}(x,y)$ to have the equality, since $u_{xx} + u_{yy}$ depend on x and y, but once obtained the equality you can write the right hand term depending on ρ and θ and using $\sqrt{x^2+y^2} = \rho$, $x = \rho \cos\theta$, $y = \rho \operatorname{sen}\theta$ one derives that the laplacian in polar coordinates is

$$\tilde{u}_{\rho\rho}(\rho,\theta) + \frac{1}{\rho}\tilde{u}_{\rho}(\rho,\theta) + \frac{1}{\rho^2}\tilde{u}_{\theta\theta}(\rho,\theta)$$

I.2 Computing u'' (when possible) we get

$$u''(x) = \alpha(\alpha-1)|x|^{\alpha-2}$$

for $\alpha \geq 2$ $\quad u'' : \mathbb{R} \to \mathbb{R}$ and in this case u is subharmonic ($-u'' \leq 0$)

for $\alpha \in [1,2)$ $\quad u'' : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $-u'' < 0$ in $(-\infty,0)$ and in $(0,+\infty)$

therefore is subharmonic in $(-\infty,0)$ and is subharmonic in $(0,+\infty)$

To see that is subharmonic in $\mathbb{R}$ it is sufficient to evaluate the mean value of u in $\partial(x-r,x+r) = \{x-r, x+r\}$ and see that

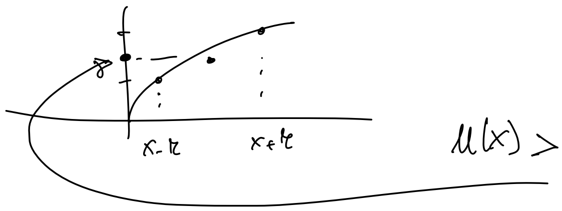
$$u(x) \leq \frac{u(x-r)+u(x+r)}{2}$$ since u is convex

for $\alpha < 1$ we have that

$$u(x) \leq \fint_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y) = \frac{u(x+r)+u(x-r)}{2}$$

only for $x = 0$

otherwise one could have the reverse inequality as in the picture below.

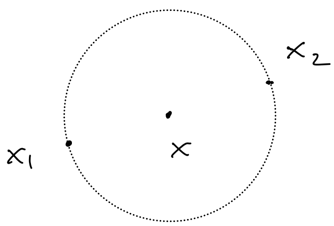


$$u(x) > \frac{u(x+r)+u(x-r)}{2}$$

I.3

Suppose u convex.

Consider $x_1, x_2 \in \Omega$ and $x := \frac{x_1+x_2}{2}$



Then $u(x) \leq \frac{1}{2}\left(u(x_1) + u(x_2)\right)$.

Integrating and since

$$u(x) = \fint_{\partial B_r(x)} \underbrace{u(x)}\, d\mathcal{H}^{n-1}(y) \quad \text{we have}$$

! $u(x)$, not $u(y)$

Since this holds for every pair (x_1, x_2) with $x_1, x_2 \in \partial B_r(x)$ and with $x = \frac{x_1+x_2}{2}$ we derive that

$$u(x) \leq \frac{1}{2}\fint_{\partial B_r(x)} u(x_1)\, d\mathcal{H}^{n-1}(x_1) + \frac{1}{2}\fint_{\partial B_r(x)} u(x_2)\, d\mathcal{H}^{n-1}(x_2)$$

$$= \fint_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y)$$

I.4 (Since u is harmonic ($u \in C^2$ and $\Delta u = 0$ in Ω)

we have that

$$u(x) = \fint_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y)$$

then

$$|u(x)|^p = \left| \fint_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y) \right|^p \overset{\text{Jensen}}{\leq}$$

$$\leq \fint_{\partial B_r(x)} |u(y)|^p\, d\mathcal{H}^{n-1}(y)$$

$$(=) \quad \sigma_n r^{n-1} |u(x)|^p \leq \int_{\partial B_r(x)} |u(y)|^p\, d\mathcal{H}^{n-1}(y)$$

Integrating between 0 and R w.r.t. r we have

$$\sigma_n \frac{R^n}{n} |u(x)|^p \leq \int_0^R dr \int_{\partial B_r(x)} |u(y)|^p\, d\mathcal{H}^{n-1}(y) =$$

$$= \int_{B_R(x)} |u(y)|^p\, dy \leq C$$

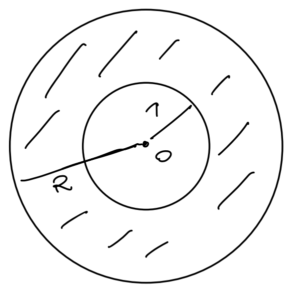
$\forall R$ since $u \in L^p(\mathbb{R}^n)$

Taking the limit for $R \to +\infty$ we get

$$\lim_{R \to +\infty} \sigma_n \frac{R^n}{n} |u(x)|^p \le C$$ and this is possible only if $u(x) = 0$.

This holds for each $x \in \mathbb{R}^n$, then $u \equiv 0$.

I. 9 Argue in annuli and send $R \to +\infty$



I. 10 Consider the function

$$v(x) = a \log |x| + b$$

which is harmonic in $B_{r_2} \setminus \overline{B_{r_1}}$.

Choose a and b in such a way that

$$\begin{cases} a \log r_1 + b = M(r_1) = \max\limits_{\partial B_{r_1}(0)} u \\ a \log r_2 + b = M(r_2) \end{cases}$$

We derive
$$a = \frac{M(r_2) - M(r_1)}{\log r_2 - \log r_1}$$

$$b = \frac{M(r_1)\log r_2 - M(r_2)\log r_1}{\log r_2 - \log r_1}$$

Then $u - v$ is sub-harmonic in $A = B_{r_2} \setminus \overline{B_{r_1}}$

and $u - v \leq 0$ in ∂A

since $u \leq M(r_j)$ in $\partial B_{r_j}(0)$ $j = 1, 2$,

While $v = M(r_j)$ in $\partial B_{r_j}(0)$, $j = 1, 2$.

By the maximum principle $u \leq v$ in A

and in particular

$$u(x) \leq v(x) = a \log r + b \quad \text{for } |x| = r$$

Then

$$M(r) \leq \frac{M(r_2) - M(r_1)}{\log r_2 - \log r_1} \log r +$$

$$+ \frac{M(r_1)\log r_2 - M(r_2)\log r_1}{\log r_2 - \log r_1}$$

$$= \frac{M(r_2)\left(\log r - \log r_1\right)}{\log r_2 - \log r_1} + \frac{M(r_1)\left(\log r_2 - \log r\right)}{\log r_2 - \log r_1}$$

L/N.1 and L.2 See ↑ (last lesson)

L.3 Given $u \in C^2(\Omega)$ we know that

$$u(x) = \fint_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y) \qquad \forall\ B_r(x) \subset\subset \Omega$$

Now consider the solution u of the problem

$$\begin{cases} -\Delta v = 0 & \text{in } B_R(x_0) \\ v = \varphi & \text{in } \partial B_R(x_0) \end{cases}$$

$$u \in C^2(B_R(x_0)) \cap C^0(\overline{B_R(x_0)})$$

By this

$$u(x_0) = \fint_{\partial B_r(x_0)} u(y)\, d\mathcal{H}^{n-1}(y) \qquad \forall\ r \in (0,R).$$

But the function (see the proof of the first theorem in I - PROPERTIES OF HARMONIC FUNCTIONS)

$$g(r) = \fint_{\partial B_r(x_0)} u(y)\, d\mathcal{H}^{n-1}(y) \qquad \text{is continuous in } [0,R]$$

and in particular

$$\lim_{r \to R} g(r) = g(R). \qquad (\bullet)$$

On the other side

$$g(r) = u(x_0) \qquad \forall \ r \in [0, R) \qquad (\circ\circ)$$

and then, by $(\circ)$ and $(\circ\circ)$, $g(R) = u(x_0)$, i.e.

$$u(x_0) = \fint_{\partial B_R(x_0)} u(y)\, d\mathcal{H}^{n-1}(y)$$

Then we can observe two things:

1°) the mean value property holds also in the boundary of the ball $B_R(x_0)$

2°) $$u(x_0) = \fint_{\partial B_R(x_0)} \varphi(y)\, d\mathcal{H}^{n-1}(y)$$

(then for the exercise $u(0) = \fint_{\partial B_1(0)} \varphi(y)\, d\mathcal{H}^{n-1}(y)$).

Observe that by formula (3) of L - GREEN FUNCTIONS ...
the solution is given by

$$u(x) = \int_{\partial B_R(0)} \frac{1}{\sigma_n R} \frac{R^2 - |x|^2}{|y-x|^n} \varphi(y)\, d\mathcal{H}^{n-1}(y)$$

that evaluated for $x = 0$ gives

$$u(0) = \frac{1}{\sigma_n R} \int_{\partial B_R(0)} \frac{R^2}{|y|^n} \varphi(y)\, d\mathcal{H}^{n-1}(y) = \fint_{\partial B_R(0)} \varphi(y)\, d\mathcal{H}^{n-1}(y).$$

N. 1 Consider $\Omega \overset{\circ}{\subseteq} \mathbb{R}^2$ and $(\bar{z}, \bar{w}) \in \partial\Omega$.

By assumption we have that there is a new system of coordinates s.t. $(\bar{z}, \bar{w})$ corresponds to $(0,0)$, a neighbourhood V of 0, a function

$$u : V \to \mathbb{R}$$

and a rectangle $R = V \times I$ such that $((x,y) \in R)$

$$\partial\Omega \cap R = \{ (x,y) \in R \mid y = u(x), \quad x \in V \}$$

Examples

and

$$\Omega \cap R = \{ (x,y) \in \mathbb{R}^2 \mid y < u(x), \; x \in V \}$$

Now, if $u \in C^2$, we have that

$$u(x) = u(0) + u'(0) + \frac{u''(0)}{2} x^2 + o(x^2)$$

We have that by assumption $u(0) = 0$ (it is

always possible to assume that taking $\tilde{u}(x) = u(x) - u(0)$)
and, for the sake of simplicity, we can assume

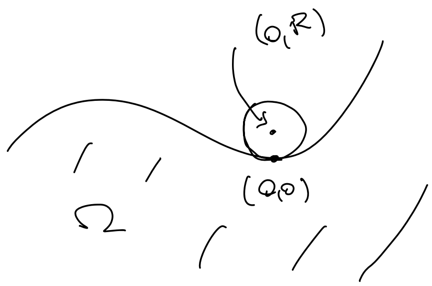
$$u'(0) = 0$$

(also this is always possible rotating, if necessary, the system of coordinates).

To find an exterior ball means to find R such that (in the new coordinates) the ball centred in $(0,R)$ and radius R is external to Ω and

$$\partial B_R(0,R) \cap \partial\Omega = \{(0,0)\}$$

This is always true in cases like that in example (1), but not obvious in cases like that in example (2)



$$x^2 + (y-R)^2 = R^2$$

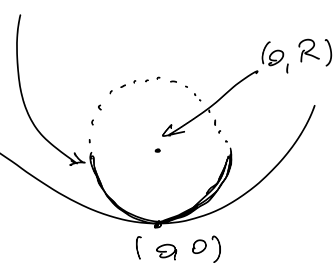
$$(y-R)^2 = R^2 - x^2$$

$$y(x) = R - \sqrt{R^2 - x^2}$$

In particular this means that

$$u(x) < y(x) \quad \forall\, x \in [-R, R],\ x \neq 0$$

$$u(0) = y(0)$$



i.e. $\quad u(x) = \frac{1}{2}u''(0)\,x^2 + o(x^2) < R - \sqrt{R^2 - x^2} \qquad x \neq 0$

$$\Leftrightarrow \quad \sqrt{R^2 - x^2} < R - \frac{u''(0)}{2}x^2 + o(x^2)$$

$$\Leftrightarrow \quad R^2 - x^2 < R^2 + \frac{(u''(0))^2}{4}x^4 - R\,u''(0)\,x^2 + o(x^2)$$

$$\Leftrightarrow \quad \frac{(u''(0))^2}{4}x^4 + \left(1 - R\,u''(0)\right)x^2 + o(x^2) > 0$$

$\left(\frac{u''(0)}{2}\right)^2 x^4 \geq 0$, then to be sure this inequality to be true we need

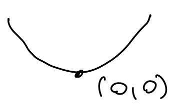
$1 - R\,u''(0) > 0$. Choosing $R < \frac{1}{u''(0)}$ if $u''(0) > 0$ we conclude.

If $u''(0) < 0$ we are in example (1) and $1 - R\,u''(0) > 0$ for every R.

If $u''(0) = 0$ $\quad 1 - R\,u''(0) > 0$ for every R.

N.2 Consider the set $(\alpha \in (1,2))$

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid y < |x|^\alpha\} \quad (C^1 \text{ but not } C^2)$$



Then there not exists $R > 0$ such that

$$|x|^\alpha < R - \sqrt{R^2 - x^2} \qquad \text{for every } x \in [-R, R],\ x \neq 0$$

$$\Leftrightarrow \quad \sqrt{R^2 - x^2} < R - |x|^\alpha$$

$$\Leftrightarrow \quad R^2 - x^2 < R^2 + |x|^{2\alpha} - 2R|x|^\alpha$$

$$\Leftrightarrow \quad |x|^{2\alpha} + x^2 - 2R|x|^\alpha > 0$$

This means $\quad |x|^\alpha \left(|x|^\alpha + |x|^{2-\alpha} - 2R \right) > 0$

for every $x \in [-R, R]$, $x \neq 0$

but this is impossible since, taking the limit for $x \to 0$, one gets $|x|^\alpha \left(|x|^\alpha + |x|^{2-\alpha} - 2R \right)$ is negative for x small enough.