INTRODUCTION

Notation: given $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ we will denote the partial derivative of u with respect to x_i ($i \in \mathbb{N}$ between 1 and n) in one of the following ways:

$$\frac{\partial u}{\partial x_i} \qquad \partial_{x_i} u \qquad \partial_i u \qquad D_{x_i} u \qquad D_i u \qquad u_{x_i} \qquad \left(u \in C^1(\Omega) \right)$$

and by ∇u or Du its gradient

By $D^k u$ one denotes the set of all derivatives or order k $\quad \left(u \in C^k(\Omega) \right)$

Example: $k = 1 \qquad Du = \left(D_1 u, \; \dots \; D_n u \right)$

$$k = 2 \qquad D^2 u = \left\{ D_{ij} u \;\middle|\; i,j = 1, \dots n \right\}$$

$$D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

α multi-index $\qquad \alpha \in \mathbb{R}^n \qquad \alpha = (k_1, \dots k_n)$

$$k_j \in \mathbb{N}$$

$$|\alpha| = \sum_{j=1}^{n} k_j$$

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \qquad \left(u \in C^{|\alpha|}(\Omega) \right)$$

Def Given $\Omega \overset{\circ}{\subseteq} \mathbb{R}^n$ and

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \dots \times \mathbb{R}^{n^k} \longrightarrow \mathbb{R}$$

a k-th order partial differential equation (PDE) is an expression of the form

$$F(x, u, Du, D^2u, \dots D^k u) = 0 \qquad \circledast$$

and the gradient of F with respect to $D^k u$ is not identically zero, i.e. k is the highest order of differentiation appearing in $\circledast$.

An equation like that in $\circledast$ is

LINEAR if F is linear with respect to u and all its derivatives, i.e. if

$$F(x, \alpha u + \beta v, D(\alpha u + \beta v), \dots D^k(\alpha u + \beta v)) =$$

$$= \alpha F(x, u, Du, \dots D^k u) + \beta F(x, v, \dots D^k v)$$

for every $\alpha, \beta \in \mathbb{R}$, $u, v \in C^k(\Omega)$

Among the others usually there is a distinction on the types of nonlinearity

SEMILINEAR if F is linear only with respect to $D^k u$ and the coefficients of the derivatives of order k depend only on the variable x

(Some authors say semilinear an equation $F=0$ when F nonlinear only with respect to u)

QUASILINEAR if F is linear only with respect to $D^k u$ and the coefficients of the derivatives or order k depend (may depend) on the terms

$$u, Du, D^2u, \dots D^{k-1}u$$

FULLY NONLINEAR if F is not linear, semilinear, quasilinear

Examples:

semilinear

$$F(x,u,Du, \dots D^k u) = \sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + \\ + f(x,u,Du, \dots D^{k-1}u)$$

quasilinear

$$F(x,u,Du,\dots D^k u) = \sum_{|\alpha|=k} a_\alpha(x,u,\dots D^{k-1}u)\, D^\alpha u(x) + f(x,u,Du,\dots D^{k-1}u)$$

Questions and problems:

given a PDE $F=0$ one can ask

1. is there at least one solution ?
   If yes, is there only one, more than one or infinite ?

2. Suppose $F=0$ has a solution (more than one)
   Adding some other information (boundary data)
   is this solution unique (in some sense) ?

Example: $y' = y$ (ODE)

$$y(x) = c\, e^x \qquad c \in \mathbb{R}$$

$$\begin{cases} y' = y \\ y(0) = \alpha \end{cases}$$ the only solution is $y(x) = \alpha e^x$

3. Estimates on the solution depending on the data
   Is the dependence of the data continuous ?

(example: $y_\alpha(x) = \alpha e^x$, $y_\beta(x) = \beta e^x$

$$|y_\alpha(x) - y_\beta(x)| \leq |\alpha - \beta| \, e^x$$)

4. Properties of the solution

5. Regularity of the solution

One could expect that every solution of

$$F(x, u, Du, \dots D^k u) = 0 \qquad \text{in } \Omega$$

is of class $C^k(\Omega)$, but not always a solution is so regular. For instance consider the equation

$$u_y = 0 \qquad \text{in } \mathbb{R}^2$$

Clearly the function $f(x,y) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$

is a solution and, more in general, every $f = f(x)$ is a solution even if $f \notin C^1(\mathbb{R}^2)$.

"Enlarging" the space where to look for a solution (= taking a space that contains also functions less regular than C^k)
helps to find a solution, but the price

to be paid is a lower, at least a priori, regularity for the solution.
Then, after proving existence and uniqueness, the problem is to gain some regularity (which depends on the data and F).

We will not treat these problems, but we will discuss general properties of the (regular) solutions of some particular equations

Some examples

TRANSPORT EQUATION

$$u_t + b \cdot Du = 0$$

$$b \in \mathbb{R}^n \qquad b = (b_1, \dots b_n)$$

$$u : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

$$Du = (\partial_{x_1} u, \dots \partial_{x_n} u)$$

$$u_t + b_1 u_{x_1} + \dots + b_n u_{x_n} = 0$$

LAPLACE AND POISSON EQUATIONS

$$u : \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

L.E. $\qquad -\Delta u = 0$

P.E. $\qquad -\Delta u = f$

HEAT EQUATION $\qquad u_t - \Delta u = 0$

WAVE EQUATION $\qquad u_{tt} - \Delta u = 0$

These last three equations (laplace, heat, wave) are the equations we will mostly treat in this course