CLASSIFICATION OF LINEAR (AND QUASI-LINEAR) SECOND ORDER PDE

For the sake of simplicity we start by the simplest case: we consider linear equation in $n = 2$.

Consider the operator

$$Lu = \underbrace{a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy}}_{\text{principal part}} + b_1 u_x + b_2 u_y + cu$$

(because the order of differentiation is the highest order appearing in the equation)

The coefficients a_{ij}, b_i, c depend only on x and y

(if they depended also on u, u_x and/or u_y the equation $Lu = 0$ would be quasi-linear).

We are supposing the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ to be symmetric. Indeed one could consider

$$\tilde{a}_{11} u_{xx} + \tilde{a}_{12} u_{xy} + \tilde{a}_{21} u_{yx} + \tilde{a}_{22} u_{yy}$$

as principal part, but, if $u \in C^2$,

$u_{xy} = u_{yx}$ and then we could replace $\tilde{a}_{12}$ and $\tilde{a}_{21}$ by $\dfrac{\tilde{a}_{12} + \tilde{a}_{21}}{2} =: a_{12}$

If $\det A > 0$ the equation is said ELLIPTIC

if $\det A = 0$ " " " " PARABOLIC

if $\det A < 0$ " " " " HYPERBOLIC

Now we consider the more general case (quasi-linear)

$$\circledast \quad \sum_{i,j=1}^{n} a_{ij}(x, u, Du)\, u_{x_i x_j} = f(x, u, Du) \qquad (x \in \mathbb{R}^n)$$

and, as before, we suppose $A = (a_{ij})_{i,j=1}^{n}$ to be symmetric. Then A has real eigenvalues $\lambda_1, \dots \lambda_n$ $\left(\lambda_j = \lambda_j(x, u(x), Du(x)) \right)$

$$\left[\quad \exists\, U \text{ unitary such that } U^{-1} A U = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \right]$$

Now we give a definition of elliptic, parabolic and hyperbolic for the above equation $\circledast$.

We say "a" and not "the" because in the literature different definitions may be found and at the end of the present section we give some different examples.

Denote by $P \in \mathbb{N}$ the number of positive eigenvalues of A and by $N \in \mathbb{N}$ the number of negative ones. Then

- the equation is said <u>ELLIPTIC</u> at x and u if

  $P = n$ or $N = n$

- the equation is said <u>HYPERBOLIC</u> at x and u if

  $P + N = n$ with $P = 1$ or $N = 1$

- the equation is said <u>PARABOLIC</u> at x and u if

  $P + N < n$

and <u>NORMAL PARABOLIC</u> if $P = 0$ and $N = n-1$ or vice versa

Examples :

- $-u_{xx} - u_{yy} = 0$ in $\mathbb{R}^2$ is elliptic
- $-\Delta u = 0$ in $\mathbb{R}^n$ is elliptic
- $u_{xx} - u_{yy} = 0$ in $\mathbb{R}^2$ is hyperbolic
- $u_{tt} - \Delta u = 0$ in $\mathbb{R}^{n+1}$ " "
- $u_{xx} + 2u_{xy} + u_{yy} = 0$ in $\mathbb{R}^2$ is parabolic
- $u_{xx} + u_{yy} = 0$ in $\mathbb{R}^3$ is "

  (but we can see this as a "family" in the parameter z of elliptic equations in $\mathbb{R}^2$)

- $u_{yy} + u_x = 0$ $u_{yy} - u_x = 0$ in $\mathbb{R}^2$ are parabolic

  $(a_{12} = a_{11} = b_2 = c = 0$ in L)

- $u_t - \Delta u = 0$ in $\mathbb{R}^{n+1}$ is parabolic

- $u_{xx} + u_{yy} - u_{zz} - u_{tt} = 0$ in $\mathbb{R}^4$?

With our definition this is not classifiable

$\Rightarrow$ there are some equations not classifiable.

[For some authors this is ultra hyperbolic]

An equation may change its nature depending on the region where you look at it!

Example: the TRICOMI equation

$$u_{xx} - x\, u_{yy} = 0 \quad \text{in } \mathbb{R}^2$$

is elliptic where $x < 0$
hyperbolic where $x > 0$

(in $\{x = 0\}$ is parabolic, since we are in $\mathbb{R}^2$)

This is an example of mixed type equation

REMARK ~~Attention~~! We say that an equation like that in ⊛ is elliptic, parabolic or hyperbolic at x and AT u if ....

Example: the equation $u\, u_{xx} - u_{yy} = 0$ in $\mathbb{R}^2$

is elliptic if $u < 0$

is hyperbolic if $u > 0$

Therefore the nature of a partial differential eq. (in general) may depend not only by the point (or the region) where we look at the equation, but also on the solution itself!

Just for completeness (or curiosity)

The division, when possible, of equations in elliptic, parabolic, hyperbolic is not uniquely defined, in particular the definition of parabolic and hyperbolic (the elliptic one is the only uniquely defined).

Here we report some of the possible different definitions.

Denoting as before by A the matrix defining the principal part of the operator, by P and N the number of positive and negative eigenvalues, n the dimension (and $n-P-N$ the dimension of the kernel of A) we have that the equation

$$F(x,u,u_{x_k},u_{x_k x_l}) = \sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} - f(x,u,Du) = 0$$

is

		(1)	(2)
PARABOLIC	$P+N<n$	$P+N<n$	$P+N<n$
NORMAL PARABOLIC			$P=0$, $N=n-1$ or vice versa
HYPERBOLIC	$P+N=n$ $P\geq 1$, $N\geq 1$	$P=1$, $N=n-1$ or vice versa	$P+N=n$ $P\geq 1$, $N\geq 1$
NORMAL HYPERBOLIC			$P=1$, $N=n-1$ or vice versa
ULTRA HYPERBOLIC		$P+N=n$ $P\geq 2$, $N\geq 2$	

	(3)	(4)
PARABOLIC	$P+N<n$	if $F=0$ can be written as $u_{x_1} = \sum_{i,j=2}^{n} a_{ij}\, u_{x_i} u_{x_j} - \tilde{f}(x,u,Du)$ with $\tilde{A} = \begin{pmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & \\ a_{n2} & \dots & & a_{nn} \end{pmatrix}$ elliptic
~~NORMAL PARABOLIC~~		
PROPERLY HYPERBOLIC	$P=1$, $N=n-1$ or vice versa	
HYPERBOLIC		$P=1$, $N=n-1$ or vice versa
ULTRA HYPERBOLIC	$P+N=n$ $P\geq 1$, $N\geq 1$	

1. PARTIAL DIFFERENTIAL EQUATIONS of Applied Mathematics
E. ZAUDERER [Chapter 3]

2. INTRODUCTION TO THE THEORY OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS (in italian)

W. MORETTI [Chapter 1]

3. PDE PRIMER (book freely downloadable)
R.E. SHOWALTER [Chapter 1]

4. PARTIAL DIFFERENTIAL EQUATIONS
J. JOST [Introduction]

!! Notice that the equation

$u_t - \Delta u = 0$ is parabolic and

$u_{tt} - \Delta u = 0$ is hyperbolic

according to each of the previous definition.