SOME RECALLS: REGULARITY OF FUNCTIONS AND SETS

Let Ω be an open and bounded set of $\mathbb{R}^n$, $k \in \mathbb{N}$.

Def We denote by $C^k(\Omega)$ the set of functions that are continuous with their derivatives till order k.

We denote $C^k(\overline{\Omega})$ the set of functions $C^k(\Omega_0)$ for some $\Omega_0 \supsetneq \overline{\Omega}$.

Def $\Omega \subseteq \mathbb{R}^n$ is an open set of class C^k (or lipschitz) ($n \geq 2$, $k \geq 0$) if $\forall\, x_0 \in \partial\Omega$ $\exists$

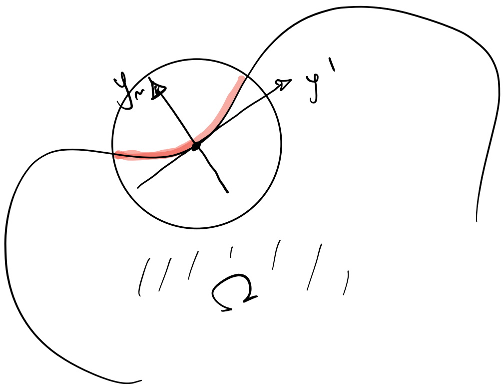
- a system of coordinates $(y', y_n) = (y_1, \dots, y_{n-1}, y_n)$ with the origin $(0', 0)$ in x_0
- a ball $B = B_r(x_0) \subseteq \mathbb{R}^n$,
- a neighbourhood $V \subseteq \mathbb{R}^{n-1}$ of $0'$ and a function $u : V \to \mathbb{R}$, $u \in C^k(V)$ (u lipschitz continuous)

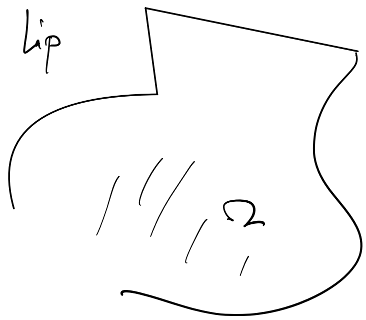
such that

1. $$\partial\Omega \cap B = \left\{ (y', y_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} y_n = u(y') \\ y' \in V \end{array} \right\}$$

2. $$\Omega \cap B = \left\{ (y', y_n) \in \mathbb{R}^n \;\middle|\; y_n < u(y'),\ y' \in V \right\}$$

Comments: condition 1. means that $\partial\Omega$ is, locally, the graph of a function of class C^k





Condition 2. We want the possibility to find a ch such a way that Ω is (locally) placed on one (and only one) side of $\partial\Omega$.

The definition given above is equivalent to the following. $(k \geq 1)$

Def Ω is an open set of class C^k $(n \geq 2)$ if for every $x_0 \in \partial\Omega$ $\exists$ an open neighborhood U of x_0 and $f \in C^k(U)$ such that $|\nabla f(x_0)| \neq 0$ and $\partial\Omega \cap U = \{x \in U \mid f(x) = 0\}$.

Ex : prove the equivalence

(hint : $f(y_1, \dots y_{n-1}, y_n) = y_n - u(y_1, \dots y_{n-1})$ or

$$f \circ \phi (y_1, \dots y_n) = y_n - u(y_1, \dots y_n)$$

where ϕ rototranslation

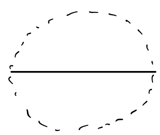
$|\nabla f(x_0)| \neq 0 \implies$ there is a neighbourhood of x_0 (say U)

for which $|\nabla f(x)| \neq 0 \quad x \in \partial\Omega \cap U$

and then $f < 0$ on one side

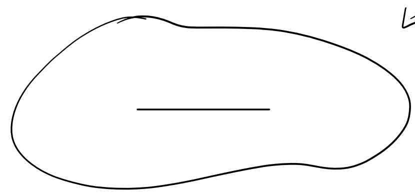
$f > 0$ on the other

Example

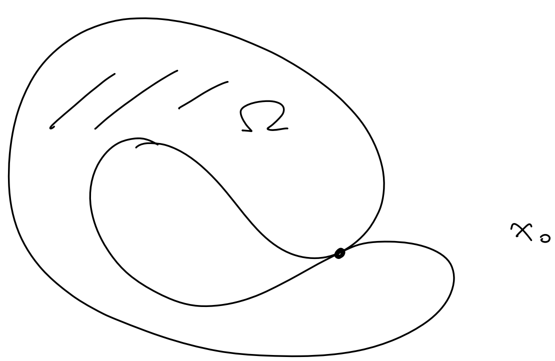


this set is clearly regular

but if Ω is (the inner part without the segment)



$\partial\Omega$ is not regular in the sense defined above.



this is not regular

Sometimes to say Ω of class C^k one says Ω open set with $\partial\Omega$ of class C^k.

Recalls

Let V be a vector field, $V : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$,

The divergence of V (shortly $\operatorname{div} V$) is the operator defined by

$$\operatorname{div} V = \sum_{i=1}^{n} \frac{\partial V_i}{\partial x_i}(x) = \frac{\partial V_1}{\partial x_1}(x) + \dots + \frac{\partial V_n}{\partial x_n}(x)$$

why?

Theorem Consider $\Omega \subseteq \mathbb{R}^n$, $\partial\Omega$ C^1, $V \in C^1(\overline{\Omega})$. Then

$$\int_\Omega \operatorname{div} V(x)\, dx = \int_{\partial\Omega} (V(x), \nu(x))\, d\mathcal{H}^{n-1}(x)$$

where ν denotes the outer normal to $\partial\Omega$

$(\cdot,\cdot)$ the scalar product in $\mathbb{R}^n$

$\mathcal{H}^{n-1}$ the $n-1$ Hausdorff measure on $\partial\Omega$

REMARK For $u \in C^2(\Omega)$ we have

$$\Delta u = \operatorname{div}(\nabla u)$$

Now consider $u \in C^2(\overline{\Omega})$, $v \in C^1(\overline{\Omega})$, Ω open subset of $\mathbb{R}^n$, $\partial\Omega$ of class C^1. Then

$$\int_\Omega \operatorname{div}(v(x)\nabla u(x))\,dx =$$

$$= \int_\Omega v(x)\,\Delta u(x)\,dx + \int_\Omega (\nabla v(x), \nabla u(x))\,dx =$$

$$= \int_{\partial\Omega} v(x)\,\frac{\partial u}{\partial \nu}(x)\,d\mathcal{H}^{n-1}(x)$$