THE CONTINUITY EQUATION

Consider $E(t) \subset E$, $t \in I$, E measurable, $E(t)$ measurable $\forall t$

$\partial E(t)$ be of class C^1 (uniformly in t)

Consider $\rho \in C^1(E \times I)$

We want to compute $\displaystyle \frac{d}{dt} \int_{E(t)} \rho(x,t)\, dx$

Given a family of $n \times n$ matrices $t \mapsto A(t)$ depending on a parameter $t \in I$, $t \mapsto A(t)$ $C^1(I)$ (i.e. $(A(t))_{ij} = a_{ij}(t)$ and $t \mapsto a_{ij}(t)$ $C^1(I)$ $\forall t$)

one has that

$$\frac{d}{dt}\left(\det A(t)\right) = \det A(t) \cdot \operatorname{tr}\left(A^{-1}(t) \frac{d}{dt} A(t)\right)$$

where $\displaystyle \left(\frac{d}{dt} A(t)\right)_{ij} = \frac{d}{dt} A_{ij}(t)$

proof: the result is given without proof, even if for the sake of completeness we put it at the end of the section.

We now go on computing $\frac{d}{dt} \int_{E(t)} \rho(x,t)\, dx$.

Assumptions

We assume that $E(t) = G_t(E_0)$

$G_t(x) = G(t,x)$ $\quad$ G_t bijection uniformly C^1

G_t^{-1} uniformly C^1

G C^1 with respect to t and

$|DG(x,\cdot)| = |\det JG_t(x)|$ $\quad$ C^1 in t

$E(t) = G_t(E_0)$

$$\frac{d}{dt} \int_{E(t)} \rho(y,t)\, dy = \qquad \left(y = G_t(x) \right)$$

$$= \frac{d}{dt} \int_{E_0} \rho\left(G_t(x), t\right) |DG_t(x)|\, dx =$$

$$= \int_{E_0} \frac{\partial}{\partial t} \left(\rho\left(G_t(x), t\right) |DG_t(x)| \right) dx =$$

$$= \int_{E_0} \sum_{i=1}^{n} \frac{\partial \rho}{\partial y_i}(G_t(x), t) \, \frac{\partial (G_t)_i}{\partial t}(x) \, |DG_t(x)| \, dx \; +$$

$$+ \int_{E_0} \frac{\partial \rho}{\partial t}(G_t(x), t) \, |DG_t(x)| \, dx \; +$$

$$+ \int_{E_0} \rho(G(x,t), t) \cdot \frac{\partial}{\partial t} |DG(x,t)| \, dx$$

$$2^\circ \text{ addend} = \int_{E(t)} \frac{\partial \rho}{\partial t}(z,t) \, dz$$

Recall that : $G_t(x)$, $G_t^{-1}(y)$ $\qquad \left(DG_t(G_t^{-1}(y))\right)^{-1} = DG_t^{-1}(y)$

$$3^\circ \text{ addend} \overset{*}{=}$$

$$= \int_{E_0} \rho(G_t(x), t) \, |DG_t(x)| \, \operatorname{tr}\left((DG_t(x))^{-1} \frac{\partial}{\partial t} DG_t(x) \right) dx =$$

$$= \int_{E_0} \rho(G_t(x), t) \, |DG_t(x)| \, \operatorname{tr}\left(DG_t^{-1}(G_t(x)) \cdot \frac{\partial}{\partial t} DG_t(x) \right) dx =$$

Now $\left(DG_t^{-1}\right)_{ij}(G_t(x)) = \frac{\partial (G_t^{-1})_i}{\partial y_j}(G_t(x))$

$$\frac{\partial}{\partial t}\left(DG_t\right)_{jk}(x) = \frac{\partial}{\partial t}\frac{\partial (G_t)_j}{\partial x_k}(x,t)$$

then $\left(DG_t^{-1}(G_t(x)) \cdot \frac{\partial}{\partial t} DG_t(x)\right)_{ik} =$

$$= \sum_{j=1}^{m} \frac{\partial (G_t^{-1})_i}{\partial y_j}(G_t(x)) \, \frac{\partial^2 (G_t)_j}{\partial t \, \partial x_k}(x)$$

$$(\text{for } i=k) = \sum_{j=1}^{m} \left(\frac{\partial}{\partial t}\frac{\partial (G_t)_j}{\partial x_i}(x)\right)\frac{\partial (G_t^{-1})_i}{\partial y_j}(G_t(x))$$

$$= \sum_{j=1}^{m} \frac{\partial}{\partial x_i}\frac{\partial (G_t)_j}{\partial t}(x) \, \frac{\partial (G_t^{-1})_i}{\partial y_j}(G_t(x))$$

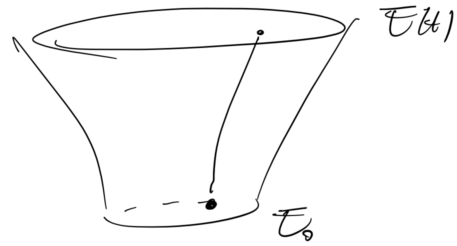
$$\operatorname{tr}\left(DG_t^{-1}(G_t(x)) \cdot \frac{\partial}{\partial t} DG_t(x)\right) =$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial}{\partial x_i}\frac{\partial (G_t)_j}{\partial t}(x,t) \, \frac{\partial (G_t^{-1})_i}{\partial y_j}(G_t(x))$$

Replacing $G_t(x)$ with y we get

$$\sum_{j=1}^{m} \frac{\partial}{\partial y_j}\left(\frac{\partial (G_t)_j}{\partial t}(G_t^{-1}(y))\right) = \operatorname{div} V(y,t)$$

where $V_j(y,t) := \frac{\partial (G_t)_j}{\partial t}(G_t^{-1}(y), t)$



Then the 1^{st} addend and the 3^{rd} addend give

$$\int_{E_0} \sum_{i=1}^{n} \frac{\partial \rho}{\partial y_i}(G_t(x), t) \, \frac{\partial (G_t)_i}{\partial t}(x,t) \, |DG_t(x)| \, dx \; +$$

$$+ \int_{E_0} \rho(G_t(x), t) \, |DG_t(x)| \, \operatorname{tr}\left((DG_t(x))^{-1} \circ \frac{\partial}{\partial t} DG_t(x) \right) dx =$$

$$= \int_{E(t)} \sum_{i=1}^{n} \frac{\partial \rho}{\partial y_i}(y,t) \, \frac{\partial (G_t)_i}{\partial t}(y,t) \, dy \; +$$

$$+ \int_{E(t)} \rho(y,t) \operatorname{div} V(y,t) \, dy =$$

$$= \int_{E(t)} \operatorname{div}\left(\rho(y,t) \, V(y,t) \right) dy$$

V vector field denoting the velocity of $x_0 \in E_0$ at time t

Then we conclude that

$$\frac{d}{dt} \int_{E(t)} \rho(x,t)\, dx =$$

$$= \int_{E(t)} \frac{\partial \rho}{\partial t}(y,t)\, dy + \int_{E(t)} \operatorname{div}\left(\rho(y,t)\, V(y,t)\right) dy$$

Now suppose ρ is the density of a fluid

($\rho(\cdot, t)$ density at time t)

$E(t)$ region occupied by the fluid at time t

examples: - ρ density of a compressible gas
- " " a liquid (water in the ground) that is incompressible

Now whatever the region $E(t)$ the mass is preserved, that is

$$\int_{E(t)} \rho(x,t)\, dx = \text{constant}$$

then $$\frac{d}{dt} \int_{E(t)} \rho(x,t)\, dx = 0$$

and since this holds for every choice of $E(t)$

we get $$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho V) = 0}$$

This is called <u>equation of continuity</u> or of <u>conservation of mass</u>.

Let us see some examples.

TRANSPORT EQUATION

$n = 1$ $\qquad v = b \in \mathbb{R}$ $\qquad$ (constant)

$$\frac{\partial \rho}{\partial t} + b \frac{\partial \rho}{\partial x} = 0$$

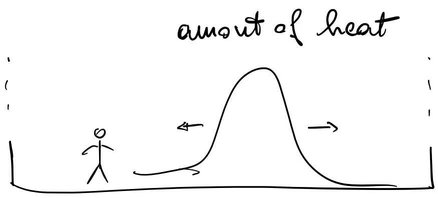
$n > 1$ $\qquad v$ constant $\qquad (v = (v_1, \dots, v_n) \in \mathbb{R}^n)$

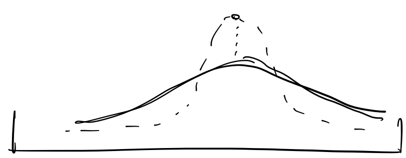
$$\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0$$

HEAT EQUATION

u temperature

the Fourier's law $\qquad uv = -k \nabla u$ $\qquad k \in \mathbb{R}$, $k > 0$ (conductivity)





$$\frac{\partial u}{\partial t} - k \,\mathrm{div}(\nabla u) = \boxed{\frac{\partial u}{\partial t} - k \Delta u = 0}$$

LAPLACE EQUATION

ρ density of an incompressible fluid (a liquid)

Darcy's law $V = -k \nabla p$ (fluid in a porous medium) where p is the pressure

$k > 0$

$\rho_t = 0$ $\quad \operatorname{div}(\rho k \nabla p) = \rho k \Delta p$

(because of incompressibility, ρ constant)

$$-\Delta p = 0$$

WAVE EQUATION

transport in two directions, for instance

$u_t + u_x = 0$ $\qquad u_t - u_x = 0$

$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u = 0$ $\qquad \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0$

$$\boxed{u_{tt} - u_{xx} = 0}$$

can be factorized as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0$$

A more general equation is

$$(k > 0 \,,\; b \in \mathbb{R}^n \,,\; c \in \mathbb{R})$$

$$a\,\frac{\partial u}{\partial t} \underbrace{- k\,\Delta u}_{\text{diffusion}} + \underbrace{b \cdot \nabla u}_{\text{advection or transport}} + \underbrace{c\,u}_{\text{reaction}} = 0$$

THE NEXT THINGS WERE NOT DONE IN CLASS

THE PROOF IS GIVEN FOR THE SAKE OF COMPLETENESS

PROOF OF

$$\frac{d}{dt}\left(\det A(t)\right) = \det A(t) \cdot \operatorname{tr}\left(A^{-1}(t)\, \frac{d}{dt} A(t)\right)$$

dim: consider the matrix $Id + H$

column vectors

$$\det(Id + H) = \det(e_1 + H_1, \dots e_n + H_n) =$$

because of multilinearity

$$= \det(e_1, e_2 + H_2, e_3 + H_3, \dots) +$$
$$+ \det(H_1, e_2 + H_2, e_3 + H_3, \dots) =$$

$$= \det(e_1, e_2, e_3 + H_3, \dots) + \det(e_1, H_2, e_3 + H_3, \dots)$$
$$+ \det(H_1, e_2, e_3 + H_3, \dots) + \det(H_1, H_2, e_3 + H_3, \dots)$$

$$= \dots$$

Observe that

$$\det(H_1, e_2, e_3, \dots e_n) = h_{11}$$
$$\det(e_1, H_2, e_3, \dots e_n) = h_{22}$$
$$\vdots$$
$$\det(e_1, e_2, \dots e_{n-1}, H_n) = h_{nn}$$

whose sum is the trace of H

$$\det(H_1, H_2, e_3, \dots e_n) = h_{11} h_{22} - h_{12} h_{21}$$

Then

$$\det(Id + H) = \det Id + \operatorname{tr} H +$$

$+$ terms of order higher than 1 in h_{ij}

$$\text{"}=\text{"} \quad 1 + \operatorname{tr} H + o(H)$$

Recall: $f : \mathbb{R} \to \mathbb{R}$

if $\quad f(x) = f(x_0) + \alpha + o(|x - x_0|)$

then $\quad \alpha = f'(x_0)(x - x_0)$

$F : \mathbb{R}^n \to \mathbb{R}$

if $\quad F(x) = F(x_0) + \alpha + o(|x - x_0|)$

then $\quad \alpha = (\nabla F(x_0), x - x_0)$

$$F(x_0 + v) = F(x_0) + \alpha + o(|x - x_0|)$$

$$\Rightarrow \quad \alpha = \frac{\partial F}{\partial v}(x_0) = dF_{x_0} v$$

Then, if we denote by $F(A)$ the function $\det A$,

we have that $\frac{\partial F}{\partial H}(Id) = dF_{Id}\, H = \operatorname{tr} H$

$$F(Id + H) = 1 + \operatorname{tr} H + o(H)$$

$$\det(A + H) = \det\left(A\,(Id + A^{-1}H)\right) =$$

$$= \det A \cdot \det\left(Id + A^{-1}H\right) =$$

$$= \det A \left(1 + \operatorname{tr} A^{-1}H + \underbrace{o(A^{-1}H)}_{= o(H)}\right)$$

$$F(A+H) = \det A + \det A \cdot \operatorname{tr} A^{-1}H + o(H)$$

$$\frac{\partial F}{\partial H}(A) = dF_A(H) = \det A \quad \operatorname{tr} A^{-1}H$$

$$\frac{d}{dt} F(A(t)) = \det A(t) \quad \operatorname{tr}\left(A(t)^{-1} \frac{d}{dt} A(t)\right) \quad \circledast$$

$$\parallel$$

$$\frac{d}{dt}\left(\det A(t)\right)$$