QUICK RECALLS ON CONVOLUTION

Consider two functions f, g

$$f \in L^1(\mathbb{R}^n), \quad g \in L^p(\mathbb{R}^n) \quad p \in [1, +\infty]$$

one can define the "convolution" product between f and g as follows:

$f * g$ is a new function in $L^p(\mathbb{R}^n)$ defined as

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

1. $*$ is commutative (EX), that is

$$\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy = \int_{\mathbb{R}^n} g(x-y)\, f(y)\, dy$$

2. given $f \in L^1(\mathbb{R}^n)$ one has that

$$g \in L^p(\mathbb{R}^n) \Rightarrow f * g \in L^p(\mathbb{R}^n), \ \|f*g\|_p \leq \|f\|_1 \|g\|_p$$

$$g \in C^0(\mathbb{R}^n) \Rightarrow f * g \in C^0(\mathbb{R}^n)$$

$$\left. \begin{array}{l} g \in C^k_c(\mathbb{R}^n) \\ f \in L^1_c(\mathbb{R}^n) \end{array} \right\| \Rightarrow f * g \in C^k_c(\mathbb{R}^n)$$

L^1_c: $L^1(\mathbb{R}^n)$ with compact support

3. (Ex) $f * (g * h) = (f * g) * h$

MOLLIFIERS

Def We say that the sequence $\{\rho_j\}_{j\in\mathbb{N}}$ is an approximate identity if

$$\rho_j(x) = j^n \rho(jx), \quad j \in \mathbb{N} \quad \text{and} \quad \rho \in C^\infty(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} \rho(x)\,dx = 1.$$

These functions (ρ_j) are also referred to as mollifiers.

REMARK Usually, but not always, ρ is chosen

$$\rho \in C_c^\infty(\mathbb{R}^n)$$

and, for the sake of simplicity, often

$$\operatorname{supp}(\rho) = B_1(0).$$

A typical example of such a function is

$$\rho(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

even if $\int \rho \, dx \neq 1$ (but one can renormalise).

<u>Ex</u> Prove that $\rho \in C^\infty(\mathbb{R}^n)$

<u>Theorem</u> Let $f \in C^0(\mathbb{R}^n)$. Then

$$\rho_n * f \xrightarrow[n]{} f$$ uniformly on compact sets.

<u>Theorem</u> Let $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < +\infty$.

Then $\rho_n * f \xrightarrow[n]{} f$ in $L^p(\mathbb{R}^n)$.

[For <u>convolution</u> see, e.g. H. BREZIS
Functional Analysis]

One can define the convolution also between a function and a distribution T.

DISTRIBUTIONS (A BRIEF INTRODUCTION)

[see, e.g., W. RUDIN, Functional Analysis]

Given $\Omega \subseteq \mathbb{R}^n$ one defines

$$\mathcal{D}(\Omega) = \{ \phi : \Omega \to \mathbb{R} \mid \phi \in C_c^\infty(\Omega) \}$$

We say that a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ converges to ϕ in $\mathcal{D}(\Omega)$ if

1) there exists a compact set $K \subseteq \Omega$ such that

$$\operatorname{supp} \phi_k \subseteq K \quad \forall\ k \in \mathbb{N}$$

2) $D^\alpha \phi_k \longrightarrow D^\alpha \phi$ uniformly in Ω (in K) for every multi-index α

With this definition we can endow $\mathcal{D}(\Omega)$ with a topology.

Then we define the set (the vectorial space) of distributions $\mathcal{D}'(\Omega)$ as the dual space of $\mathcal{D}(\Omega)$, i.e. the space of linear and continuous functionals on $\mathcal{D}(\Omega)$.

$$T \in \mathcal{D}'(\Omega) \Rightarrow T : \mathcal{D}(\Omega) \to \mathbb{R}$$

$$\lim_{k \to +\infty} \langle T, \phi_k \rangle = \langle T, \phi \rangle$$

for every $\phi \in \mathcal{D}(\Omega)$ and
for every sequence $\{\phi_k\}_{k \in \mathbb{N}} \subseteq \mathcal{D}(\Omega)$
converging to ϕ according to the previous definition.

Who belongs to the set $\mathcal{D}'(\Omega)$?

For instance, $L^1_{loc}(\Omega) \subseteq \mathcal{D}'(\Omega)$

if $f \in L^1_{loc}(\Omega)$ we can define $T_f \in \mathcal{D}'(\Omega)$ as

$$\langle T_f, \phi \rangle := \int_\Omega f \phi \, dx \qquad (\text{Ex Verify } T_f \text{ is continuous})$$

$\mathcal{D}'(\Omega)$ is constructed around a special object, called Dirac delta.

1st fact a distribution T admits whatever derivative

Given $T \in \mathcal{D}'(\Omega)$ one can define $D^\alpha T \in \mathcal{D}'(\Omega)$

α multi-index, as follows: (Ex verify)

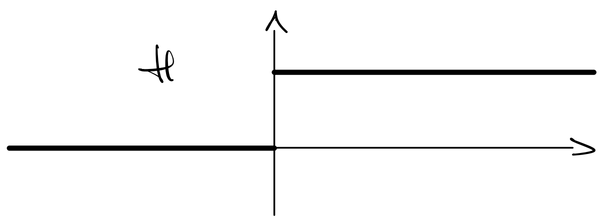
$$\langle D^\alpha T, \phi \rangle := (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle$$

2nd fact the function $H : \mathbb{R} \to \mathbb{R}$, $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$

belongs to $L^1_{loc}(\mathbb{R})$, therefore is a distribution and is differentiable:

$$\langle T_H, \phi \rangle := \int_0^{+\infty} \phi(x)\, dx$$

$$\langle T_H', \phi \rangle = -\int_{\mathbb{R}} H(x)\, \phi'(x)\, dx = -\int_0^{+\infty} \phi'(x)\, dx = \phi(0)$$



the first "derivative" of H is the distribution so defined:

$$\langle \delta, \phi \rangle := \phi(0)$$

In general (in $\mathbb{R}^n$ and for $x_0 \in \mathbb{R}^n$)

one denotes by δ_{x_0} the distribution in $\mathcal{D}'(\mathbb{R}^n)$ defined by

$$\langle \delta_{x_0}, \phi \rangle := \phi(x_0)$$

Notice that δ_{x_0} can be differentiated :

$$\langle D^\alpha \delta_{x_0}, \phi \rangle = (-1)^{|\alpha|} D^\alpha \phi(x_0)$$

Ex : find a function defined in $\mathbb{R}^2$ whose some derivative (of some order) is δ ($\delta = \delta_{(0,0)}$)

Ex compute the derivative D_x of the distributions

$$\tilde{H}(x,y) := \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$H_f(x,y) := \begin{cases} f(y) & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\tilde{H}_f(x,y) := \begin{cases} f(x,y) & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad f \in C^1([0,+\infty) \times \mathbb{R})$$

Ex Given a distribution T
show that there exists $D_v T$ ($v \in \mathbb{R}^n$, $|v| = 1$)
where $D_v \phi$ denotes the directional derivative of ϕ in the direction v and

$$\langle D_v T, \phi \rangle = - \langle T, D_v \phi \rangle$$

Ex compute the derivative D_v of the distributions

$$H_v(x,y) := \begin{cases} 1 & \text{if } (x,y) \in A \\ 0 & \text{if } (x,y) \notin A \end{cases}$$

where $v = (v_1, v_2)$, $|v| = 1$, and

$$A = \{ (x,y) \in \mathbb{R}^2 \mid \langle (x,y), (v_1, v_2) \rangle \geq 0 \}$$

One can endow $\mathcal{D}'(\Omega)$ with the following topology (the weak $*$ topology induced by $\mathcal{D}(\Omega)$):

given $\{T_n\}_{n \in \mathbb{N}}$ a sequence of distribution we say that $T_n \to T$ in the sense of distribution if

$$\lim_{n \to +\infty} \langle T_n, \phi \rangle = \langle T, \phi \rangle \quad \forall \phi \in \mathcal{D}(\Omega)$$

Consequence: this in fact means that

$$\lim_{n \to +\infty} \langle T_n, D^\alpha \phi \rangle = \langle T, D^\alpha \phi \rangle \quad \forall \phi \in \mathcal{D}(\Omega)$$
$$\forall \alpha \text{ multi-index}$$

Let δ be the Dirac delta in dimension 1.

One can find a sequence of distributions T_n, associated to some function f_n, i.e. $T_n = T_{f_n}$, such that

$$T_n \to \delta \quad \text{in the sense defined above.}$$

We can find f_n in such a way that

$$\|f_n\|_{L^1(\mathbb{R})} = 1 \quad \forall n \in \mathbb{N}$$

For example
$$f_n(x) = \begin{cases} n^2\left(x+\frac{1}{n}\right) & \text{if } x \in \left[-\frac{1}{n}, 0\right] \\ -n^2\left(x-\frac{1}{n}\right) & \text{if } x \in \left[0, 1/n\right] \\ 0 & \text{otherwise} \end{cases}$$

or, more simply,
$$f_n = \begin{cases} \frac{n}{2} & x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases}$$

In both cases (but not only in these cases!)

one has that $T_{f_n} \to \delta$

Let us verify that in the second case:

$$\langle T_{f_n}, \phi \rangle = \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2}\,\phi(x)\,dx = \frac{1}{\frac{1}{2n}} \int_{-1/n}^{1/n} \phi(x)\,dx \longrightarrow \phi(0)$$

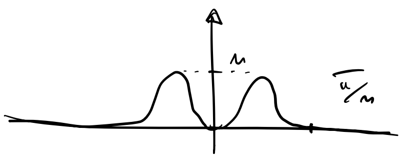
(if not clear Ex: prove that for a continuous ϕ
$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \phi(x)\,dx = \lim_{\varepsilon\to 0} ⨍_{(-\varepsilon,\varepsilon)} \phi(x)\,dx = \phi(0) \quad)$$

Observe that $\int_{\mathbb{R}} f_n(x)\,dx = 1 \quad \forall n \in \mathbb{N}$

Ex $\left(\int_0^{\pi} \text{sen}^2 x \, dx = \frac{\pi}{2} \right)$

Consider the sequence $f_n(x) = \frac{1}{\pi} \begin{cases} n\,\text{sen}^2(nx) & x \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right] \\ 0 & \text{otherwise} \end{cases}$

$n \in \mathbb{N}^*$



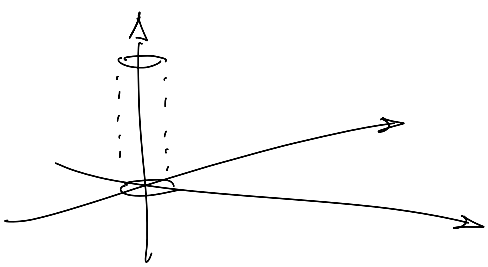
May $\{T_{f_n}\}_n$ converge to δ ? $(\delta = \delta_0)$

Also in dimension n one can find a sequence (in fact infinite) which converges to $\delta = \delta_{(0,\dots,0)}$.

For example consider

$$f_k(x) = \begin{cases} \frac{k^n}{\omega_n} & \text{in } B_{1/k}(0) \\ 0 & \text{otherwise} \end{cases} \qquad \omega_n := |B_1(0)|$$

Lebesgue measure in $\mathbb{R}^n$



$\omega_1 = 2$

$\omega_2 = \pi \qquad \omega_3 = \frac{4}{3}\pi$

and the sequence of distributions

$\{T_{f_k}\}_{k\in\mathbb{N}}$ defined as

$$\langle T_{f_k}, \phi \rangle_{\mathcal{D}'\times\mathcal{D}} := \int_{\mathbb{R}^n} f_k(x)\,\phi(x)\,dx$$

Ex Verify that $T_{f_k} \longrightarrow \delta$ in $\mathcal{D}'(\mathbb{R}^n)$

Def We say that the sequence $\{h_j\}_{j\in\mathbb{N}}$ is an approximate identity if

$$h_j(x) = j^n h(jx), \; j\in\mathbb{N} \text{ and } h\in\mathcal{D}(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} h(x)\,dx = 1$$

REMARK Observe that it is not required $h \geq 0$.

For instance consider $\phi \in C^\infty(\mathbb{R})$, $\phi \geq 0$, $\text{supp}\,\phi = [-\frac{3}{2}\pi, \frac{3}{2}\pi]$, the function

$$f(x) = \begin{cases} \cos x & x \in [-3/2\,\pi, 3/2\,\pi] \\ 0 & \text{otherwise} \end{cases}$$

Consider $h = f * \phi \in C^\infty_c(\mathbb{R})$ (but possibly both positive and negative).

This can define an approximate identity

REMARK In fact one could also consider h also not compactly supported such that

$$\int_{\mathbb{R}^n} h(x)\,dx = 1$$

As an example consider:

$$h(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4} \qquad \text{(gaussian)}$$

and

$$h_j(x) = \frac{j}{\sqrt{4\pi}} e^{-\frac{j^2 x^2}{4}} > 0 \quad \forall x \in \mathbb{R}$$

Instead of an integer $j \in \mathbb{N}$ one can consider a positive parameter $\varepsilon > 0$ (usually going to zero). In this way taking ε at the place of $\sqrt{\frac{1}{j}}$ one gets

$$h_\varepsilon(x) = \frac{1}{\sqrt{4\pi\varepsilon}} e^{-\frac{x^2}{4\varepsilon}}$$

or $(t = \varepsilon)$

$$h_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \qquad \left(\begin{array}{l}\text{see parabolic eq.s} \\ \text{and EXERCISES}\end{array}\right)$$

CONVOLUTION OF A DISTRIBUTION WITH A FUNCTION

One can define the convolution between a distribution T and a function g as follows.

First we define:

$$\check{g}(x) := g(-x)$$

$$\tau_x g(y) := g(y-x)$$

Then, for $g \in \mathcal{D}(\mathbb{R}^n)$, we define (for each $x \in \mathbb{R}^n$)

$$T * g\,(x) := \langle T, \tau_x \check{g} \rangle$$

Theorem Consider $T \in \mathcal{D}'$, $\phi \in \mathcal{D}$, $\{\varphi_j\}_{j \in \mathbb{N}}$ to be an approximate identity. Then

(a) $T * \phi \in C^\infty(\mathbb{R}^n)$

(b) $\lim_{j \to +\infty} T * \varphi_j = T$ in $\mathcal{D}'(\mathbb{R}^n)$.

Proof: No proof

EX Take for granted that $T * \varphi_j$ is a function in L^1_{loc} for each $j \in \mathbb{N}$ and point (b) of the previous theorem. Show point (a)

[Hint : show first $T * \phi$ is continuous]

Now for a given $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we want to compute

$$\delta * \varphi$$

Consider an approximate identity $\{h_j\}_{j\in\mathbb{N}}$ or simply the sequence

$$f_k = \begin{cases} \frac{1}{\omega_n} k^n & \text{in } B_{1/k}(0) \\ 0 & \text{otherwise} \end{cases}$$

$$\left(|B_{1/k}(0)| = \omega_n \left(\frac{1}{k}\right)^n \right)$$

and the sequence of distributions T_{f_k}. Then

$T_{f_k} \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$.

Let's compute $\delta * \varphi = \lim_{k\to+\infty} T_{f_k} * \varphi$

$$\lim_{k \to +\infty} T_{f_k} * \varphi = \lim_{k \to +\infty} \int_{B_{1/k}(0)} f_k(y)\, \varphi(x-y)\, dy =$$

$$= \lim_{k \to +\infty} \fint_{B_{1/k}(0)} \varphi(x-y)\, dy = \lim_{k \to +\infty} \fint_{B_{1/k}(0)} \tau_x \check{\varphi}(y)\, dy$$

$$= \tau_x \check{\varphi}(0) = \varphi(x)$$

that is

$$\boxed{\delta * \varphi = \varphi} \quad \text{for every } \varphi \in \mathcal{D}(\mathbb{R}^n)$$