Example (Hadamard) Take $\Omega = B_1(0)$ in $\mathbb{R}^2$ and (expressed in polar coordinates) the two functions

NOT SHOWN IN CLASS

$$g(\theta) = \sum_{k=1}^{+\infty} \frac{\cos(k!\,\theta)}{k^2} \qquad 0 < \rho < 1$$

$$u(\rho,\theta) = \sum_{k=1}^{+\infty} \frac{\rho^{k!}\cos(k!\,\theta)}{k^2}$$

These series are totally, and then uniformly, converging (also in $\overline{B_1(0)}$).

Then we can differentiate the series defining u term by term. For instance

$$u_\rho = \sum_{k=1}^{\infty} k!\,\rho^{k!-1}\frac{\cos(k!\,\theta)}{k^2}.$$

Notice that this series does not converge for $\rho = 1$. Nevertheless it converges for every $\rho \in [0,1)$ and then it totally converges in every ball $\overline{B_r(0)}$ with $r \in (0,1)$.

Then we can further differentiate with respect to ρ in $B_1(0)$ since each point in $B_1(0)$ is contained in a ball $B_r(0)$ for a suitable r.

Then we have

[See APPENDIX A below for some details]

$$u_\rho = \sum_{k=1}^{\infty} k! \, \rho^{k!-1} \frac{\cos(k!\theta)}{k^2}$$

$$u_{\rho\rho} = \sum_{k=1}^{+\infty} k!\,(k!-1)\,\rho^{k!-2} \frac{\cos(k!\theta)}{k^2}$$

$$u_\theta = \sum_{k} \left(-\rho^{k!} \frac{\operatorname{sen}(k!\theta)}{k^2} \right) k!$$

$$u_{\theta\theta} = -\sum_{k=1}^{+\infty} \rho^{k!} (k!)^2 \frac{\cos(k!\theta)}{k^2}$$

One can express the Laplacian with respect to the polar coordinates [see EX below and "Solutions to exercises"]

EX The Laplacian of u, expressed in polar coordinates, is $u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta}$.

Then: $\quad u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta} =$

$$= \sum_{k=1}^{\infty} \frac{\cos(k!\theta)}{k^2} \left(k!(k!-1)\rho^{k!-2} + k!\rho^{k!-2} - (k!)^2 \rho^{k!-2} \right)$$

$$= \sum_{k=1}^{+\infty} k! \underbrace{\left(k!-1+1-k! \right)}_{=0} \frac{\cos(k!\theta)}{k^2} \rho^{k!-2}$$

$$= 0$$

but compute $\displaystyle\iint_{B_1(0)} |Du(x,y)|^2 \, dx\, dy = \int_0^{2\pi} d\theta \int_0^1 \rho \, |Du(\theta,\rho)|^2 d\rho$

$$|Du(\theta,\rho)|^2 = \left((u_\rho, u_\theta), (u_\rho, u_\theta) \right) =$$

$$= \sum_{k=1}^{\infty} \left[(k!)^2 \left(\rho^{k!-1}\right)^2 \frac{1}{k^4} \cos^2(k!\theta) + (k!)^2 \left(\rho^{k!}\right)^2 \frac{1}{k^4} \operatorname{sen}^2(k!\theta) \right] =$$

$$= \sum_{k=1}^{+\infty} \left[(k!)^2 \left(\rho^{k!-1}\right)^2 \frac{1}{k^4} + (k!)^2 \left[\left(\rho^{k!}\right)^2 - \left(\rho^{k!-1}\right)^2 \right] \frac{\operatorname{sen}^2(k!\theta)}{k^4} \right]$$

$$\int_0^1 \rho \, \frac{(k!)^2}{k^4} \left(\rho^{k!-1}\right)^2 d\rho =$$

$$= \frac{(k!)^2}{k^4} \left. \frac{\rho^{2k!}}{2k!} \right|_0^1 = \frac{k!}{2k^4}$$

and then $\displaystyle \int_0^{2\pi} d\theta \int_0^1 \rho \sum \frac{(k!)^2}{k^4} \left(\rho^{k!-1}\right)^2 d\rho = +\infty$

As regards the series

$$\sum (k!)^2 \rho^{2k!-2} \left(\rho^2 - 1\right) \frac{\operatorname{sen}^2(k!\theta)}{k^4}$$

we have that

$$\int_0^1 \rho \, (k!)^2 \rho^{2k!-2} \left(\rho^2 - 1\right) d\rho =$$

$$= \int_0^1 (k!)^2 \left(\rho^{2k!+1} - \rho^{2k!-1}\right) d\rho =$$

$$= (k!)^2 \left. \left(\frac{\rho^{2k!+2}}{2k!+2} - \frac{\rho^{2k!}}{2k!} \right) \right|_0^1 =$$

$$= (k!)^2 \left(\frac{1}{2k!+2} - \frac{1}{2k!} \right) =$$

$$= (k!)^2 \frac{2k! - 2k! - 2}{(2k!+2)(2k!)} =$$

$$= -2 \frac{(k!)^2}{(2k!+2)\,2k!} = -\frac{k!}{2k!+2}$$

and then

$$\left| \int_0^{2\pi} d\theta \int_0^1 \rho \sum (k!)^2 \rho^{2k!-2} (\rho^2 - 1) \frac{\operatorname{sen}^2(k!\theta)}{k^4}\, d\rho \right|$$

$$\leq \int_0^{2\pi} d\theta \sum_{k=1}^{+\infty} \frac{k!}{2k!+2} \frac{1}{k^4} \leq \pi \sum_{k=1}^{+\infty} \frac{1}{k^4}$$

Conclusion: the function u is harmonic, but

$$\int_{B_1(0)} |\nabla u|^2 \, dx\,dy = +\infty$$

THIS PART CONTAINS SOME RECALLS USEFUL TO FOLLOW THE EXAMPLE OF HADAMARD

APPENDIX A   RECALLS ABOUT SERIES OF FUNCTIONS

1) Given a series of functions $\sum_{n=1}^{\infty} f_n(x)$, $x \in I$, I open interval

if $$\sum_{n=1}^{+\infty} \sup_{x \in I} |f_n(x)| < +\infty$$

then $\sum f_n(x)$ uniformly converges in I.

2) Given $f_n \in C^1(I)$ and $f : I \to \mathbb{R}$ such that

$\sum_{n=1}^{\infty} f_n$ uniformly converges to f in I

$\sum_{n=1}^{\infty} f_n'$ " " (to someone) in I

then for each $x \in I$ one has

$$\sum_{n=1}^{\infty} f_n'(x) = \frac{d}{dx} \sum_{n=1}^{+\infty} f_n(x) .$$

The assumptions are not sharp.

These results may be extended to functions of two (or more) variables.