THE LAPLACE EQUATION

The laplace operator is defined as

$$\Delta u(x) := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) \qquad (u \in C^2)$$

Even if the equation $-\Delta u = 0$ is often called LAPLACE EQUATION and $-\Delta u = f$ POISSON EQUATION we will not always distinguish the two cases.

Some examples:

- E electric vector field generated by a distribution of charge whose density is ρ

  $$\operatorname{div} E = c\rho \quad \text{(one of Maxwell's equation)}$$

  if E admits a potential ($E = -\nabla u$ for some u, u electric potential)

  $$-\operatorname{div} \nabla u = -\Delta u = c\rho$$

- $-\Delta u = f$ in Ω u describes the position of an elastic membrane subject subject to a force f (f "little")

(and to some conditions in $\partial\Omega$),
for instance the membrane of a drum

- if $f \equiv 0$ and u is a solution of
$-\Delta u = 0$ u is called harmonic

- two possible generalisations of $-\Delta$ are

$$L_1 u = - \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

$$L_2 u = - \operatorname{div}(a(x) \cdot Du) =$$

$$= - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right)$$

with the $n \times n$ matrix a having n positive eigenvalues
(they could be all negative, but usually
a is chosen in such a way that is
positive definite)

a usually is symmetric
otherways one could substitute a with

$$\tilde{a} , \quad \tilde{a}_{ij} := \frac{a_{ij} + a_{ji}}{2}$$

indeed

$$a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + a_{ji} \frac{\partial^2 u}{\partial x_j \partial x_i} = 2 \tilde{a}_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

since $\dfrac{\partial^2 u}{\partial x_i \partial x_j} = \dfrac{\partial^2 u}{\partial x_j \partial x_i}$

L_2 is elliptic since, if $a_{ij} \in C^1(\overline{\Omega})$,

one can write

$$L_2 u = - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial a_{ij}}{\partial x_i} \right) \frac{\partial u}{\partial x_j}$$

$$= \underbrace{L_1 u}_{\text{principal part of } L_2} - b \cdot Du \qquad b_j = \sum_{i=1}^{n} \frac{\partial a_{ij}}{\partial x_i}$$

If $a = Id$ L_1 and L_2 coincide with $-\Delta$

Now we see some typical problems, i.e. the Laplace equation coupled with some additional information.

Typical problems are the following:

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \text{boundary condition in } \partial\Omega \end{cases}$$

The classical boundary conditions for the Laplace (and in general elliptic) equation in $\mathbb{R}^n$, $n \geq 2$ are the following:

i) $u = g$ in $\partial\Omega$ (Dirichlet conditions)

ii) $\frac{\partial u}{\partial \nu} = g$ in $\partial\Omega$ (Neumann conditions)

$\frac{\partial u}{\partial \nu}$ normal derivative $\left((Du, \nu) \quad \nu \text{ outer normal to } \partial\Omega \right)$

iii) $\alpha \frac{\partial u}{\partial \nu} + \beta u = g$ (Robin conditions)

$\alpha, \beta > 0$

VARIATIONAL NATURE OF (SOME) HARMONIC FUNCTIONS

(DIRICHLET'S PRINCIPLE)

Lemma Consider $u \in L^1_{loc}(\Omega)$. If

$$\int_\Omega u \varphi \, dx = 0 \quad \forall \varphi \in C^\infty_c(\Omega)$$

then $u = 0$ in Ω.

Proof: We prove the result for $u \in C^0(\Omega)$ and, by contradiction, suppose $u \not\equiv 0$.

In particular there is $x_0 \in \Omega$ such that $u(x_0) \neq 0$. Suppose $u(x_0) > 0$. By continuity there is a ball $B = B_\varepsilon(x_0)$ such that $u > 0$ in B. Then, taking $\varphi \in C^\infty_c(B)$, $\varphi > 0$, we would get

$$\int_\Omega u \varphi \, dx > 0 ,$$

which is impossible ↯.

($u \in L^1_{loc}$ needs some approximations or convolution with suitable functions).

<u>RECALL</u> Consider $f : E \times I \longrightarrow \mathbb{R}$ $(x \in E, t \in I)$

$E \subseteq \mathbb{R}^n$, I open interval, such that

for every $x \in E$ the function

$f(x, \cdot)$ is differentiable in t and

$$f, \frac{\partial f}{\partial t} \in C^0(E \times I) \cap L^1(E \times I).$$

Then, once defined $g(t) := \int_E f(x,t)\,dx$,

one has that g is differentiable and

$$\left.\frac{d}{dt}\right|_{t=s} g(t) = g'(s) = \int_E \frac{\partial f}{\partial t}(x,s)\,dx$$

i.e.

$$\left.\frac{d}{dt}\right|_{t=s} \int_E f(x,t)\,dx = \int_E \frac{\partial f}{\partial t}(x,s)\,dx$$

REMARK The assumptions can be weakened, but for our purposes this is sufficient.

Consider Ω an open subset of $\mathbb{R}^n$ with boundary of class C^1, Ω bounded, a function $f \in C^0(\overline{\Omega})$, a function $g \in C^0(\partial\Omega)$, and the set of admissible functions

$$\mathcal{A}(\Omega; g) = \left\{ u \in C^1(\Omega) \cap C^0(\overline{\Omega}) \;\middle|\; u = g \text{ in } \partial\Omega \text{ and } \int_\Omega |Du|^2 dx < +\infty \right\}$$

and define $\mathcal{F}u := \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega f u \, dx$

Theorem (Dirichlet's principle)

i) Suppose $\mathcal{F}$ admits a minimum point $u \in \mathcal{A}(\Omega; g)$ and $u \in C^2(\Omega)$.

Then $-\Delta u = f$ in Ω.

ii) Conversely, if $u \in C^2(\overline{\Omega})$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases}$$

then $\mathcal{F}(u) = \min\limits_{\mathcal{A}(\Omega; g) \cap C^2(\overline{\Omega})} \mathcal{F}$

Remark The statements are equivalent in $\mathcal{A}(\Omega; g) \cap C^2(\overline{\Omega})$ (in point i) we assume something less)

proof: ii) Consider $v \in \mathcal{A}(\Omega; g) \cap C^2(\overline{\Omega})$. Then

$$0 = \int_\Omega (-\Delta u - f)(u - v)\, dx = -\int_{\partial\Omega} \overset{=0}{(u-v)} (Du, \nu)\, d\mathcal{H}^{n-1} + \int_\Omega (Du, D(u-v))\, dx - \int_\Omega f(u-v)\, dx$$

and then

$$\int_\Omega [|Du|^2 - fu]\, dx = \int_\Omega [(Du, Dv) - fv]\, dx \leq$$

$$\leq \frac{1}{2} \int_\Omega [|Du|^2 + |Dv|^2]\, dx - \int_\Omega fv\, dx$$

by which the thesis.

i) Consider $\varphi \in C_c^\infty(\Omega)$. Then for every $\varepsilon \in \mathbb{R}$ $u + \varepsilon\varphi \in \mathcal{A}(\Omega; g)$ and consequently

$$\mathcal{F}(u) \leq \mathcal{F}(u + \varepsilon\varphi)$$

If we define $F(\varepsilon) := \mathcal{F}(u + \varepsilon\varphi)$ (u and also φ fixed)

We have that $F(0) \leq F(\varepsilon) \quad \forall\, \varepsilon \in \mathbb{R}$

Moreover F is differentiable and then $F'(0) = 0$.

Let us compute F':

$$F'(\varepsilon) = \frac{d}{d\varepsilon}\left[\frac{1}{2}\int_\Omega |D(u+\varepsilon\varphi)|^2 dx - \int_\Omega f(u+\varepsilon\varphi)\,dx\right] =$$

$$= \frac{d}{d\varepsilon}\left(\frac{1}{2}\int \left(D(u+\varepsilon\varphi), D(u+\varepsilon\varphi)\right) dx - \int fu - \varepsilon\int f\varphi\right)$$

$$= \frac{d}{d\varepsilon}\left[\frac{1}{2}\int_\Omega \left[|Du|^2 + 2\varepsilon(Du, D\varphi) + \varepsilon^2 |D\varphi\right] dx + \right.$$

$$\left. - \int_\Omega fu - \varepsilon\int_\Omega f\varphi\, dx\right] =$$

$$= \int_\Omega (Du, D\varphi)\,dx + \varepsilon\int_\Omega |D\varphi|^2 dx - \int_\Omega f\varphi\, dx$$

$$\Leftarrow) \quad F'(0) = \int_\Omega (Du, D\varphi)\,dx - \int_\Omega f\varphi$$

$$\mathrm{div}(\varphi Du) = (D\varphi, Du) + \varphi\,\Delta u$$

$$\Rightarrow \int_\Omega (Du, D\varphi)\,dx = -\int_\Omega \varphi\,\Delta u\,dx + \int_\Omega \mathrm{div}(\varphi Du)\,dx =$$

$$= -\int_\Omega \varphi \Delta u \, dx + \underbrace{\int_{\partial\Omega} \varphi (Du, \nu) \, d\mathcal{H}^{n-1}}_{\substack{=0 \\ \text{since } \varphi \in C_c^\infty(\Omega)}}$$

Then

$$0 = F'(0) = \int_\Omega (-\Delta u - f)\varphi \, dx$$

(! no problem at the boundary since φ is compactly supported, so we do not need $u \in C^1(\overline{\Omega})$ or $u \in C^2(\overline{\Omega})$)

Since this holds for every $\varphi \in C_c^\infty(\Omega)$, by the previous lemma we have that

$$-\Delta u = f \quad \text{in } \Omega .$$

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REMARK Even if we have seen the classification of (some) second order PDEs in elliptic, parabolic and hyperbolic for $n \geq 2$ if you consider (in $n = 1$) the functional

$$Fu = \frac{1}{2} \int_a^b |u'(x)|^2 - \int_a^b fu \, dx$$

you obtain the same conclusions as before.
In this sense the "nature" of $-u'' = f$
is (improperly) elliptic.

Def When a PDE is obtained by looking for
stationary points of a functional $\mathcal{F}$
we call that PDE the
Euler - Lagrange equation of $\mathcal{F}$.

Therefore $-\Delta u = f$ in Ω is the Euler - Lagrange
equation corresponding to the functional

$$\mathcal{F}u = \int_\Omega \left[\frac{1}{2} |Du|^2 - fu \right] dx$$

In particular $u \in \mathcal{A}(\Omega; g) \cap C^2(\overline{\Omega})$ is a
minimum point for $\mathcal{F}$ in $\mathcal{A}(\Omega; g) \cap C^2(\overline{\Omega})$
if and only if u solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases} \qquad (P)$$

The functional $\mathcal{F}$ not always has a minimum (an absolute minimum) in $\mathcal{A}(\Omega; g)$, even if we look in $\mathcal{A}(\Omega; g) \cap C^2(\Omega)$ (and do not require $u \in C^1(\overline{\Omega})$ or $u \in C^2(\overline{\Omega})$).

- first, there are no reasons for which $\mathcal{F}$ should have a minimum in $C^2(\Omega)$ since $\mathcal{F}$ depends only on first derivatives;
- second, if g oscillates "too much" and (P) has a solution u, the energy of u,

  i.e. $\int_\Omega |\nabla u|^2 dx$, might be $+\infty$,

  even with $f \equiv 0$, as in an example due to Hadamard (for those who want the example is available in the "OPTIONAL MATERIAL")

The function of the example belongs to $C^2(\Omega) \cap C^0(\overline{\Omega})$, but, as we will see, not to $C^1(\overline{\Omega})$.

ENERGY METHOD FOR UNIQUENESS

Consider the problem (D) (D for Dirichlet boundary conditions)

$$(D) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases}$$

Suppose to know that problem (D) has (at least) one solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$ and moreover $u \in C^1(\overline{\Omega})$. Then this is unique.

$$\left(u \in C^1(\overline{\Omega}) \Rightarrow \int_\Omega |\nabla u|^2 dx < +\infty \quad \text{and} \quad \frac{\partial u}{\partial \nu} \in C^0(\partial\Omega) \right)$$

proof: suppose to have two solutions u_1 and u_2

$$\begin{cases} -\Delta u_j = f & \text{in } \Omega \\ u_j = g & \text{in } \partial\Omega \end{cases} \qquad j = 1, 2$$

Consider $u = u_2 - u_1$: u satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

Multiply by u the equation and integrate.
You get

$$\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u \, d\mathcal{H}^{n-1} = 0$$

Since $u = 0$ in $\partial\Omega$ the second integral is null and then we get $\int_\Omega |\nabla u|^2 dx = 0$,
by which u is constant in each connected component of Ω.
Since moreover $u = 0$ in $\partial\Omega$
we derive that $u \equiv 0$, i.e. $u_1 = u_2$. //

Ex Apply the energy method to study uniqueness for the Neumann and Robin problems (N) and (R)

$$(N) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{in } \partial\Omega \end{cases}$$

$$(R) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ \alpha \frac{\partial u}{\partial \nu} + \beta u = g & \text{in } \partial\Omega \end{cases} \qquad \alpha, \beta > 0$$

? Have they a unique solution ?

REMARK Observe that by uniqueness of the solution of problem (D) (after doing **Ex** above also of problem (N)) the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g_1 & \text{in } \partial\Omega \\ \frac{\partial u}{\partial \nu} = g_2 \end{cases}$$

is ill posed and, in general, not solvable.

REMARK A compatibility condition about (N).

Suppose u is a solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{in } \partial\Omega \end{cases}$$

Multiply by v the equation. Since

$$-\int_\Omega \left[v \Delta u + (Du, Dv) \right] dx = -\int_\Omega \operatorname{div}(v\, Du)\, dx = -\int_{\partial\Omega} v (Du, \nu)\, d\mathcal{H}^{n-1}(x)$$

taking $v \equiv 1$ we derive the compatibility condition

$$-\int_\Omega \underbrace{\Delta u}_{= -f}\, dx = -\int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}}_{= g}\, d\mathcal{H}^{n-1} \qquad \text{i.e.}$$

$$\int_\Omega f\, dx = -\int_{\partial\Omega} g\, d\mathcal{H}^{n-1}$$

This is a necessary condition to solve (N).