FUNDAMENTAL SOLUTION

Our goal is to find a special solution for the equation

$$-\Delta u = 0$$

in a subset of $\mathbb{R}^n$ (precisely in $\mathbb{R}^n \setminus \{0\}$)

This function will be called fundamental solution.

It will be clear later why the name "fundamental".

First we see a general fact.

FACT – The equation $\Delta u = 0$ is rotations invariant, that is if

$$\Delta u(x) = 0 \quad \forall x \in \mathbb{R}^n \quad \text{then}$$

$$\Delta v(x) = 0 \quad \forall x \in \mathbb{R}^n \quad \text{where} \quad v(y) = u(My)$$

where M is a unitary matrix

proof: let M be an invertible matrix such that

$$M^{-1} = M^t \qquad \text{and write} \quad x = My$$

Now define $v(y) := u(My)$ where $\Delta u(x) = 0$
$\forall x \in \mathbb{R}^n$

$$\frac{\partial v}{\partial y_i}(y) = \frac{\partial}{\partial y_i}\left(u\left(M_1 \cdot y, \dots M_n \cdot y\right)\right) =$$

$$\left(M_k \cdot y = m_{k1} y_1 + \dots + m_{kn} y_n \right)$$

$$= \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}(My)\, m_{ki}$$

$$\frac{\partial^2 v}{\partial y_i^2}(y) = \sum_{h,k=1}^{n} \frac{\partial^2 u}{\partial x_h \partial x_k}(My)\, m_{ki}\, m_{hi}$$

$$m_{hi} = \left(M^{-1}\right)_{ih}$$

$$\Delta v(y) = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial y_i}(y) = \sum_{h,k=1}^{n} \frac{\partial^2 u}{\partial x_h \partial x_k}(My) \underbrace{\sum_{i=1}^{n} M_{ki} \left(M^{-1}\right)_{ih}}_{= \delta_{kh}}$$

$$= \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}(My) =$$

$$= \Delta u(My) = 0$$

This in particular means that if u is harmonic and M is a rotation (which is unitary) then v is harmonic, where $v(y) := u(Ry)$.

This in particular holds if u is radial.

! This does not mean that the solutions of $-\Delta u = 0$ are (all) radial

HARMONIC FUNCTIONS AND HOLOMORFIC FUNCTIONS

In fact, in $n = 2$ for instance, the theory of harmonic functions is connected with the holomorfic functions $f : \mathbb{C} \to \mathbb{C}$.

Consider u and v respectively the real and the imaginary part of f (holomorphic), i.e.

$$f(z) = u(x,y) + i\,v(x,y) \qquad z = x + iy \qquad u, v : \mathbb{R}^2 \to \mathbb{R}$$

Computing the derivative of f in $z_0 = (x_0, y_0)$ we have

$$\lim_{(h,k) \to (0,0)} \frac{f(z_0 + h + ik) - f(z_0)}{h + ik} = f'(z_0)$$

$$k = 0 \qquad \lim_{h \to 0} \frac{u(x_0 + h, y_0) + i\,v(x_0 + h, y_0) - u(x_0, y_0) - i\,v(x_0, y_0)}{h}$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) = f'(z_0)$$

$$h = 0 \quad \dots \quad f'(z_0) = \frac{1}{i} \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

Then

$$u_x + i\,v_y = v_y + \frac{1}{i} u_y \quad \Rightarrow \quad \left| \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right.$$

$\Rightarrow$ (since u and v C^∞) $\quad u_{xx} + u_{yy} = 0$

$$v_{xx} + v_{yy} = 0$$

That is : the real and imaginary part of a holomorfic function f are harmonic

For example, the functions $f(z) = z^m$ $(m \in \mathbb{N})$ and $g(z) = e^{i\alpha z}$ are holomorfic in $\mathbb{C}$.

Writing $z = x+iy$ one gets

$m = 0 \qquad u(x,y) = 1$

$m = 1 \qquad x+iy \qquad u(x,y) = x \qquad v(x,y) = y$

$m = 2 \qquad (x+iy)^2 = x^2-y^2+2i\,xy$

$$u(x,y) = x^2-y^2 \qquad v(x,y) = xy$$

$m = 3 \qquad (x+iy)^3 = x^3+3ix^2y - 3xy^2 + i^3y^3$

$$u(x,y) = x^3 - 3xy^2$$

$$v(x,y) = 3x^2y - y^3$$

$\vdots$

or $\quad u(x,y) = e^{\alpha x}\cos \alpha y \,, \quad v(x,y) = e^{\alpha x} \operatorname{sen} \alpha y$.

Notice that taking $h(z) = \log z$

we derive that

$$u(x,y) = \log|z| = \log\rho = \log\sqrt{x^2+y^2}$$

is harmonic in $\mathbb{R}^2 \setminus \{(0,0)\}$

and $v(x,y) = \theta$ harmonic in $\mathbb{R}^2 \setminus S$

S half-line starting in $(0,0)$

Therefore we have a lot (infinite) non-radial harmonic functions.

More generally, without writing the Laplacian in new coordinates, we look for a radial solution to

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^n \quad (\text{in } \mathbb{R}^n \setminus \{0\})$$

for every $n \geq 2$, i.e. we look for a solution u such that $u(x) = v(|x|)$ for some v

$$|x| = \sqrt{x_1^2 + \dots + x_n^2} =: r(x)$$

$$\frac{\partial r}{\partial x_i} = \frac{1}{2|x|} 2x_i = \frac{x_i}{|x|} \qquad \text{and then}$$

$$\frac{\partial u}{\partial x_i}(x) = \frac{dv}{dr}(r(x)) \frac{x_i}{|x|}$$

$$\frac{\partial^2 u}{\partial x_i^2}(x) = \frac{d^2 v}{dr^2}(r(x)) \frac{x_i^2}{|x|^2} + \frac{dv}{dr}(r(x)) \frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|}\right) =$$

$$= v''(r) \frac{x_i^2}{|x|^2} + v'(r) \frac{|x|^2 - x_i^2}{|x|^3}$$

e) $\Delta u(x) = \dfrac{v''(r)}{r^2} \sum_{i=1}^{n} x_i^2 + \dfrac{v'(r)}{r^3} (n-1) r^2$

$$= v''(r) + \frac{(n-1)}{r} v'(r) = 0$$

Notice that for $n = 2$ we get the equation obtained before when we explicitely wrote the Laplacian.

<u>RECALL</u> To solve $y' + a y = 0$ one has to compute one primitive of a, $A(t) = \int a(t)\,dt$, and then a generic solution will be

$$y(t) = c\, e^{-\int a(t)dt}$$

Then

$$v''(r) + \frac{(n-1)}{r} v'(r) = 0$$

$$w'(r) + \frac{n-1}{r} w(r) = 0$$

$\int \frac{1}{r} dr = \log |r| + k \quad (r>0)$ and choose $k = 0$

Then

$$w(r) = a\, e^{-(n-1)\log r} = a \frac{1}{r^{n-1}}, \quad a \in \mathbb{R}$$

and finally ($v' = w$)

$$v(r) = \begin{cases} a \log r + b & \text{if } n = 2 \\ \dfrac{a}{2-n} r^{2-n} + b & \text{if } n \geq 3 \end{cases}$$

$$a, b \in \mathbb{R}$$

We define

$$E(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \dfrac{1}{(n-2)|S^{n-1}|} \dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3 \end{cases}$$

$$S^{n-1} = \{ x \in \mathbb{R}^n \mid |x| = 1 \}$$

EX Verify that E and $\nabla E \in L^1_{loc}(\mathbb{R}^n)$.

Curiosity Denote by σ_n the quantity $|S^{n-1}|$
and by ω_n " " $|B_1|$

$\left(|S^{n-1}| := |S^{n-1}|_{\mathcal{H}^{n-1}} \right), \quad |B_1| := |B_1(0)|_{\mathcal{L}^n},$

being $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure,
$\mathcal{L}^n$ the n-dimensional Lebesgue measure)

Given $r>0$ and $B_r \subseteq \mathbb{R}^n$ one then has
$|B_r| = \omega_n r^n$ and $|\partial B_r| = \sigma_n r^{n-1}$.

Then

$$|\partial B_r| = \frac{d}{dr} |B_r|$$

and in particular $\sigma_n = n\,\omega_n$.

proof (EX) Use the formula

See, e.g., EVANS PDE Appendix C

$$\int_{B_r(x_0)} f(x)\,dx = \int_0^r d\rho \int_{\partial B_\rho(x_0)} f \, d\mathcal{H}^{n-1}$$

that holds for every $f \in C^0\left(\overline{B_r(x_0)}\right)$
and take $f \equiv 1$.

The remark made at the beginning, i.e. $\Delta u = 0$ even with a rotation of coordinates, justifies the research of a radial solution in dimension n.

Rmk Suppose you want to solve, for some f,

$$P(D)u = f$$

$P(D)$ is a differential polynomial of degree N and $f \in C_c^\infty(\mathbb{R}^n)$ (it would be sufficient $f \in C_c^N$)

(P is a polynomial in the n variables $\xi_1, \dots \xi_n$ and D is $(D_1, \dots D_n) = \left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}\right)$)

Suppose there exists a function E such that

$$P(D)E = \delta \quad \text{(Dirac delta in the origin)}$$

in the sense of distributions.

Consider the function $E * f$ and observe that

$$P(D)(E * f) = (P(D)E) * f =$$

$$= \delta * f = f$$

Def E is said fundamental solution for the differential operator $P(D)$.

The interest to find the fundamental solution is clear: once one knows E one can solve (theoretically)

$$P(D)u = f \qquad \text{whatever } f \in \mathcal{D}(\mathbb{R}^n)$$

simply taking $E * f$.

Now, which is the meaning of $P(D)u = \delta$ for some $u \in L^1_{loc}(\mathbb{R}^n)$?

We have seen that a distribution T is infinitely differentiable and $D^\alpha T$ is the distribution acting as follows

$$\langle D^\alpha T, \phi \rangle \overset{def}{=} (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle_{\mathcal{D}' \times \mathcal{D}} \quad \forall \phi \in \mathcal{D}(\Omega)$$

In particular we are now interested in the solutions of $-\Delta u = \delta$.

Def (distributional solutions)

We will say that a function $u \in L^1_{loc}(\mathbb{R}^n)$ verifies the equation

$$-\Delta u = T \qquad \text{for some } T \in \mathcal{D}'(\mathbb{R}^n)$$

in the distributional sense if for every $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$-\int_{\mathbb{R}^n} u(x)\, \Delta\phi(x)\, dx = \langle T, \phi \rangle_{\mathcal{D}' \times \mathcal{D}} \qquad \forall\, \phi \in \mathcal{D}(\mathbb{R}^n).$$

In particular

$$-\Delta u = \delta \qquad \text{in the distributional sense}$$

if

$$-\int_{\mathbb{R}^n} u(x)\, \Delta\phi(x)\, dx = \phi(0) \qquad \forall\, \phi \in \mathcal{D}(\mathbb{R}^n).$$

and u is harmonic in the distributional sense

if

$$-\int_{\mathbb{R}^n} u(x)\, \Delta\phi(x)\, dx = 0 \qquad \forall\, \phi \in \mathcal{D}(\mathbb{R}^n).$$

If we know that $u, \nabla u \in L^1_{loc}(\mathbb{R}^n)$ then

$$-\int_{\mathbb{R}^n} u\, \Delta\phi\, dx = \int_{\mathbb{R}^n} \nabla u \cdot \nabla\phi\, dx.$$

Now we verify that $E(x,y) = -\frac{1}{2\pi} \log |(x,y)|$ is the fundamental solution for $-\Delta$ in $\mathbb{R}^2$.

EX If you want you can verify the analogous fact in $\mathbb{R}^n$, $n = 3$.

By definition

$$-\Delta E = \delta \quad \Leftrightarrow \quad -\int_{\mathbb{R}^2} E(x,y)\, \Delta\phi(x,y)\, dx\,dy = \phi(0,0)$$

But notice that

$$-\int_{\mathbb{R}^2} E\, \Delta\phi\, dx\,dy = \int_{\mathbb{R}^2} \nabla E \cdot \nabla\phi\, dx\,dy \qquad \left(\begin{array}{c} E \in L^1_{loc} \\ \nabla E \in L^1_{loc} \end{array} \right)$$

Then we compute the term on the right hand side.

$$\nabla \log |(x,y)| = \nabla \log \sqrt{x^2+y^2} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

$$\phi(\rho\cos\theta, \rho\,\text{sen}\,\theta) =: \tilde{\phi}(\rho,\theta)$$

$$\frac{\partial}{\partial\rho} \tilde{\phi}(\rho,\theta) = \partial_x\phi(\rho\cos\theta, \rho\,\text{sen}\,\theta)\cos\theta + \\ + \partial_y\phi(\rho\cos\theta, \rho\,\text{sen}\,\theta)\,\text{sen}\,\theta$$

$$\frac{\partial}{\partial\theta}\tilde{\Phi}(\rho\,\theta) = \partial_x\Phi(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)\,(-\rho\,\mathrm{sen}\,\theta) + $$
$$+ \partial_y\Phi(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)\,\rho\cos\theta$$

Call A the jacobian matrix $A = \begin{pmatrix} \cos\theta & \mathrm{sen}\,\theta \\ -\rho\,\mathrm{sen}\,\theta & \rho\cos\theta \end{pmatrix}$

of the map $F(\rho,\theta) = (\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)$

in such a way ~~that~~

$$\nabla_{\rho,\theta}\tilde{\Phi}(\rho,\theta) = \underbrace{\begin{pmatrix} \cos\theta & \mathrm{sen}\,\theta \\ -\rho\,\mathrm{sen}\,\theta & \rho\cos\theta \end{pmatrix}}_{=A} \nabla_{x,y}\Phi(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)$$

and $\nabla_{x,y}\Phi(F(\rho,\theta)) = A^{-1}\nabla_{\rho,\theta}\tilde{\Phi}(\rho,\theta)$ where

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} \rho\cos\theta & -\mathrm{sen}\,\theta \\ \rho\,\mathrm{sen}\,\theta & \cos\theta \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} \rho\cos\theta & -\mathrm{sen}\,\theta \\ \rho\,\mathrm{sen}\,\theta & \cos\theta \end{pmatrix}.$$

Then

$$\iint_{\mathbb{R}^2} \nabla\log|(x,y)|\cdot\nabla\phi(x,y)\,dx\,dy = \iint_{\mathbb{R}^2} \underbrace{\frac{1}{x^2+y^2}(x,y)}_{\nabla E}\cdot\underbrace{(\partial_x\Phi, \partial_y\Phi)}_{\nabla\phi}\,dx\,dy =$$

$$= \int_0^{+\infty} d\rho \int_0^{2\pi} d\theta \; \overset{\text{Jacobian of } A}{\rho} \; \underbrace{\frac{1}{\rho^2} \left(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta \right)}_{\nabla E} \; \underbrace{\overbrace{\frac{1}{\rho} \begin{pmatrix} \rho\cos\theta & -\mathrm{sen}\,\theta \\ \rho\,\mathrm{sen}\,\theta & \cos\theta \end{pmatrix}}^{A^{-1}} \nabla_{\rho,\theta} \tilde{\phi}}_{\nabla\phi} =$$

$$= \int_0^{+\infty} d\rho \int_0^{2\pi} d\theta \; \frac{1}{\rho} \left(\cos\theta, \mathrm{sen}\,\theta \right) \left(\rho\cos\theta\, \partial_\rho \tilde{\phi} - \mathrm{sen}\,\theta\, \partial_\theta \tilde{\phi} ,\; \rho\,\mathrm{sen}\,\theta\, \partial_\rho \tilde{\phi} + \cos\theta\, \partial_\theta \tilde{\phi} \right) =$$

$$= \int_0^{+\infty} d\rho \int_0^{2\pi} d\theta \left[\rho\cos^2\theta\, \partial_\rho \phi - \mathrm{sen}\,\theta\cos\theta\, \partial_\theta \phi + \rho\,\mathrm{sen}^2\theta\, \partial_\rho \tilde{\phi} + \mathrm{sen}\,\theta\cos\theta\, \partial_\theta \tilde{\phi} \right] \frac{1}{\rho} =$$

$$= \int_0^{+\infty} d\rho \int_0^{2\pi} d\theta \; \partial_\rho \tilde{\phi}(\rho,\theta) =$$

$$= \int_0^{2\pi} d\theta \left(\tilde{\phi}(+\infty,\theta) - \tilde{\phi}(0,\theta) \right) = -2\pi\, \phi(0,0)$$

since $\tilde{\phi}(0,\theta) = \phi(0,0)$ (for every θ)

where $\displaystyle\int_0^{+\infty} d\rho \, \dots = \lim_{c \to +\infty} \int_0^{c} \dots =$

$$\lim_{c \to +\infty} \int_0^{2\pi} d\theta \left(\tilde{\phi}(c,\theta) - \tilde{\phi}(0,\theta) \right) = -2\pi \, \phi(0,0)$$

this is 0 for c big enough, since ϕ is compactly supported.