SOME PROPERTIES OF HARMONIC FUNCTIONS

A function $u : \Omega \to \mathbb{R}$, $\Omega \overset{\circ}{\subset} \mathbb{R}^n$, $u \in C^0(\Omega)$ is said

subharmonic if $u(x) \leq \fint_{\partial B_R(x)} u(y)\, d\mathcal{H}^{n-1}(y)$ and

superharmonic if $u(x) \geq \fint_{\partial B_R(x)} u(y)\, d\mathcal{H}^{n-1}(y)$

for every $x \in \Omega$, $R > 0$, $B_R(x) \subset\subset \Omega$.

Example: $n = 1$ $\quad u(x) = |x|$ is subharmonic.
$\qquad u(x) = -|x|$ is superharmonic.

THE MEAN VALUE PROPERTY

Theorem Let $u \in C^2(\Omega)$. Then if $-\Delta u = (\leq \ \geq)\ 0$ for every ball $B_R(x) \subset\subset \Omega$ the following hold:

i) $u(x) = (\leq \ \geq) \fint_{\partial B_R(x)} u(y)\, d\mathcal{H}^{n-1}(y)$

ii) $u(x) = (\leq \ \geq) \fint_{B_R(x)} u(y)\, dy$

proof: i) suppose, for instance, $-\Delta u \leq 0$

consider $r \in (0, R]$ and define

$$g(r) = \fint_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y)$$

see that g is continuous: write

$$g(\rho) = \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B_\rho(x)} u(y)\, d\mathcal{H}^{n-1}(y) = \frac{\cancel{\rho^{n-1}}}{\sigma_n \cancel{\rho^{n-1}}} \int_{\partial B_1(0)} u(x+\rho z)\, d\mathcal{H}^{n-1}(z)$$

$$y = x + \rho z \qquad |z| = 1 \qquad d\mathcal{H}^{n-1}(y) = \rho^{n-1}\, d\mathcal{H}^{n-1}(z)$$

by uniform continuity of u in $\partial B_1(0)$

$$\lim_{\rho \to r} \int_{\partial B_1(0)} u(x+\rho z)\, d\mathcal{H}^{n-1}(z) = \int_{\partial B_1(0)} u(x+rz)\, d\mathcal{H}^{n-1}(z)$$

Now we compute g' (we see that g is also differentiable):

$$\sigma_n \frac{dg}{dr}(r) = \frac{d}{dr} \int_{\partial B_1(0)} u(x+rz)\, d\mathcal{H}^{n-1}(z) = \left[\begin{array}{l} r \mapsto u(x+rz) \text{ is} \\ \text{differentiable} \\ \text{for every } x, z \\ u \in C^1(\overline{B_1(0)}) \end{array}\right.$$

$$= \int_{\partial B_1(0)} (\nabla u(x+rz), z)\, d\mathcal{H}^{n-1}(z) \qquad (z = \nu)$$

If we define $v(z) := u(x+rz)$ we get

$$\frac{\partial v}{\partial z_i}(z) = r\frac{\partial u}{\partial y_i}(x+rz), \qquad \frac{\partial^2 v}{\partial z_i^2}(z) = r^2\frac{\partial^2 u}{\partial y_i}(x+rz)$$

Then

$$\sigma_n \frac{dg}{dr}(r) = \int_{\partial B_1(0)} \frac{1}{r}\underbrace{\big(\nabla v(z), z\big)}_{=\frac{\partial v}{\partial \nu}(z)}\, d\mathcal{H}^{n-1}(z) =$$

$$= \frac{1}{r}\int_{B_1(0)} \Delta v(z)\,dz = \frac{1}{r}\int_{B_1(0)} r^2 \Delta u(x+rz)\,dz =$$

$$= r\int_{B_1(0)} \Delta u(x+rz)\,dz = \frac{1}{r^{n-1}}\int_{B_r(x)} \Delta u(y)\,dy \geq 0$$

Then g is differentiable and monotone,

$$g'(r) \geq 0 \qquad \big(g'(r) = 0 \quad \text{or} \quad g'(r) \leq 0\big)$$

By continuity of g we have

$$\lim_{r\to 0^+} g(r) = u(x) \quad \text{and} \quad g(r) \leq g(R) \quad \forall r \leq R$$

by which we conclude.

ii) suppose $-\Delta u \le 0$. Integrating (by i))

(•) $u(x) \le \fint_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y)$ between 0 and R

we obtain

$$\frac{R^n}{n} u(x) = \int_0^R r^{n-1} u(x)\, dr \overset{(\bullet)}{\le}$$

$$\le \frac{1}{\sigma_n} \int_0^R dr \int_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y) = \frac{1}{\sigma_n} \int_{B_R(x)} u(y)\, dy$$

by which

$$u(x) = \frac{n}{R^n \sigma_n} \int_{B_R(x)} u(y)\, dy .$$

Recalling that (lection 4) $\sigma_n = n\,\omega_n$ we conclude. //

EX Compute for which $\alpha \in \mathbb{R}$ the function $u(x) = |x|^\alpha$ is subharmonic (u defined in its natural domain) in $\mathbb{R}$ ($n = 1$)

MAXIMUM PRINCIPLE

Theorem (maximum principle)

Consider Ω an open, bounded and connected subset of $\mathbb{R}^n$, $u \in C^0(\overline{\Omega})$. Then

i) if $u(x) \leq \fint_{\partial B_R(x)} u(y)\, d\mathcal{H}^{n-1}(y)$ $\forall x \in \Omega$ and $B_R(x) \subset\subset \Omega$

then either u is constant

or $u(x) < \max_{\partial\Omega} u$ $\forall x \in \Omega$;

ii) if $u(x) \geq \fint_{\partial B_R(x)} u(y)\, d\mathcal{H}^{n-1}(y)$ $\forall x \in \Omega$ and $B_R(x) \subset\subset \Omega$

then either u is constant

or $u(x) > \min_{\partial\Omega} u$ $\forall x \in \Omega$.

Remark. Sometimes the following is called

Strong maximum principle

WITHOUT PROOF

Let u satisfy the mean value property in $\Omega \subseteq \mathbb{R}^n$, open and connected (not necessarily bounded). Then if u attains its maximum or its minimum at $x_0 \in \Omega$ then u is constant in Ω.

Weak maximum principle

If moreover Ω is bounded, $u \in C^0(\overline{\Omega})$ and u not constant then

$$\min_{\partial\Omega} u < u(x) < \max_{\partial\Omega} u \qquad \forall x \in \Omega.$$

<u>Remark</u> As a consequence in particular we have:

if u is subharmonic $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, (a)

if u is superharmonic $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$, (b)

if $u(x) = \fint_{\partial B_R(x)} u(y)\, d\mathcal{H}^{n-1}(y)$ or if u is harmonic

both (a) and (b) hold and moreover

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u| .$$

Proof: i) suppose that $u(x) \leq \fint_{\partial B_R(x)} u(y)\, d\mathcal{H}^{n-1}(y)$

for every $x \in \Omega$ and ball $B_R(x) \subset\subset \Omega$.

Consider $x_0 \in \overline{\Omega}$ such that $u(x_0) = M = \max_{\overline{\Omega}} u$ and suppose u is not identically equal to $u(x_0)$. Suppose by contradiction that $x_0 \in \Omega$. Then for every ball $B_R(x_0) \subset \Omega$ we have

$$u(y) \leq u(x_0) \qquad \forall\, y \in B_R(x_0)$$

by which

$$\int_{\partial B_R(x_0)} (u(y) - u(x_0))\, d\mathcal{H}^{n-1}(y) \leq 0$$

and, since u is subharmonic the same integral has to be non-negative. Then

$$\int_{\partial B_R(x_0)} (u(y) - u(x_0))\, d\mathcal{H}^{n-1}(y) = 0 \qquad \text{for every } B_R(x_0).$$

Since $u \leq u(x_0)$ there exists at least one value $r > 0$ such that

$$u \equiv u(x_0) \quad \text{in } B_r(x_0).$$

Then we have shown that the set

$$L_M := \{ x \in \Omega \mid u(x) = M \} \qquad \text{is open.}$$

By the continuity of u L_M is also closed in Ω.

Since Ω is connected we have that $L_M = \Omega$.

But we excluded u constant, then $x_0 \notin \Omega$

$(x_0 \in \partial\Omega)$. //

Now we see the proof of the maximum principles for subharmonic function $u \in C^2(\Omega)$ satisfying $-\Delta u \leq 0$ without using the mean value property, but only $\Delta u \geq 0$.
In the following Ω is bounded.

proof: suppose first $\Delta u > 0$ in Ω. Then there cannot exist an internal maximum point x_0, since otherwise we would have

$$\Delta u(x_0) \leq 0 .$$

In the general case: consider $v(x) = e^{x_1}$ and consider $u_\varepsilon = u + \varepsilon v$, which satisfies $\Delta u_\varepsilon > 0$ for $\varepsilon > 0$.

Then

$$\sup_{\Omega} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon$$

and since this holds for every $\varepsilon > 0$ we conclude. //

UNIQUENESS VIA MAXIMUM PRINCIPLE

We have already seen (by energy method) the uniquess of the solution of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases} \qquad \begin{array}{l} f \in C^0(\Omega) \\ g \in C^0(\partial\Omega) \end{array}$$

The maximum principle gives another way to prove it.
(We do not yet know weather a solution exists, but if it exists it is unique).

proof : consider u_1, u_2 two solutions. Then $w := u_1 - u_2$ satisfies $-\Delta w = 0$ in Ω, $w = 0$ in $\partial\Omega$.
From the maximum principle we derive

$$\max_{\overline{\Omega}} |u_1 - u_2| = \max_{\partial\Omega} |u_1 - u_2| = 0 .$$

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Corollary (comparison) Consider Ω bounded and $g_1, g_2 \in C^0(\partial\Omega)$. Suppose u_i is a solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g_i & \text{in } \partial\Omega \end{cases} \qquad i = 1,2 .$$

If $g_1 \geq g_2$ in $\partial\Omega$ $\Rightarrow$ $u_1 \geq u_2$ in Ω,
and if there is $x_0 \in \partial\Omega$ for which $g_1(x_0) > g_2(x_0)$

$$\Rightarrow \quad u_1 > u_2 \quad \text{in } \Omega .$$

proof: by linearity $v := u_1 - u_2$ is a solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g_1 - g_2 & \text{in } \partial\Omega \end{cases}$$

and by the maximum principle either v is constant or

$$v(x) > \min_{\partial\Omega} (g_1 - g_2) \geq 0 . \qquad //$$

EX In the same assumptions prove (stability and continuous dependence on the datum) that

$$\max_{\overline{\Omega}} |u_1 - u_2| = \max_{\partial\Omega} |g_1 - g_2| .$$

GREEN'S IDENTITIES

Consider Ω bounded and open set of $\mathbb{R}^n$ of class C^1.

Consider $u, v \in C^2(\bar{\Omega})$. It is easy to observe

$$\text{I} \quad \int_\Omega v \Delta u \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1}$$

$$\text{II} \quad \int_\Omega (v \Delta u - u \Delta v) \, dx = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) d\mathcal{H}^{n-1}$$

Notice that, in particular, if u is harmonic in Ω taking in I

- $v \equiv 1$ we get $\displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1} = 0$ (1)

(already seen as compatibility condition of Neumann problem: $-\Delta u = f$ in Ω and $\frac{\partial u}{\partial \nu} = g$ in $\partial\Omega$

$\Rightarrow -\displaystyle\int_{\partial\Omega} g \, d\mathcal{H}^{n-1} = \int_\Omega f \, dx$)

- $v = u$ we get $$\int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1} = \int_{\Omega} |\nabla u|^2 dx$$

THE STOKES IDENTITIES (OR IDENTITY)

This identity is an implicit representation formula of a generic function $u \in C^2(\overline{\Omega})$.

This representation involves the fundamental solution of $-\Delta$ and for this reason sometimes it is referred to as "identities", one in dimension 2, one in dimension $n \geq 3$.

Consider Ω bounded and of class C^1.

For every $u \in C^2(\overline{\Omega})$ and $x \in \Omega$ it holds:

$$\boxed{u(x) = \int_{\partial\Omega} \left(E^x \frac{\partial u}{\partial \nu} - u \frac{\partial E^x}{\partial \nu} \right) d\mathcal{H}^{n-1} - \int_{\Omega} E^x(y) \, \Delta u(y) \, dy}$$

being E the fundamental solution of $-\Delta$

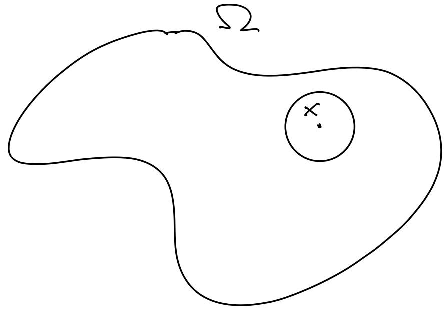
and $E^x(y) := E(y-x) = E(x-y)$.

proof: for $n \geq 3$ (EX $n = 2$)

fix $x \in \Omega$ and consider $B_\varepsilon(x)$

ε will go to zero, therefore we can suppose

$$B_\varepsilon(x) \subset \Omega$$



Now we apply the II Green identity with $v(y) = E(y-x) =$

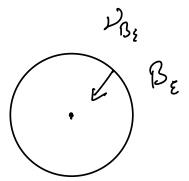
$$= \frac{1}{(n-2)\,\sigma_n} \; \frac{1}{|y-x|^{n-2}}$$

in the set $\Omega \setminus B_\varepsilon(x)$.

Since v is harmonic (in $\Omega \setminus B_\varepsilon(x)$) we get:

$$\int_{\Omega \setminus B_\varepsilon(x)} E^x(y)\, \Delta u(y)\, dy = \int_{\partial\Omega} \left(E^x \frac{\partial u}{\partial \nu_\Omega} - u \frac{\partial E^x}{\partial \nu_\Omega} \right) d\mathcal{H}^{n-1} +$$

$$+ \int_{\partial B_\varepsilon(x)} \left(E^x \frac{\partial u}{\partial \nu_{B_\varepsilon}} - u \frac{\partial E^x}{\partial \nu_{B_\varepsilon}} \right) d\mathcal{H}^{n-1}$$



Observe that: $\nabla E^x(y) = \dfrac{1}{(n-2)\,\sigma_n} (2-n) \dfrac{y-x}{|y-x|^n}$

and $\nu_{B_\varepsilon} = \dfrac{x-y}{|x-y|}$ so that

in ∂B_ε

$$\frac{\partial E^x}{\partial \nu_{B_\varepsilon}}(y) = \frac{1}{(n-2)\,\sigma_n} (2-n) \frac{1}{|y-x|^{n-1}} = \frac{1}{\sigma_n} \frac{1}{\varepsilon^{n-1}} \qquad (2)$$

Then

$$\int_{\partial B_\varepsilon(x)} \left(E^x \frac{\partial u}{\partial \nu_{B_\varepsilon}} - u \frac{\partial E^x}{\partial \nu_{B_\varepsilon}} \right) d\mathcal{H}^{n-1} =$$

$$= \int_{\partial B_\varepsilon(x)} \frac{1}{(n-2)\sigma_n} \left(\frac{1}{\varepsilon^{n-2}} \nabla u(y) \cdot \frac{(x-y)}{|x-y|} - \frac{n-2}{\varepsilon^{n-1}} u(y) \right) d\mathcal{H}^{n-1}(y)$$

$$\left| \frac{1}{(n-2)\sigma_n} \frac{1}{\varepsilon^{n-2}} \int_{\partial B_\varepsilon(x)} \nabla u(y) \cdot \frac{x-y}{|x-y|} \, d\mathcal{H}^{n-1}(y) \right| \leq$$

$$\leq \frac{1}{(n-2)\sigma_n} \max_{B_1(x)} |\nabla u| \frac{1}{\varepsilon^{n-2}} \varepsilon^{n-1} \sigma_n \xrightarrow[\varepsilon \to 0]{} 0$$

$$\frac{1}{\sigma_n} \frac{1}{\varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)} u(y) \, d\mathcal{H}^{n-1}(y) = \fint_{\partial B_\varepsilon(x)} u(y) \, d\mathcal{H}^{n-1} \xrightarrow[\varepsilon \to 0]{} u(x)$$

u is continuous

$$\left[\fint u(y) = \fint \left[(u(y) - u(x)) + u(x) \right] = u(x) + \fint \left[u(y) - u(x) \right] \right.$$

$$\text{and} \quad \left| \fint (u(y) - u(x)) \right| \leq \max |\nabla u| \frac{1}{\varepsilon^{n-1}\sigma_n} \left| \int_{\partial B_\varepsilon} |x-y| \, d\mathcal{H}^{n-1} \right|$$

$$\left. \lesssim \frac{1}{\varepsilon^{n-1}} \varepsilon^n \to 0 \right]$$

We conclude recalling that $E^x \in L^1(\Omega)$ and $\Delta u \in C^0(\overline{\Omega})$.

Then, taking the limit for $\varepsilon \to 0^+$, we finally get

$$\int_{\Omega} E^x(y)\, \Delta u(y)\, dy = \int_{\partial\Omega} \left(E^x \frac{\partial u}{\partial \nu} - u \frac{\partial E^x}{\partial \nu} \right) d\mathcal{H}^{n-1} - u(x)$$

ν outer normal to Ω

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Remark Under the previous assumptions, i.e. Ω bounded and $\partial\Omega$ of class C^1, $u \in C^2(\overline{\Omega})$, if moreover u is harmonic one has

$$\boxed{u(x) = \int_{\partial\Omega} \left(E^x \frac{\partial u}{\partial \nu} - u \frac{\partial E^x}{\partial \nu} \right) d\mathcal{H}^{n-1},} \qquad (3)$$

if u is compactly supported in Ω

$$\boxed{u(x) = - \int_{\Omega} E^x(y)\, \Delta u(y)\, dy} \qquad (4)$$

Observe that, being $x \in \Omega$, the term on the right hand side of (3) is differentiable infinitely many times with respect to x.

Now, using the Stokes identity, we see a different representation for a C^2 function.

Corollary Consider $x \in \mathbb{R}^n$, $r > 0$ and $u \in C^2(\overline{B_r(x)})$. Then

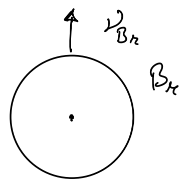
$$u(x) = \fint_{\partial B_r(x)} u \, d\mathcal{H}^{n-1} - \int_{B_r(x)} (E^x - E_r) \Delta u \, dy \qquad (6)$$

where $$E_r := \frac{1}{(n-2)\sigma_n} \frac{1}{r^{n-2}} \quad \text{for } n \geq 3$$

$$E_r := -\frac{1}{2\pi} \log r \quad \text{if } n = 2$$

proof: We see the proof for $n \geq 3$ (Ex $n = 2$).

By (2) we have



$$\frac{\partial E^x}{\partial \nu}(y) = -\frac{1}{\sigma_n r^{n-1}} = -\frac{1}{|\partial B_r(x)|_{n-1}}$$

and by the divergence theorem

$$\int_{\partial B_r(x)} E^x \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1} = E_r \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1} = E_r \int_{B_r(x)} \Delta u \, dy .$$

Using these informations in the Stokes identity

$$u(x) = \int_{\partial B_r(x)} \left(E^x \frac{\partial u}{\partial \nu} - u \frac{\partial E^x}{\partial \nu} \right) d\mathcal{H}^{n-1} - \int_{B_r(x)} E^x(y) \, \Delta u(y) \, dy$$

We conclude that

$$u(x) = \int_{B_r(x)} \left(E_r - E^x(y) \right) \Delta u(y) \, dy + \frac{1}{\mathcal{H}^{n-1}(\partial B_r(x))} \int_{\partial B_r(x)} u(y) \, d\mathcal{H}^{n-1}(y)$$

CONSEQUENCE Suppose $u \in C^2(\Omega)$. Then

$$u \underset{(=)}{\overset{(\geq)}{\leq}} \fint_{\partial B_r(x)} u \, d\mathcal{H}^{n-1} \quad \begin{array}{l} \forall x \in \Omega \text{ and} \\ \forall r > 0 \text{ s.t.} \\ \forall B_r(x) \subset\subset \Omega \end{array} \quad (\Leftrightarrow) \quad -\Delta u \underset{(=)}{\overset{(\geq)}{\leq}} 0 .$$

$(\Leftarrow)$ already seen.

$(\Rightarrow)$ By the previous corollary we have that

$$\int_{B_r(x)} (E^x - E_r) \Delta u \, dy \geq 0 \qquad \forall B_r(x) \subset\subset \Omega .$$

We can argue by contradiction since Δu is continuous: if $\exists x_0$ s.t. $\Delta u(x_0) < 0$ we would have a ball $B_r(x_0)$ in which $\Delta u < 0$ and since $E^x - E_r \geq 0$

and $E^x - E_r > 0$ in $B_r(x_0)$ $\left(= 0 \text{ only in } \partial B_r(x_0) \right)$
we would have the above integral negative.

The equivalence between the reverse inequalities ($\geqslant$)
are shown in an analogous way; the case
for harmonic functions comes from the other two. //