# AN EXAMPLE OF FUNCTION $f \in C^0$ FOR WHICH THE SOLUTION $u$ OF $-\Delta u = f$ IS NOT $C^2$

Consider $n = 2$, $p(x,y) = xy$, $\eta \in C_c^\infty(\mathbb{R}^2)$

with $\eta \equiv 1$ in $B_1(0,0)$ and $\eta \equiv 0$ outside of $B_2(0,0)$

Consider the two sequences:

$$t_k = 2^k, \qquad c_k = \frac{1}{k} \qquad \text{for } k \in \mathbb{N}^*$$

and define

$$f(x,y) := \sum_{k=1}^{+\infty} c_k \, \Delta(\eta p)\,(t_k(x,y))$$

We show that $f \in C^0(\mathbb{R}^2)$, but

$$-\Delta u = f \quad \text{has no a } C^2 \text{ solution.}$$

1°) $\Delta(\eta p) = p\,\Delta\eta + 2\nabla\eta\cdot\nabla p + \eta\,\Delta p =$

$= p\,\Delta\eta + 2\,\nabla\eta\cdot\nabla p \qquad (\Delta p = 0)$

2°) $W_k(x,y) := \Delta(\eta p)(t_k(x,y))$ is supported in $B_{\frac{1}{2^{k-1}}}(0,0)$

3°) $w_k(0,0) = 0$ in $B_{\frac{1}{2^k}}(0,0)$

4°) By 2nd and 3rd points we have that the series defining $f$ is in fact a finite sum in every $(x,y) \neq (0,0)$ since there is $\bar{k} \in \mathbb{N}$ such that $|(x,y)| > \dfrac{1}{2^{\bar{k}-1}}$ and then $f$ turn out to be $\displaystyle\sum_{k=1}^{\bar{k}} c_k w_k$ in $(x,y)$.

Moreover observe that:

$$w_k \equiv 0 \quad \text{in} \quad B_{\frac{1}{2^k}}(0,0)$$

since $\Delta\eta$ and $\nabla\eta$ are zero in $B_1(0,0)$

and then $\Delta\eta(t_k x, t_k y) = 0$ if $|t_k(x,y)| \leq 1$

$$\Updownarrow$$

$$|(x,y)| \leq 1/2^k$$

Then we have uniform converge of

$$\sum c_k w_k$$

and being $w_k$ continuous the limit (i.e. $f$) is a continuous function.

5°) $\Delta(\eta p)\,(t_k(x,y)) = \frac{1}{t_k^2}\,\Delta\left(\eta p\,(t_k x, t_k y)\right)$

Since the sum is in fact finite in each $(x,y)$ we have

$$f(x,y) = \sum_{k=1}^{+\infty} c_k\,\Delta(\eta p)\,(t_k(x,y))$$

Moreover the series

$$\sum_{k=1}^{+\infty} \frac{1}{t_k^2}\,c_k\,\eta p\,(t_k(x,y))$$

is absolutely converging, and the uniformly converging ( since $\left|\frac{c_k}{t_k^2}\,\eta p\right| \leq \max_{B_1} |p|\,\frac{1}{k\,2^{2k}}$

Then

$$\Delta \sum_{k=1}^{+\infty} \frac{1}{t_k^2}\,c_k\,\eta p\,(t_k(x,y)) =$$

$$= \sum_{k=1}^{+\infty} \frac{c_k}{t_k^2}\,\Delta\left(\eta p\,(t_k(x,y))\right) =$$

$$= \sum_{k=1}^{+\infty} c_k \, \Delta(\eta p)\,(t_k x, t_k y)$$

$6^\circ$) Then we have the solution of $\quad -\Delta u = f$,

i.e. $\quad u = -\sum \frac{c_k}{t_k^2}\, \eta(t_k x, t_k y)\, p(t_k x, t_k y)$

$$\frac{\partial u}{\partial x} = -\sum \frac{c_k}{t_k^2} \Big[ \eta_x(t_k x, t_k y)\, t_k\, p(t_k x, t_k y) + \eta(t_k x, t_k y)\, t_k\, p_x(t_k x, t_k y) \Big]$$

$$\frac{\partial^2 u}{\partial y \partial x} = -\sum \frac{c_k}{t_k^2} \Big[ \eta_{xy}(t_k x, t_k y)\, t_k^2\, p(t_k x, t_k y) + \eta_x(t_k(x,y))\, t_k^2\, P_y(t_k(x,y)) + \eta_y(t_k(x,y))\, t_k^2\, P_x(t_k(x,y)) + \eta(t_k(x,y))\, t_k^2\, P_{xy}(t_k(x,y)) \Big]$$

and

$$u_{xy}(0,0) = -\sum c_k\, \eta(0,0)\, P_{xy}(0,0) = -\sum c_k = -\infty$$

since $P_{xy} = 1$