SOLUTION OF THE DIRICHLET PROBLEM

Consider Ω open and bounded of class C^1.
Given $f \in C^0(\Omega)$ and $\varphi \in C^0(\partial\Omega)$ we define the Dirichlet problem of finding $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ s.t.

$$(D) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases}.$$

We will also consider the Poisson equation for the Dirichlet problem: find $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ s.t.

$$(PD) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases}.$$

We will use the Stokes identity (u a generic function in $C^2(\bar{\Omega})$, $x \in \Omega$)

$$u(x) = \int_{\partial\Omega} \left(E^x \frac{\partial u}{\partial \nu} - u \frac{\partial E^x}{\partial \nu} \right) d\mathcal{H}^{n-1} - \int_{\Omega} E^x(y)\, \Delta u(y)\, dy \quad (1)$$

where we recall that E is the fundamental solution of $-\Delta$ and $E^x(y) := E(y-x) = E(x-y)$.

Consider the auxiliary problem

$$(\mathcal{A}D) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = E^x & \text{in } \partial\Omega \end{cases}$$

and suppose for the moment this has solution, denoted by ϕ^x.

The role of the function ϕ^x will be to suitably correct the fundamental solution in such a way to find a "kernel" for the problem (D) (and (PD)).

Now if we suppose problem (PD) has a solution and denote it by u, ϕ^x the solution of $(\mathcal{A}D)$,

use the second Green identity

$$\int_\Omega (v_1 \Delta v_2 - v_2 \Delta v_1)\, dx = \int_{\partial\Omega} \left(v_1 \frac{\partial v_2}{\partial \nu} - v_2 \frac{\partial v_1}{\partial \nu}\right) d\mathcal{H}^{n-1}$$

with $v_1 = u$, $v_2 = \phi^x$, we get

$$\int_\Omega \phi^x(y)\, \Delta u(y)\, dy + \int_{\partial\Omega} \left(u \frac{\partial \phi^x}{\partial \nu} - \phi^x \frac{\partial u}{\partial \nu}\right) d\mathcal{H}^{n-1} = 0$$

If we insert u in the Stokes identity (1) and sum to that we obtain

$$u(x) = -\int_{\Omega} (E^x - \Phi^x) \underbrace{\Delta u}_{\overset{\text{IF}}{=} -f} \, dy - \int_{\partial\Omega} \underbrace{u}_{= \varphi} \left(\frac{\partial E^x}{\partial \nu} - \frac{\partial \Phi^x}{\partial \nu} \right) d\mathcal{H}^{n-1}$$

one could hope to have found a formula to represent the solution of (PD)

Then if $u \in C^2(\overline{\Omega})$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases}$$

the function u has the following representation

$$u(x) = \int_{\Omega} G^x f \, dy - \int_{\partial\Omega} \varphi \frac{\partial G^x}{\partial \nu} \, d\mathcal{H}^{n-1} \qquad (2)$$

If we define $G(x,y) := G^x(y)$ and

$P(x,y) := -\dfrac{\partial G^x}{\partial \nu_y}(y)$ $\quad \nu_y \perp \partial\Omega$ in y

the representation becomes

$$u(x) = \int_{\Omega} G(x,y) f(y) \, dy + \int_{\partial\Omega} P(x,y) \varphi(y) \, d\mathcal{H}^{n-1}(y) \qquad (2)'$$

The function $P(x,y) := -\dfrac{\partial G^x}{\partial \nu}(y)$ $\quad y \in \partial\Omega$, $x \in \Omega$

will be called Poisson kernel.

The function $G^x(y) := E^x(y) - \Phi^x(y)$ is called

Green function for the Laplacian in Ω.

We will now focus our attention on the Dirichlet problem

$$(D) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases} \qquad \varphi \in C^0(\partial\Omega)$$

Proposition $G(x,y) = G(y,x)$ (without proof)

(This also means that $\Phi^x(y) = \Phi^y(x) =: \Phi(x,y)$)

Remark. Since $\Delta G^x = 0$ for $y \neq x$, i.e.

$\Delta_y G(x,y) = 0$ for $y \neq x$, by the previous proposition we also have $\Delta_x G(x,y) = 0$ for $x \neq y$.

Proposition The function $\Omega \ni x \longmapsto \dfrac{\partial G^x}{\partial \nu}(y)$

($y \in \partial\Omega$ fixed, $\nu = \nu(y)$ normal to $\partial\Omega$ in y)

is harmonic.

proof: since $\Delta_x G(x,y) = \Delta_x G^x(y) = 0$ for $x \neq y$

if we consider $x \in \Omega$ e $y \in \partial\Omega$ we have $$\Delta_x \frac{\partial G^x}{\partial \nu(y)}(y) = \Delta_x \langle \nabla_y G(x,\bar{y}), \nu(\bar{y}) \rangle =$$

$$= \langle \nabla_y \Delta_x G(x,\bar{y}), \nu(\bar{y}) \rangle = 0$$

since $$\Delta_x G(x,\bar{y}) = 0 \qquad (x \in \Omega, \ \bar{y} \in \partial\Omega).$$

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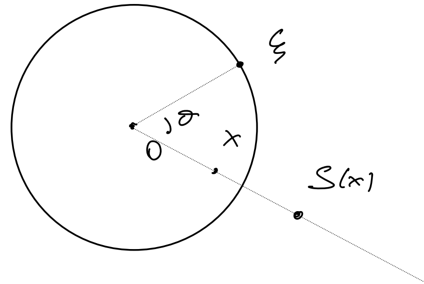
Remark Notice that the role of the "corrector" ϕ^x may be simbolically explained by

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega \\ G^x = 0 & \text{on } \partial\Omega \end{cases}$$

GREEN FUNCTION FOR A BALL

We consider the "spheric inversion":
consider the map defined for $x \neq 0$

$$S(x) := \frac{x}{|x|^2} R^2 \quad (\text{Kelvin transform})$$



Notice that S maps

$$B_R(0) \setminus \{0\} \quad \text{in} \quad \mathbb{R}^n \setminus \overline{B_R(0)} \,,$$

$$\mathbb{R}^n \setminus \overline{B_R(0)} \quad \text{in} \quad B_R(0) \setminus \{0\} \,,$$

$$\partial B_R(0) \quad \text{in} \quad \partial B_R(0) \,.$$

For sake of simplicity we denote by

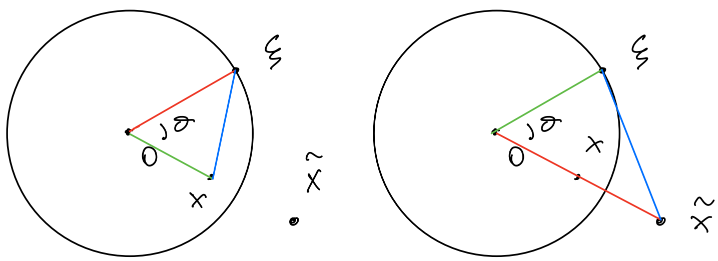
$\tilde{x} \in \mathbb{R}^n \setminus \overline{B_R(0)}$ the point $S(x)$ for $x \in B_R(0) \setminus \{0\}$

Notice that the two triangles $\overset{\triangle}{\xi 0 x}$ and $\overset{\triangle}{\xi 0 \tilde{x}}$ are similar. Indeed they have the angle θ in common

and
$$\frac{|x|}{|\xi|} = \frac{|\xi|}{|\tilde{x}|} \iff \frac{|x|}{R} = R \, \frac{|x|^2}{|x|} \frac{1}{R^2} = \frac{|x|}{R}$$

$\frac{\text{green}}{\text{blue}}$

The proportionality factor is from left to right $\frac{R}{|x|}$



Then $|\xi - \tilde{x}| = \frac{R}{|x|}\,|\xi - x|$ (•)

Define (for each $x \in B_R(0) \setminus \{0\}$)

$$\phi(x,y) = \phi^x(y) := \begin{cases} \dfrac{1}{(n-2)\,\sigma_n}\left(\dfrac{R}{|x|}\right)^{n-2}\dfrac{1}{|\tilde{x} - y|^{n-2}} & n \geq 3 \\[2ex] -\dfrac{1}{2\pi}\log\left(|\tilde{x} - y|\,\dfrac{|x|}{R}\right) & n = 2 \end{cases}$$

$$= E\left(\frac{|x|}{R}(y - \tilde{x})\right)$$

Clearly, since $\tilde{x} \notin B_R(0)$ (i.e. its pole $\tilde{x}$ lies outside B_R),

$$\Delta \phi^x = 0 \quad \text{in } B_R(0) \quad \text{for every } x.$$

Moreover, by (•), for $y \in \partial B_R(0)$ we also have

$$\phi^x(y) = E\left(\frac{|x|}{R}(y - \tilde{x})\right) = E\left(\frac{|x|}{R}\,\frac{R}{|x|}(y - x)\right) = E^x(y)$$

i.e. ϕ^x satisfies (AD). By uniqueness of the solution (already seen by energy method or following by the maximum principle), we conclude that <u>ϕ^x is the (only!) solution of $(\star D)$!</u>

Now inserting ϕ^x in $G^x(y) = E^x(y) - \phi^x(y)$ we find (in fact, $G(x,y) = G(x-y) = G(y-x)$ in this case)

$$G(x,y) = E^x(y) - \phi^x(y) = E(y-x) - \phi^x(y) =$$

$$= \begin{cases} \dfrac{1}{(n-2)\,\sigma_n}\left[\dfrac{1}{|x-y|^{n-2}} - \left(\dfrac{R}{|x|}\right)^{n-2}\dfrac{1}{|\tilde{x}-y|^{n-2}}\right] & \text{if } n \geq 3 \\[2ex] \dfrac{1}{2\pi}\left[\log\left(|\tilde{x}-y|\,\dfrac{|x|}{R}\right) - \log|x-y|\right] & \text{if } n = 2 \end{cases}$$

the Green function for the Dirichlet problem in the ball $B_R(0)$.

The last thing to do to represent u (the solution of (D)) is to compute the normal derivative of G.

First we compute the gradient of G^x
(we already computed ∇E^x in (2), I - lection 6):

$$\nabla G^x(y) = \frac{1}{(n-2)\,\sigma_n}\left[(2-n)\,\frac{y-x}{|y-x|^n} - (2-n)\left(\frac{R}{|x|}\right)^{n-2}\frac{y-\tilde{x}}{|y-\tilde{x}|^n}\right]$$

and for $y \in \partial B_R(0)$

$$= -\frac{1}{\sigma_n}\left[\frac{y-x}{|y-x|^n} - \left(\frac{R}{|x|}\right)^{n-2}\frac{y-\frac{x}{|x|^2}R^2}{\left(\frac{R}{|x|}\,|y-x|\right)^n}\right] =$$

$$= -\frac{1}{\sigma_n}\,\frac{1}{R^2}\,\frac{R^2-|x|^2}{|y-x|^n}\,y$$

Notice that for $n=2$ we have

$$\nabla E^x(y) = -\frac{1}{2\pi}\,\frac{y-x}{|y-x|^2}$$ by which

$$\nabla G^x(y) = \frac{1}{2\pi}\left[\frac{y-\tilde{x}}{|y-\tilde{x}|^2} - \frac{y-x}{|y-x|^2}\right] = \left(\text{for } y \in \partial B_R(0)\right)$$

$$= \frac{1}{2\pi}\,\frac{1}{R^2}\,\frac{|x|^2-R^2}{|y-x|^2}\,y = -\frac{1}{2\pi}\,\frac{1}{R^2}\,\frac{R^2-|x|^2}{|y-x|^2}\,y$$

So we have $\nabla G^x(y) = -\frac{1}{\sigma_n}\,\frac{1}{R^2}\,\frac{R^2-|x|^2}{|y-x|^n}\,y$

for every $n \geq 2$ $\quad (\sigma_2 = 2\pi)$.

Since $\frac{\partial G^x}{\partial \nu}(y) = \left\langle \nabla G^x(y), \frac{y}{|y|}\right\rangle \quad \left(y \in \partial B_R(0)\right)$

we derive

$$\frac{\partial G^x}{\partial \nu}(y) = -\frac{1}{\sigma_n}\,\frac{1}{R}\,\frac{R^2-|x|^2}{|y-x|^n}$$

So we have a candidate to be the solution of (D) if $\Omega = B$

If $u \in C^2(B_R(0)) \cap C^0(\overline{B_R(0)})$ solves

$$(D)\quad \begin{cases} -\Delta u = 0 & \text{in } B_R(0) \\ u = \varphi & \text{in } \partial B_R(0) \end{cases}$$

then

$$\boxed{u(x) = \int_{\partial B_R(0)} P(x,y)\,\varphi(y)\,d\mathcal{H}^{n-1}(y)} \qquad (3)$$

where $P(x,y) := \frac{1}{\sigma_n R}\,\frac{R^2-|x|^2}{|y-x|^n}$ is called Poisson kernel $(n \geq 2)$

EXERCISE Show that

$$\int_{\partial B} \frac{1}{\sigma_n R}\,\frac{R^2-|x|^2}{|y-x|^n}\,d\mathcal{H}^{n-1}(y) = 1$$

Solution: by (3) we have a representation for the solution, provided we know the solution and know that this is $C^2(\overline{B})$.

Now observe that

$u \equiv 1$ in $\overline{B}$ is "a" solution of $\begin{cases} -\Delta u = 0 & B \\ u = 1 & \partial B \end{cases}$

We know that if a solution exists this is unique.
Then $u \equiv 1$ in $\overline{B}$ is the solution.
By (3) we get the thesis.

Theorem (Dirichlet problem for a ball).

The function u defined in (3) belongs to $C^0(\overline{B_R(0)})$, is harmonic in $B_R(0)$ and $u = \varphi$ in $\partial B_R(0)$.

proof: We recall that for a generic open bounded Ω of class C^1 the function

$$\Omega \ni x \longmapsto \frac{\partial G^x}{\partial \nu}(y) = -P(x,y)$$

is harmonic if $y \in \partial\Omega$. Then

$$\Delta u(x) = \Delta_x u(x) = \Delta_x \int_{\partial B_R(0)} \frac{1}{\sigma_n R} \frac{R^2 - |x|^2}{|y-x|^n} \varphi(y) \, d\mathcal{H}^{n-1}(y) =$$

$$= \int_{\partial B_R(0)} \varphi(y) \Delta_x \left(\frac{1}{\sigma_n R} \frac{R^2 - |x|^2}{|y-x|^n} \right) d\mathcal{H}^{n-1}(y) = 0$$

for $x \in B_R(0)$.

We have to show that

$$\lim_{\substack{x \to \bar{x} \\ x \in B_R(0)}} u(x) = \varphi(\bar{x}) \qquad \bar{x} \in \partial B_R(0)$$

Fix $\varepsilon > 0$: by continuity of φ we can find $\delta > 0$ s.t.

$$|\varphi(y) - \varphi(\bar{x})| < \varepsilon \quad \forall y \in \Sigma_\delta(\bar{x}) := \left\{ y \in \partial B_R(0) \,\middle|\, |y - \bar{x}| < \delta \right\}$$

We now use (previous exercise)

$$\varphi(\bar{x}) = \int_{\partial B} \frac{1}{\sigma_n R} \frac{R^2 - |x|^2}{|y - x|^n} \varphi(\bar{x}) \, d\mathcal{H}^{n-1}(y)$$

Then

$$u(x) - \varphi(\bar{x}) = \frac{1}{\sigma_n R} \int_{\partial B} \big(\varphi(y) - \varphi(\bar{x})\big) \frac{R^2 - |x|^2}{|y - x|^n} \, d\mathcal{H}^{n-1}(y)$$

$$= \frac{1}{\sigma_n R} \left[\int_{\partial B \setminus \Sigma_\delta} \cdots + \int_{\Sigma_\delta} \cdots \right]$$

$$\left| \int_{\Sigma_\delta} \big(\varphi(y) - \varphi(\bar{x})\big) \frac{R^2 - |x|^2}{|y - x|^n} \, d\mathcal{H}^{n-1}(y) \right| < \varepsilon$$

while

$$\left| \int_{\partial B_R \setminus \Sigma_\delta} \big(\varphi(y) - \varphi(\bar{x})\big) \frac{R^2 - |x|^2}{|y - x|^n} \, d\mathcal{H}^{n-1}(y) \right| \leq \qquad \left(\text{since } P > 0\right)$$

$$\leq 2 \max_{\partial B_R} |\varphi| \int_{\partial B_R \setminus \Sigma_\delta} \frac{R^2 - |x|^2}{|y-x|^n} \, d\mathcal{H}^{n-1}(y)$$

and
$$\lim_{\substack{x \to \bar{x} \\ x \in B_R(0)}} \int_{\partial B_R(0) \setminus \Sigma_\delta} \frac{R^2 - |x|^2}{|y-x|^n} \, d\mathcal{H}^{n-1}(y) = \left(\begin{array}{c} \text{by Lebesgue} \\ \text{dominated} \\ \text{convergence} \\ \text{theorem} \end{array} \right)$$

$$= \int_{\partial B_R(0) \setminus \Sigma_\delta} \lim_{\substack{x \to \bar{x} \\ x \in B_R(0)}} \frac{R^2 - |x|^2}{|y-x|^n} \, d\mathcal{H}^{n-1}(y) = 0$$

since $|\bar{x}| = R$
and $|y - \bar{x}| \geq \delta$

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What about problem (PD)?
Id est, does formula (2) (or (2)') furnish the solution of problem (PD)? Not exactly.

If $f \in C^0(\Omega)$ does $\Omega \ni x \mapsto \int_\Omega G(x,y) f(y)\, dy$ belong to $C^2(\Omega)$?

Notice that (E the fundamental solution)

$$|DE|(x) \leq \frac{c}{|x|^{n-1}} \quad \text{and} \quad |D^2 E|(x) \leq \frac{c}{|x|^n}$$

Since $\frac{1}{|x|^n}$ is not summable differentiating under the sign of integral is delicate and require more regularity than simple continuity on f.

Indeed it is possible to find $f \in C^0(\mathbb{R}^n)$ such that $-\Delta u = f$ has no C^2 solutions [see Ex 4.9 GILBARG-TRUDINGER]

EXERCISE consider $f \in C^0((a,b)) \cap L^1(a,b)$.
Is it possible to find C^2 solutions of
$$-u'' = f \quad \text{in } (a,b) \quad ?$$

Another reason (beyond $|D^2E|$ not integrable) why $f \in C^0(\Omega)$ is not necessarly the laplacian of a function $u \in C^2(\Omega)$ is that Δu does not contain all information about second derivatives (for instance, knowing Δu does not give us any information about $\frac{\partial^2 u}{\partial x_1 \partial x_2}$) unless we are in $\mathbb{R}^1$, i.e. $n = 1$ (see the previous exercise).

Therefore we could have $\frac{\partial^2 u}{\partial x_i \partial x_i}$ continuous for each $i \in \{1, \dots n\}$ without having the existence or the continuity of some $\frac{\partial^2 u}{\partial x_i \partial x_j}$ $(i \neq j)$.

SOME CONSEQUENCES OF POISSON'S FORMULA

Theorem $u \in C^2(\Omega)$ is harmonic

$$\Updownarrow$$

$u \in C^0(\Omega)$ and $u(x) = \fint_{B_R(x)} u(y)\,dy \quad \forall B_R(x) \subset\subset \Omega$

proof: $(\Downarrow)$ already seen

$(\Uparrow)$ fix u satisfying assumptions and let $\tilde{u}_B$ be the solution of

$(B = B_R(x))$
$$\begin{cases} -\Delta w = 0 & \text{in } B_R(x) \\ w = u & \text{in } \partial B_R(x) \end{cases}$$

Then $\tilde{u}_B$ is harmonic, $C^\infty(B_R(x))$ and satisfies the mean value property in $B_R(x)$.

Since $u = \tilde{u}_B$ in $\partial B_R(x)$ we have that

$$v = u - \tilde{u} = 0 \quad \text{in } \partial B_R(x)$$

On the other side $v \in C^0(\overline{B_R(x)})$ and (by linearity) v satisfies the mean value property;

by the maximum principle we have that

$$\max_{\overline{B_R(x)}} |v| = \max_{\partial B_R(x)} |v| = 0 ,$$

Since this holds for every $x \in \Omega$ and every ball $B_R(x) \subset\subset \Omega$ we conclude that $v \equiv 0$ for every $\tilde{u}_B$, i.e. $\tilde{u}_B \equiv u \ \forall B$. //

Theorem (Harnack inequality 1)

Consider u harmonic in Ω, $u \geq 0$ in Ω. Fix a ball $B_R(x_0) \subset\subset \Omega$. Then for every $\rho \in (0,R)$ and every $x \in B_\rho(x_0)$

$$\left(\frac{R}{R+\rho}\right)^{n-2} \frac{R-\rho}{R+\rho}\, u(x_0) \leq u(x) \leq \left(\frac{R}{R-\rho}\right)^{n-2} \frac{R+\rho}{R-\rho}\, u(x_0) \qquad (4)$$

proof: for simplicity suppose $x_0 = 0$. B denotes $B_R(0)$

$$u(x) = \int_{\partial B} \frac{1}{\sigma_n R} \frac{R^2 - |x|^2}{|y-x|^n}\, u(y)\, d\mathcal{H}^{n-1}(y) \le \qquad \left(\text{Poisson kernel}\right)$$

$$\le \frac{R^2-|x|^2}{\sigma_n R}\int_{\partial B}\frac{u(y)}{(|y|-|x|)^n}\, d\mathcal{H}^{n-1}(y) =$$

$$= \frac{R^2-|x|^2}{\sigma_n R}\,\frac{1}{(R-|x|)^n}\,\sigma_n R^{n-1} \fint_{\partial B} u(y)\, d\mathcal{H}^{n-1}(y) =$$

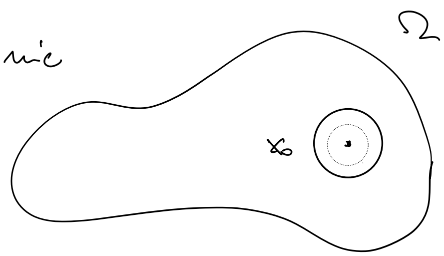
$$= \frac{R+|x|}{R-|x|}\left(\frac{R}{R-|x|}\right)^{n-2} u(0) \qquad \left(\text{Mean Value property}\right)$$

Estimating $\dfrac{1}{|y-x|^n} \ge \dfrac{1}{(|y|+|x|)^n}$ one gets the other inequality. //

Corollary In the same assumptions one has that

$$\sup_{B_\rho(x_0)} u \le \left(\frac{R+\rho}{R-\rho}\right)^n \inf_{B_\rho(x_0)} u$$

Remark Notice that u is harmonic in Ω and, a priori, is not bounded.



The constant explodes to $+\infty$ as ρ goes to R.

Theorem (Harnack inequality 2)

Consider u harmonic in Ω, $u \geq 0$ in Ω.
Then for every ω connected, $\omega \subset\subset \Omega$, there is a positive constant $c = c(n, \omega)$ such that

$$\max_{\omega} u \leq c \min_{\omega} u$$

Remark – If $u(x_0) = 0$ for some $x_0 \in \Omega \Rightarrow u \equiv 0$.

proof: consider $x, y \in \overline{\omega}$ such that (notice that $x, y \in \partial\omega$)

$u(x) = \sup_{\omega} u$, $u(y) = \inf_{\omega} u$. Since Ω is open and ω connected there is a path $\Gamma \subset \omega$ joining x and y. We can cover Γ with a finite number of balls $B_\rho(x_0), \dots B_\rho(x_N)$ in such a way that

$B_\rho(x_i) \subset B_R(x_i) \subset\subset \Omega$ for some $R > \rho$

$B_\rho(x_i) \cap B_\rho(x_{i+1}) \neq \emptyset$ for each i between 0 and $N-1$

$x_0 = x$, $x_N = y$

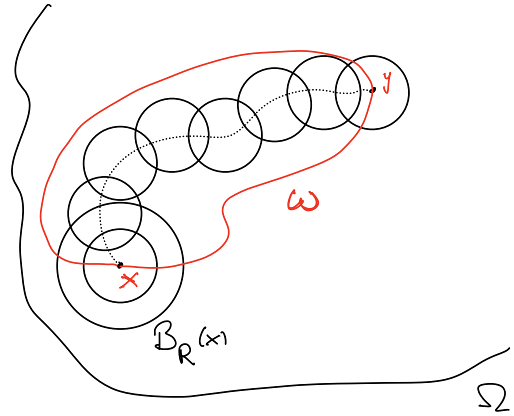
If we set $\gamma := \left(\dfrac{R+\rho}{R-\rho}\right)^n$ we derive

$$\sup_{\omega} u = \sup_{B_\rho(x_0)} u \leq$$

$$\leq \gamma \inf_{B_\rho(x_0)} u \leq$$

$$\leq \gamma \sup_{B_\rho(x_1)} u \leq \gamma^2 \inf_{B_\rho(x_1)} u \leq$$

$$\leq \gamma^2 \sup_{B_\rho(x_2)} u \leq \dots \gamma^{N+1} \inf_{B_\rho(x_N)} u = \gamma^{N+1} \inf_{\omega} u$$



Remark For every $\omega' \subset \omega$ we have

$$\max_{\omega'} u \leq c \min_{\omega'} u \quad \text{with the same constant } c.$$

We now see two results regarding sequences of harmonic functions.

Theorem Consider $\Omega \subseteq \mathbb{R}^n$ and $\{u_k\}_{k\in\mathbb{N}}$ a sequence of harmonic functions uniformly convergent on compact sets of Ω to a function u.
Then u is harmonic.

proof

$$u(x) = \lim_{k\to+\infty} u_k(x) = \lim_{k\to+\infty} \fint_{B_r(x)} u_k(y)\,dy =$$

$$\overset{u.c.}{=} \fint_{B_r(x)} \lim_{k\to+\infty} u_k(y)\,dy = \fint_{B_r(x)} u(y)\,dy. \qquad //$$

Theorem (Harnack principle) Consider $\Omega \overset{\circ}{\subseteq} \mathbb{R}^n$ connected and $\{u_k\}_{k\in\mathbb{N}}$ an increasing sequence of harmonic functions. If there is $x_0 \in \Omega$ such that $\{u_k(x_0)\}_{k\in\mathbb{N}}$ converges then $\{u_k\}_k$ uniformly converges to a harmonic function on compact sets.

proof: first observe that, by monotonicity,

$$u_n - u_m \geq 0 \quad \text{in } \Omega \quad \text{for every } n, m \in \mathbb{N} \quad (n \geq m)$$

and moreover $u_n - u_m$ is harmonic in Ω.

Now let ω be a compact set in Ω, ω connected containing x_0. By convergence of $\{u_k(x_0)\}$ and by monotonicity, for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that for every $n, m \geq N$, $n > m$,

$$0 \leq u_n(x_0) - u_m(x_0) < \varepsilon$$

and by Harnack's inequality we have the existence of $c = c(\text{dimension}, \omega)$ such that

$$\sup_{x\in\omega} \left(u_n(x) - u_m(x)\right) < c \inf_{x\in\omega} \left(u_n(x) - u_m(x)\right) < c\,\varepsilon$$

Observe how c is independent of n, m, N.
Then $\{u_k\}_k$ uniformly converges in ω and, by the previous result, the limit is a harmonic function. //

? Why in the proof do we choose ω containing x_0?

And why in the statement this is not necessary?

Theorem (Liouville) $u : \mathbb{R}^n \to \mathbb{R}$ harmonic, $u \geq 0$.
Then u is constant.

proof: write the first Harnack inequality in some ball $B_\rho(x_0) \subset B_R(x_0)$ and send $R \to +\infty$ to get that u is constant in every ball B_ρ. //

Corollary If $u: \mathbb{R}^n \to \mathbb{R}$ is harmonic, $u \leq 0$, then u is constant.

Corollary. If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic, $u \geq k$ or $u \leq k$ for some $k \in \mathbb{R}$. Then u is constant.

EX Prove that if $u : \Omega \to \mathbb{R}$ is bounded and u satisfies the mean value property then $u \in C^\infty(\Omega)$.

[hint : prove first u is continuous]