EX Prove that if $u : \Omega \to \mathbb{R}$ is bounded and u satisfies the mean value property then $u \in C^\infty(\Omega)$.

[hint : prove first u is continuous]

EXERCISE Consider $x_0 \in \Omega$, $\Omega \mathring{\subseteq} \mathbb{R}^n$, and $u : \Omega \setminus \{x_0\} \longrightarrow \mathbb{R}$ harmonic and bounded. Prove that u can be extended to a harmonic function $\tilde{u} : \Omega \to \mathbb{R}$ (i.e. $\tilde{u} \equiv u$ in $\Omega \setminus \{x_0\}$)

proof: suppose for the sake of simplicity that $x_0 = 0$ and $B_2(0) \subset \Omega$. Now solve the problem

$$\begin{cases} -\Delta w = 0 & \text{in } B_1(0) \\ w = u & \text{in } \partial B_1(0) \end{cases}$$ and call v its solution

Consider the Green function

$$(G<0) \qquad G(x) = \begin{cases} \frac{1}{2\pi} \log|x| & \text{if } n=2 \\ \frac{1}{(2-n)\sigma_n} \left(\frac{1}{|x|^{n-2}} - 1 \right) & \text{if } n \geq 3 \end{cases}$$

and define for $\varepsilon > 0$

$$u_\varepsilon(x) := v(x) - \varepsilon G(x) \qquad \text{in } \overline{B_1(0)}$$

First of all notice that $u_\varepsilon(x) = v(x) = u(x)$ in $\partial B_1(0)$.

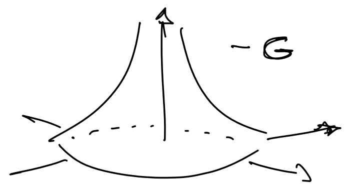
Since u (in particular) is C^1 then

$$\max_{\partial B_1(0)} |\nabla u| \leq \text{const}$$

Since $-\varepsilon G > 0$, $-\frac{\partial G}{\partial \nu}(x) < 0$ for $|x| = 1$

and $\lim_{|x| \to 0} [-G(x)] = +\infty$

$$\lim_{|x|\to 0} u_\varepsilon(x)$$



for ε sufficiently large we have that

$$u_\varepsilon(x) > u(x) \quad \text{in } B_1(0) \setminus \{0\}$$

Now we have two possibilities:

1) $u_\varepsilon > u$ in $B_1(0)$ $\forall \varepsilon > 0$

2) there is a smallest $\varepsilon_0 > 0$ for which

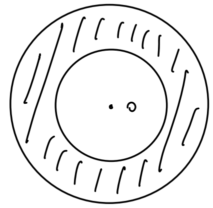
$$u_{\varepsilon_0}(x) \geq u(x) \quad \forall x \in \overline{B_1(0)} \setminus \{0\}$$

Now we exclude the second possibility: assume such an $\varepsilon_0 > 0$ exists. Since $\lim_{|x|\to 0} u_\varepsilon(x) = +\infty$ this would imply the existence of $r = r(\varepsilon_0)$ such that

$$u_{\varepsilon_0/2}(x) > u(x) \quad \text{in } B_r(0) \setminus \{0\} \left(\lim_{|x|\to 0} |u_\varepsilon| = +\infty \right)$$

and $$\min_{B_1(0) \setminus B_r(0)} \left(u_{\frac{\varepsilon_0}{2}} - u\right) < 0$$

Then $\displaystyle\min_{\partial B_r(0)\cup\partial B_1(0)} \left(u_{\varepsilon_0/2} - u \right) = 0$



But this is impossible since

$u_{\varepsilon_0/2} - u$ is harmonic in $B_1 \setminus \overline{B_r}$

but it reaches its minimum in the inner part.

Then the smallest ε_0 for which $u_{\varepsilon_0} \geq u$ in $B_1(0)\setminus\{0\}$

is $\varepsilon_0 = 0$, that is

$$v \geq u \quad \text{in } B_1(0)\setminus\{0\}$$

Taking $\tilde{u}_\varepsilon := v + \varepsilon G$ and arguing in the same way

one gets
$$v \leq u \quad \text{in } B_1(0)\setminus\{0\}.$$

Since v is harmonic we define $u(0) = v(0)$

and we are done. [Jost, p. 24]

Consequence It is <u>not</u> possible to find a solution

to
$$\begin{cases} -\Delta u = 0 & \text{in } B_1(0)\setminus\{0\} \\ u = 0 & \text{in } \partial B_1(0) \\ u(0) = 1 \end{cases}$$

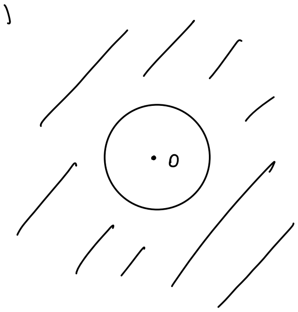
since $u \equiv 0$ is harmonic in $B_1(0)\setminus\{0\}$

and its only harmonic extension is $u(0) = 0$.

DIRICHLET PROBLEM IN EXTERIOR DOMAINS

Consider the following simple example:

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^2 \setminus B_1(0) \\ u = 0 & \text{in } \partial B_1(0) \end{cases}$$



The family of functions

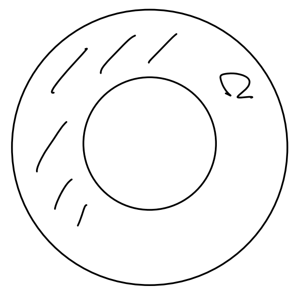
$$u_\alpha(x) = \alpha \log|x| \qquad \alpha \in \mathbb{R}$$

are all solutions. Analogous examples may be done in higher dimension $\left(u_\alpha(x) = \alpha\left(1 - |x|^{2-n}\right) \right)$.

Therefore it's clear that assigning a condition at the boundary of $B_1(0)$ is not sufficient.

On the other side for the Laplace equation in

$$\Omega = B_2(0) \setminus B_1(0)$$



we have to assign a datum in $\partial\Omega = \partial B_1(0) \cup \partial B_2(0)$.

Therefore a condition at $+\infty$ is to be expected.

We will call exterior domain a set like $\mathbb{R}^n \setminus \overline{\Omega}$ where Ω is an open and bounded subset of $\mathbb{R}^n$.

For simplicity we will assume that $0 \in \Omega$ and Ω is connected.

From now on we will assume that Ω is as above.

Theorem Let $u \in C^2(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \overline{\Omega})$ be harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$ and vanishing at $+\infty$.
If $u \geq 0$ (resp. ≤ 0) in $\partial\Omega$
$\Rightarrow$ $u \geq 0$ (resp. ≤ 0) in $\mathbb{R}^n \setminus \overline{\Omega}$.

proof: Since $\lim\limits_{|x| \to +\infty} u(x) = 0$ we can find,

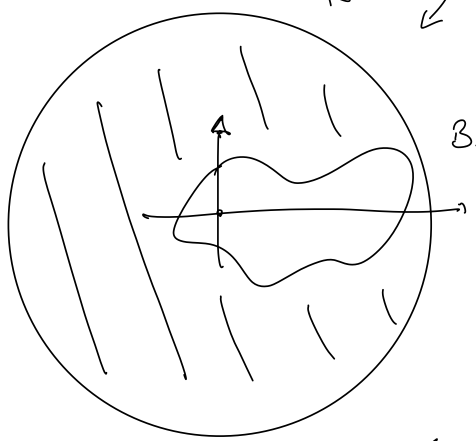
once fixed $\varepsilon > 0$, $R > 0$ such that

$$\Omega \subset B_R(0) \quad \text{and}$$
$$u \geq -\varepsilon \quad \text{in} \quad \mathbb{R}^n \setminus B_R(0)$$

and in particular $u \geq -\varepsilon$ in $\partial B_R(0)$.

From the maximum principle we derive that

$$u \geq -\varepsilon \quad \text{in} \quad (\mathbb{R}^n \setminus \overline{\Omega}) \cap B_R(0)$$



B_R

Since ε is arbitrary
(and, taking ε smaller, R can only increase)
we deduce that
$u \geq 0$ in $(\mathbb{R}^n \setminus \overline{\Omega}) \cap B_R(0)$

and then $u \geq 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$.

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Let now see the result of existence.

Theorem Consider Ω open and bounded in $\mathbb{R}^n$, $n \geq 2$, $\partial\Omega$ lip. continuous, $\varphi \in C^0(\partial\Omega)$. Then the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = \varphi & \text{in } \partial\Omega \\ \lim_{|x| \to +\infty} u(x) = 0 \end{cases}$$

has a (unique) solution in $C^2(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \overline{\Omega})$. Moreover there is a constant α such that

$$\boxed{\left(\text{only for } n \geq 3\right)} \qquad u(x) = \frac{\alpha}{|x|^{n-2}} + O\left(\frac{1}{|x|^{n-1}}\right) \quad \text{at } +\infty.$$

$$\nabla u(x) = (2-n)\,\alpha \frac{x}{|x|^n} + O\left(\frac{1}{|x|^n}\right)$$

JUST FOR CURIOSITY

WIENER CRITERION

Capacity of a set - Given a bounded open set Ω with regular boundary $\partial\Omega$, consider the solution of the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \\ u = 1 & \text{in } \partial\Omega \\ \lim\limits_{|x|\to+\infty} u(x) = 0 \end{cases}$$

the capacity of Ω is defined as (ν outer normal to Ω)

$$\operatorname{cap} \Omega = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^n \setminus \Omega} |Du|^2 \, dx$$

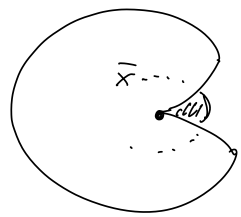
and it is the total electric charge (up to a constant factor) on the conductor $\partial\Omega$ (or Ω) held at constant potential equal to 1.

One can define capacity also for a compact set K, even with non-smooth boundary, as

$$\operatorname{cap} K := \inf_{\substack{u \in C^1_c(\mathbb{R}^n) \\ u \geq 1 \text{ on } K \\ (u = 1 \text{ on } K)}} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx$$

Consider $\lambda \in (0,1)$ and $\bar{x} \in \partial\Omega$

Define $c_j := \operatorname{cap}\left(\Omega^c \cap B_{\lambda^j}(\bar{x}) \right)$



Wiener criterion

$\bar{x} \in \partial\Omega$ is a regular point
(i.e. there exists a barrier)
if and only if

$$\sum_{j=0}^{+\infty} c_j = +\infty$$