THE DIRICHLET PROBLEM

(IN A GENERIC OPEN AND BOUNDED SET Ω :
THE PERRON'S METHOD)

We consider now $\Omega \subseteq \mathbb{R}^n$, an open and bounded set.

Start with some definitions.

$$\sigma(\Omega) := \{ u \in C^0(\Omega) \mid u \text{ is subharmonic in } \Omega \}$$

$$\Sigma(\Omega) := \{ u \in C^0(\Omega) \mid u \text{ is superharmonic in } \Omega \}$$

EX Prove that, given $v_1, \dots v_n \in \sigma(\Omega)$ and $w_1, \dots w_n \in \Sigma(\Omega)$, $c_1, \dots c_n > 0$

- $\max\{v_1, \dots v_n\} \in \sigma(\Omega)$, $\sum_{i=1}^{n} c_i v_i \in \sigma(\Omega)$
- $\min\{w_1, \dots w_n\} \in \Sigma(\Omega)$, $\sum_{i=1}^{n} c_i w_i \in \Sigma(\Omega)$
- $v \in \sigma(\Omega) \iff -v \in \Sigma(\Omega)$

Now for Ω bounded and $\varphi \in C^0(\partial\Omega)$ we define the two classes of functions

$$\sigma(\Omega;\varphi) := \{ v \in \sigma(\Omega) \cap C^0(\overline{\Omega}) \mid v|_{\partial\Omega} \leq \varphi \}$$

$$\Sigma(\Omega;\varphi) := \{ v \in \Sigma(\Omega) \cap C^0(\overline{\Omega}) \mid v|_{\partial\Omega} \geq \varphi \}$$

Observe that, since Ω is bounded, these classes are not empty: for instance, consider $v(x) = k \ \forall x \in \overline{\Omega}$ where k is a constant satisfying $k \leq \min_{\partial\Omega} \varphi$.

Now we see a particular and important subclass of subharmonic functions.

Consider $v \in \sigma(\Omega)$ and a ball $B_\rho(x_0) \subset\subset \Omega$.

Solve the following Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } B_\rho(x_0) \\ u = v & \text{in } \partial B_\rho(x_0) \end{cases}$$

and call $h_{v,x_0,\rho}$ its solution (sometimes this function is called harmonic lifting of v in $B_\rho(x_0)$).

Define

$$v_{x_0,\rho} := \begin{cases} v & \text{in } \Omega \setminus B_\rho(x_0) \\ h_{v,x_0,\rho} & \text{in } B_\rho(x_0) \end{cases} \qquad (1)$$

Then $\boxed{v \leq v_{x_0,\rho} \text{ in } \Omega}$ (2)

Indeed by the previous Ex $v - h_{v,x_0,\rho} \in \sigma(B_\rho(x_0))$ and $v - h_{v,x_0,\rho} = 0$ in $\partial B_\rho(x_0)$. Then, by the maximum principle, we have that

either $v - h_{v,x_0,\rho}$ is constant (i.e. 0)

or $v(x) - h_{v,x_0,\rho}(x) < 0 \quad \forall x \in B_\rho(x_0)$

In both cases we obtain (2).

Finally we prove:

Proposition Take $v \in \sigma(\Omega)$, then $v_{x_0,\rho} \in \sigma(\Omega)$.

proof: clearly $v_{x_0,\rho} \in C^0(\Omega)$, we only need to verify

$$v_{x_0,\rho}(x) \leq \fint_{\partial B_R(x)} v_{x_0,\rho}(y)\, d\mathcal{H}^{n-1}(y)$$

for every $x \in \Omega$, for every $B_R(x) \subset\subset \Omega$. If $x \in \Omega \setminus B_\rho(x_0)$

$$v_{x_0,\rho}(x) \underset{(1)}{=} v(x) \underset{v \text{ subharmonic}}{\leq} \fint_{\partial B_R(x)} v(y)\, d\mathcal{H}^{n-1}(y) \underset{(2)}{\leq} \fint_{\partial B_R(x)} v_{x_0,\rho}(y)\, d\mathcal{H}^{n-1}(y)$$

Now suppose $x \in B_\rho(x_0)$ and, by contradiction, there is a ball $B_R(x)$ such that

$$v_{x_0,\rho}(x) > \fint_{\partial B_R(x)} v_{x_0,\rho}\, d\mathcal{H}^{n-1} \qquad (\bullet)$$

Define the function

$$(v_{x_0,\rho})_{x,R} = \begin{cases} v_{x_0,\rho} & \text{in } \Omega \setminus B_R(x) \\ h_{v_{x_0,\rho},x,R} & \text{in } B_R(x) \end{cases}$$

By (2) we have

$$(v_{x_0,\rho})_{x,R} \geq v_{x_0,\rho} \quad \text{in } \Omega. \qquad (\bullet\bullet)$$

Moreover, since $(v_{x_0,\rho})_{x,R}$ is harmonic in $B_R(x)$, we have

$$(v_{x_0,\rho})_{x,R}(x) = \fint_{\partial B_R(x)} (v_{x_0,\rho})_{x,R} \, d\mathcal{H}^{n-1} = \fint_{\partial B_R(x)} v_{x_0,\rho} \, d\mathcal{H}^{n-1}$$

Then by (in the order) $(\bullet)$ and the last equality

$$v_{x_0,\rho}(x) > (v_{x_0,\rho})_{x,R}(x)$$

which contradicts $(\bullet\bullet)$. //

Def (exterior sphere condition)

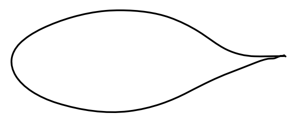
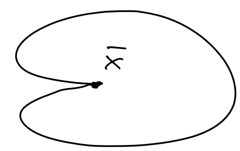
An open set $\Omega \subset \mathbb{R}^n$ satisfies the exterior sphere condition if for every $\bar{x} \in \partial\Omega$ there exists $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and a ball $B_\rho(x_0)$ such that

$$\partial B_\rho(x_0) \cap \partial\Omega = \{\bar{x}\}.$$

EX - if $\partial\Omega$ is of class C^2 then Ω satisfies the exterior sphere condition.

(prove it in $n=2$)

Ex Show that if $\partial\Omega$ is of class C^1 (but not C^2) Ω could not satisfy the ex. sphere cond. $(n=2)$

- $\Omega = [0,1]^n$ satisfies the ex. sphere cond.
-  also a point in $\partial\Omega$ in the vertex of a cusp satisfies the ex. sphere cond.
- if the cusp is directed inside then Ω <u>does not</u> verify the external sphere condition 

Theorem (Perron) Consider $\Omega \overset{\circ}{\subseteq} \mathbb{R}^n$ bounded satisfying the exterior sphere condition.

Then for every $\varphi \in C^0(\partial\Omega)$ the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases} \qquad (D)$$

has a unique solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$.

Idea: show that $\sup \sigma(\Omega;\varphi)$ is harmonic

" " $\sup \sigma(\Omega;\varphi) = \inf \Sigma(\Omega;\varphi)$

" " this function satisfies the boundary condition

Lemma 1 The function $u_\varphi := \sup \sigma(\Omega;\varphi)$ is harmonic.

proof: fix $x_0 \in \Omega$: we will show that u is harmonic in a neighboorhood of x_0

We can find a sequence $\{v_n\}_n \subseteq \sigma(\Omega;\varphi)$ such that

$$v_n(x_0) \longrightarrow u_\varphi(x_0)$$

Then we construct an increasing sequence with the same property:

$$V_n := \max\{v_1, .. v_n\} \qquad V_n \in \sigma(\Omega;\varphi)$$

Now consider a ball $B_\rho(\xi) \subset\subset \Omega$ with $x_0 \in B_\rho(\xi)$ and consider $V_{n;\xi,\rho}$ (as defined in (1)).

By the comparison principle (see I - lection 6) one has

$$V_{n;\xi,\rho} \leq V_{n+1;\xi,\rho} \qquad \text{for every } n$$

Since $V_{n;\xi,\rho} \in \sigma(\Omega;\varphi)$ we have

$$v_n(x) \leq V_n(x) \leq V_{n;\xi,\rho}(x) \leq u_\varphi(x)$$

and then in particular

$\{V_{n;\xi,\rho}\}_n$ is an increasing sequence satisfying

$$V_{n;\xi,\rho}(x_0) \xrightarrow[n\to+\infty]{} u_\varphi(x_0)$$

By monotonicity there is a function w such that

$$\lim_{n\to+\infty} V_{n;\xi,\rho} = w \qquad \text{in } \Omega.$$

By Harnack principle (unif. conv. of an increasing sequence of harmonic functions [see L - lecture I]) we have that

$$w(x_0) = u_\varphi(x_0) \qquad \text{and} \qquad w \text{ is harmonic}$$

in the compact sets contained in $B_\rho(\xi)$.

Now, to prove that $u_\varphi = w$ in $B_\rho(\xi)$, we repeat the previous argument.

Fix $\tilde{x}_0 \in B_\rho(\xi)$: as before we construct $\tilde{v}_n$, $\tilde{V}_n$, $\tilde{V}_{n;\xi,\rho}$ such that, as before, satisfy

$$\tilde{V}_{n;\xi,\rho} \leq \tilde{V}_{n+1;\xi,\rho}$$

$$\tilde{V}_{n;\xi,\rho}(\tilde{x}_0) \longrightarrow u_\varphi(\tilde{x}_0)$$

$$\tilde{V}_{n;\xi,\rho} \longrightarrow \tilde{w} \qquad \tilde{w} \text{ harmonic and}$$

$$\tilde{w}(\tilde{x}_0) = u_\varphi(\tilde{x}_0)$$

With the difference that

$$\tilde{V}_n = \max\{\tilde{v}_1, \dots \tilde{v}_n, V_n\}$$

In this way we obtain that $\boxed{\tilde{w} \geq w.}$

Then

$$u_\varphi(x_0) = w(x_0) \leq \tilde{w}(x_0)$$

and by definition of u_φ $\quad \tilde{w}(x_0) = u_\varphi(x_0)$

$$\Rightarrow \boxed{w(x_0) = \tilde{w}(x_0).}$$

So we have two functions, harmonic in $B_\rho(\xi)$, $\tilde{w}$ and w, with $\tilde{w} - w \geq 0$ and $(\tilde{w} - w)(x_0) = 0$. Since x_0 is interior to $B_\rho(\xi)$ the only possibility (by the max. princ.) is that $\tilde{w} \equiv w$ in $B_\rho(\xi)$.

In particular

$$\tilde{w}(\tilde{x}_0) = w(\tilde{x}_0) = u_\varphi(\tilde{x}_0)$$

So we have proved that u coincides with w in x_0 and in $\tilde{x}_0$. Since $\tilde{x}_0$ is arbitrary we can repeat the argument for every point $\tilde{x}_0 \in B_\rho(\xi)$ and then we conclude that

$$u_\varphi \equiv w \text{ in } B_\rho(\xi).$$

Again, since x_0 and $B_\rho(\xi)$ are arbitrary we conclude that u_φ is harmonic. //

In an analogous way one can prove

Lemma 1bis $\quad U_\varphi = \inf \Sigma(\Omega;\varphi)$ is harmonic

Other, simple but important, result is the following

Lemma $\frac{3}{2}$ $\quad u_\varphi \leq U_\varphi$

proof: consider $v \in \sigma(\Omega;\varphi)$, $w \in \Sigma(\Omega;\varphi)$.

By what observed at the beginning

$$-v \in \Sigma(\Omega;\varphi).$$

Then $w - v \in \Sigma(\Omega)$ and $w - v \geq 0$ in $\partial\Omega$.

By the maximum principle we have that

$$w - v \geq \min_{\partial\Omega}(w - v) \geq 0.$$

Then we have

$$v \leq w \quad \text{in } \overline{\Omega}.$$

Since this holds for every $v \in \sigma(\Omega;\varphi)$ and $w \in \Sigma(\Omega;\varphi)$ we conclude that

$$\sup_{v \in \sigma(\Omega;\varphi)} v \leq \inf_{w \in \Sigma(\Omega;\varphi)} w.$$

//

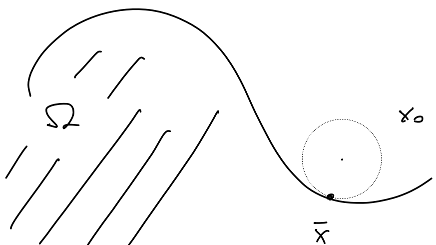
Lemma 2 $u_\varphi \in C(\bar\Omega)$, $u_\varphi|_{\partial\Omega} = \varphi$ and $u_\varphi = U_\varphi$.

proof: fix $\bar x \in \partial\Omega$. Since Ω satisfies the exterior ball condition we can find $B_R(x_0)$ $(x_0 \notin \bar\Omega)$ satisfying the condition. Consider the function

$$h(x) = \begin{cases} \frac{1}{R^{n-2}} - \frac{1}{|x-x_0|^{n-2}} & \text{if } n \geq 3 \\ \log \frac{|x-x_0|}{R} & \text{if } n = 2 \end{cases} \qquad (3)$$

This is harmonic in a neighbourhood of Ω, positive in $\partial\Omega$ except at $\bar x$, where it vanishes.

The exterior ball condition guarantees that h is positive in $\bar\Omega \setminus \{\bar x\}$ since h is radial.



For every $\varepsilon > 0$ $\exists\, \delta > 0$ such that $|\varphi(x) - \varphi(\bar x)| < \varepsilon$ if $|x - \bar x| < \delta$, $x \in \partial\Omega$.

Now we want to show that there is a constant $c_\varepsilon > 0$ such that

$$|\varphi(x) - \varphi(\bar x)| < \varepsilon + c_\varepsilon h(x) \quad \forall\, x \in \partial\Omega \qquad (*)$$

Consider $x \in \partial\Omega$ with $|x - \bar x| \geq \delta$ (if $|x-\bar x| < \delta$ it is obvious).

We have

$$|\varphi(x) - \varphi(\bar{x})| \leq 2\|\varphi\|_{L^\infty(\partial\Omega)} \leq 2\|\varphi\|_{L^\infty(\partial\Omega)} \frac{h(x)}{h_\delta}$$

where $h_\delta := \min_{\substack{x \in \partial\Omega \\ |x-\bar{x}| \geq \delta}} h(x)$ (h_δ depends on δ which depends on ε)

Then we have proved (✱). By that we have

$$\underbrace{\varphi(\bar{x}) - \varepsilon - c_\varepsilon h(x)}_{\in \sigma(\Omega;\varphi)} \leq \varphi(x) \leq \underbrace{\varphi(\bar{x}) + \varepsilon + c_\varepsilon h(x)}_{\in \Sigma(\Omega;\varphi)}$$

h is superharmonic

Then for every $x \in \overline{\Omega}$

$$\varphi(\bar{x}) - \varepsilon - c_\varepsilon h(x) \leq u_\varphi(x) \leq U_\varphi(x) \leq \varphi(\bar{x}) + \varepsilon + c_\varepsilon h(x)$$

In particular

$$|u_\varphi(x) - \varphi(\bar{x})| \leq \varepsilon + c_\varepsilon h(x)$$
$$|U_\varphi(x) - \varphi(\bar{x})| \leq \varepsilon + c_\varepsilon h(x) \qquad \forall x \in \overline{\Omega}$$

Taking the limit we get

$$\limsup_{\substack{x \to \bar{x} \\ x \in \Omega}} |\varphi(\bar{x}) - u_\varphi(x)| \leq \varepsilon$$

$\boxed{\text{since } h \text{ continuous and } \underline{\underline{h(\bar{x}) = 0}}}$

and then $\lim_{x \to \bar{x},\, x \in \Omega} u_\varphi(x) = \varphi(\bar{x})$

and in the same way $\lim_{x \to \bar{x},\, x \in \partial\Omega} U_\varphi(x) = \varphi(\bar{x})$.

We conclude that $U_\varphi - u_\varphi$ is harmonic in Ω and $U_\varphi - u_\varphi = 0$ in $\partial\Omega$.

By the maximum principle $u_\varphi \equiv U_\varphi$. //

Proof of Perron's theorem:

$u_\varphi \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is harmonic and $u_\varphi = \varphi$ in $\partial\Omega$. By uniqueness of the solution of the Dirichlet problem u_φ is the solution. //

REMARK What is really needed in the proof is not properly the exterior sphere condition, but what follows (what stressed in red in the proof):

that for every $\bar{x} \in \partial\Omega$, there exists a function $h_{\bar{x}}$ (like that defined in (3)) which is

- superharmonic in Ω
- positive in $\bar{\Omega} \setminus \{\bar{x}\}$
- continuous in $\bar{\Omega}$ and $h_{\bar{x}}(\bar{x}) = 0$

Such a function is called barrier in Ω at $\bar{x}$.

Then one can substitute the assumption of exterior sphere condition with the $\boxed{\text{barrier postulate}}$ and repeat the proof.

Therefore now we give the following definition.

Def A point $\bar{x} \in \partial\Omega$ is called regular if there exists a barrier in Ω at that point.

But we have something more!

Theorem The Dirichlet problem (D) with Ω open and bounded is solvable for $\varphi \in C^0(\partial\Omega)$ if and only if all the boundary points are regular.

proof: adapting the previous lemma 2 we have already seen the "if" part.
Assume now (D) is solvable: consider a point $\bar{x} \in \partial\Omega$ and the function

$$\psi(x) = |x - \bar{x}|$$

and consider u_ψ, i.e. the solution (which exists by assumption), of the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = \psi & \text{in } \partial\Omega \end{cases}.$$

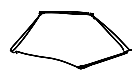
The function u_+ is clearly a barrier. //

The question now is: are we able to say what domains have all the boundary points that are regular? We cannot answer in general, but we can consider some wide classes.

Let us see some classes of open and bounded sets whose boundary is made by regular points only:
if

- Ω satisfies the exterior sphere condition
  (the barrier is that given in (3))

- $\partial\Omega$ is of class C^2 then Ω satisfies the exterior sphere condition (proof EX)

- Ω is convex 
  (it satisfies the exterior sphere condition)

Def We say that K is a cone in $\mathbb{R}^n$ if

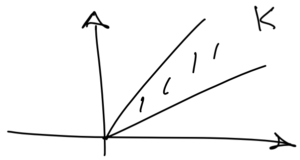
i) $x, y \in K \Rightarrow x+y \in K$

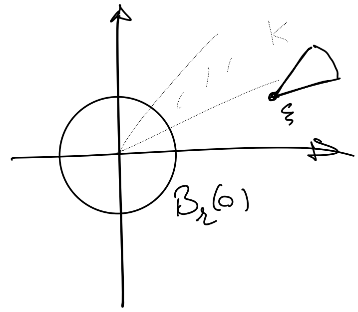
ii) $\lambda x \in K \quad \forall\, x \in K,\ \lambda \geq 0$

We define K_ξ a truncated cone with vertex ξ the set

$$K_\xi := \xi + \left(K \cap B_r(0)\right)$$

for some K cone and some ball $B_r(0)$

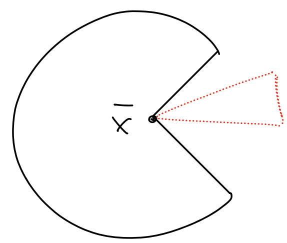




- Ω satisfies the exterior cone condition, i.e.
  for every $\bar{x} \in \partial\Omega$ there exist a truncated cone
  $K_{\bar{x}}$ satisfying $\overline{K_{\bar{x}}} \cap \overline{\Omega} = \{\bar{x}\}$

  Indeed (it is not simple,

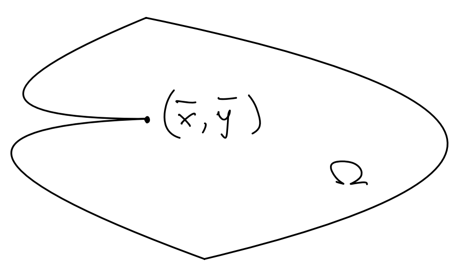
  see L. Helmes
  Introduction to potential theory
  Zaremba's Theorem)



  a barrier is given by u_ψ with $\psi(x) = |x - \bar{x}|$

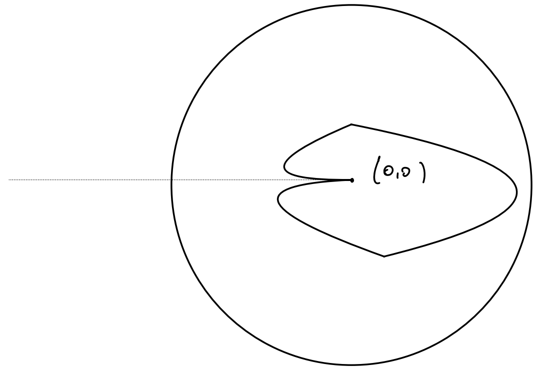
- Ω with Lipschitz continuous boundary
  (it satisfies the exterior cone condition)

- $n = 2$ In this dimension we can consider
  also cusps. For instance consider a set
  like that in figure.
  Modulo a translation
  and a homotetic
  transformation we



can suppose that $\bar{x} = 0$ and

$\Omega \subset B_1((0,0))$



looking $\mathbb{R}^2$ as $\mathbb{C}$
we can consider the complex function

$$f(z) = \frac{1}{\log z}$$

defined $(z = \rho e^{i\theta})$ for $\rho > 0$, $\theta \in (-\pi, \pi)$

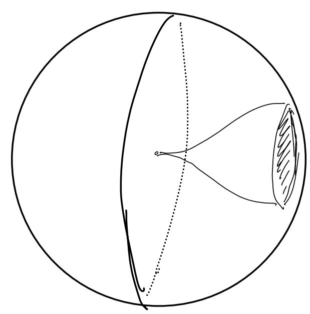
Then the function

$$h(x,y) = -\operatorname{Re} \frac{1}{\log z} = -\frac{\log \rho}{\theta^2 + \log^2 \rho}$$

$$\left(h(0,0) = \lim_{\rho \to 0} -\frac{\log \rho}{\theta^2 + \log^2 \rho} = 0 \right)$$

is a barrier for Ω at $(0,0)$.

= $n = 3$ One can construct cusps in $\mathbb{R}^3$ for which the vertex is not a regular point



[see E. DiBenedetto
Partial Differential Equations
Section 2.7.2

or S. Salsa G. Verzini
Partial Differential Equations in Action
Section 3.4.3]

Therefore in general for $n \geq 3$ not all types of cusps are admitted, unless they are exterior.