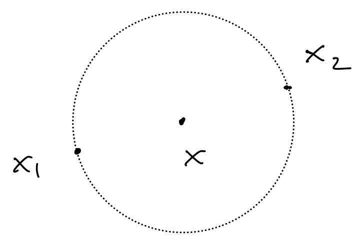
EXERCISE (EX I.3 in the EXERCISES)

Show that if $u : \Omega \to \mathbb{R}$ is convex (concave) then u is subharmonic (superharmonic).

Suppose u convex. Consider $x_1, x_2 \in \Omega$ and $x := \frac{x_1 + x_2}{2}$



Then $u(x) \leq \frac{1}{2}\big(u(x_1) + u(x_2)\big)$.

Integrating and since

$$u(x) = \int_{\partial B_r(x)} u(x)\, d\mathcal{H}^{n-1}(y)$$

we have

! $u(x)$, not $u(y)$

Since this holds for every pair (x_1, x_2) with $x_1, x_2 \in \partial B_r(x)$ and with $x = \frac{x_1 + x_2}{2}$ we derive that

$$u(x) \leq \frac{1}{2} \int_{\partial B_r(x)} u(x_1)\, d\mathcal{H}^{n-1}(x_1) + \frac{1}{2} \int_{\partial B_r(x)} u(x_2)\, d\mathcal{H}^{n-1}(x_2)$$

$$= \int_{\partial B_r(x)} u(y)\, d\mathcal{H}^{n-1}(y)$$

[Examples: $|x|^\alpha$

$ax^2 - by^2$

EXERCISE Solve, if possible: $\left(u \in C^1([a,b]) \right)$

$$\min \underbrace{\frac{1}{2}\int_a^b |u'(x)|^2 dx}_{=: F_u} \qquad \text{with} \qquad \begin{array}{l} u(a) = A \\ u(b) = B \end{array}$$

Take $\varphi \in C_c^1((a,b))$, $\qquad u + \varepsilon\varphi \qquad \varepsilon \in \mathbb{R}$

$$F(u+\varepsilon\varphi) = \frac{1}{2}\int_a^b |(u+\varepsilon\varphi)'|^2 dx$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon\varphi) = \frac{1}{2}\int_a^b 2(u'+\varepsilon\varphi')\varphi' dx \Big|_{\varepsilon=0} =$$

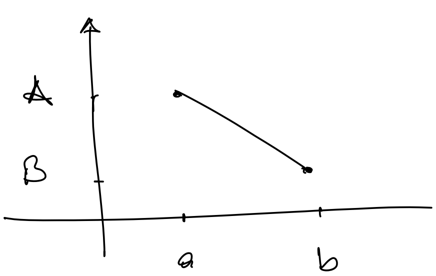
$$= \int_a^b u'(x)\varphi'(x) dx = -\int_a^b u''(x)\varphi(x) dx$$

If we impose $\displaystyle \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u+\varepsilon\varphi) = 0 \qquad \underline{\forall \varphi}$

we derive $\qquad u''(x) = 0 \quad \Rightarrow \quad u(x) = \alpha x + \beta$

$$\begin{array}{l} u(a) = A \\ u(b) = B \end{array} \qquad u(x) = \frac{B}{b-a}(x-a) + A$$

$$= \frac{B}{b-a}x + A - \frac{aB}{b-a}$$



To see that this is the minimum (and not a maximum or something else) it is sufficient to observe that $\mathcal{F}$ is convex in $C^1([a,b])$

$$\mathcal{F}\left((1-t)u+tv\right) = \int_a^b \frac{1}{2}\left((1-t)\,u'+t\,v'\right)^2 dx$$

by convexity of $x \mapsto x^2$ $\quad \leq \dfrac{1}{2}\displaystyle\int_a^b \left((1-t)\,|u'|^2 + t\,|v'|^2\right) dx$

$$= (1-t)\,\mathcal{F}u + t\,\mathcal{F}v \qquad t \in [0,1]$$

then the function $\left(u \in C^1([a,b]),\ \varphi \in C^1_c((a,b)) \text{ fixed}\right)$

$$f(\varepsilon) := \mathcal{F}(u+\varepsilon\varphi) \qquad f : \mathbb{R} \to \mathbb{R}$$

is convex

$$f(\varepsilon) := \mathcal{F}(u+\varepsilon\varphi)$$

$$f\left((1-t)\varepsilon + t\delta\right) = \int_a^b \left(u' + (1-t)\varepsilon\varphi' + t\delta\varphi'\right)^2$$

$$= \int_a^b \left((1-t)(u'+\varepsilon\varphi') + t(u'+\delta\varphi')\right)^2 dx$$

$$\leq (1-t)\int (u'+\varepsilon\varphi')^2 + t\int (u'+\delta\varphi')^2 = (1-t)\,f(\varepsilon) + t\,f(\delta)$$

Then,

$$f(\varepsilon) \geq f(0) + f'(0)\,\varepsilon$$

and if 0 is a stationary point we get

$$f(\varepsilon) \geq f(0) \qquad \forall\, \varepsilon \in \mathbb{R},$$

that is (if u is stationary for $\mathcal{F}$)

$$\mathcal{F}(u + \varepsilon\varphi) \geq \mathcal{F}u$$ and this holds $\forall\, \varphi$

For the following exercise we use the polar coordinates in $\mathbb{R}^2$,

$$\mathcal{T} : [0, +\infty) \times [0, 2\pi) \longrightarrow \mathbb{R}^2$$
$$(\rho, \theta) \longmapsto (\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)$$

See at the end of the exercise the details

$$u = u(x,y) \qquad \tilde{u}(\rho,\theta) := u \circ \mathcal{T}(\rho,\theta) =$$
$$= u(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)$$

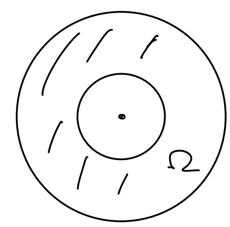
Analogous problem in dimension 2:

$$\mathcal{F}u = \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dx\,dy \qquad \Omega = B_R(0) \setminus \overline{B_r(0)}$$

min $\mathcal{F}$ among the functions u satisfying

$$u = B \quad \text{in } \partial B_R(0)$$
$$u = A \quad \text{in } \partial B_r(0)$$

Take $\varphi \in C^1_c(\Omega)$



$$\mathcal{F}u = \frac{1}{2}\iint_{\Omega} |\nabla u(x,y)|^2 \, dx\,dy =$$

$$\overset{\text{polar coordinates}}{=} \frac{1}{2}\int_0^{2\pi} d\theta \int_r^R \rho \, |u_\rho|^2 \, d\rho$$

(u instead of $\tilde{u}$)

With the additional assumption that u is radial

Then, differentiating

$$\frac{1}{2}\int_0^{2\pi} d\theta \int_r^R \rho \left(u_\rho + \varepsilon \varphi_\rho\right)^2 d\rho$$

$\varphi \in C^1_c(\Omega)$
φ radial
(so that $u + \varepsilon\varphi$ radial)

with respect to ε and evaluating for $\varepsilon = 0$, Then imposing the derivative to be zero, we get

$$0 = 2\pi \int_r^R \rho\, u_\rho\, \varphi_\rho\, d\rho = -2\pi \int_r^R (\rho u_\rho)_\rho\, \varphi\, d\rho$$

by which
$$(\rho u_\rho)_\rho = 0$$

$$\rho u_\rho = a \qquad a \in \mathbb{R}$$

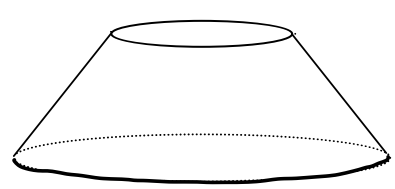
$$u_\rho = \frac{a}{\rho}$$

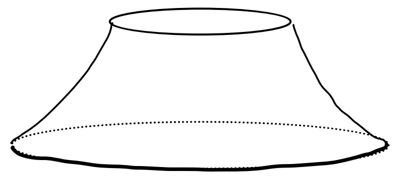
$$u(\rho) = a \log \rho + b \qquad a, b \in \mathbb{R}$$

Assigning the boundary values one finds

$$u(x,y) = \frac{(A-B)}{\log \frac{r}{R}} \log |(x,y)| + B - (A-B) \frac{\log R}{\log \frac{r}{R}} \qquad (*)$$

Then the solution is not something like that, as one could expect looking the solution in dimension 1, but something like the figure below ($r < R$, $A > B$)





Observe that, as already seen, the solution has to be radial (a stationary point for F, and in particular the minimum, has to be harmonic).

Therefore the minimum has to satisfy

$$\begin{cases} -\Delta u = 0 & \text{in} \quad B_R(0) \setminus \overline{B_r(0)} \\ u = A & \text{in} \quad \partial B_r(0) \\ u = B & \text{in} \quad \partial B_R(0) \end{cases}$$

Since the function defined in ⊛ solves this Dirichlet problem and the solution is unique we have found the solution.

This is also the minimum.

Two remarks

1. By the Dirichlet's principle, once we have found a minimum for F we know that this minimum, if C^2, is harmonic.

2. We have also solved the problem

$$\begin{cases} -\Delta u = 0 & \text{in } B_R(0,0) \setminus B_r(0,0) \\ u = A & \text{in } \partial B_r(0,0) \\ u = B & \text{in } \partial B_R(0,0) \end{cases}$$

$A, B \in \mathbb{R}$.

Ex Solve the analogous problem in $\mathbb{R}^n$, $n \geq 3$.

Ex I.10 Let $u \in C^2(\mathbb{R}^2)$ subharmonic

For $r > 0$ denote $M(r)$ the quantity $\max\limits_{x \in \partial B_r(0)} u(x)$

Prove that for each $r_1, r_2 > 0$,

$r_1 < r_2$, one has

$$M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} M(r_2)$$

for every $r \in (r_1, r_2)$.

I. 10

Consider the function

$$v(x) = a \log|x| + b$$

which is harmonic in $B_{r_2} \setminus \overline{B_{r_1}}$.

Choose a and b in such a way that

$$\begin{cases} a \log r_1 + b = M(r_1) = \max\limits_{\partial B_{r_1}(0)} u \\ a \log r_2 + b = M(r_2) \end{cases}$$

We derive
$$a = \frac{M(r_2) - M(r_1)}{\log r_2 - \log r_1}$$

$$b = \frac{M(r_1) \log r_2 - M(r_2) \log r_1}{\log r_2 - \log r_1}$$

Then $u - v$ is sub-harmonic in $A = B_{r_2} \setminus \overline{B_{r_1}}$

and $u - v \leq 0$ in ∂A

since $u \leq M(r_j)$ in $\partial B_{r_j}(0)$ $j = 1, 2$,

while $v = M(r_j)$ in $\partial B_{r_j}(0)$, $j = 1, 2$.

By the maximum principle $u \leq v$ in A

and in particular

$$u(x) \leq v(x) = a \log r + b \qquad \text{for } |x| = r$$

Then

$$M(r) \leq \frac{M(r_2) - M(r_1)}{\log r_2 - \log r_1} \log r + $$

$$+ \frac{M(r_1) \log r_2 - M(r_2) \log r_1}{\log r_2 - \log r_1}$$

$$= \frac{M(r_2)(\log r - \log r_1)}{\log r_2 - \log r_1} + \frac{M(r_1)(\log r_2 - \log r)}{\log r_2 - \log r_1}$$

EXERCISE Show that every sub-harmonic function on $\mathbb{R}^2$ which is bounded from above is constant.

Starting from the inequality of the previous exercise and assuming $u \leq c$ in $\mathbb{R}^2$ we have, taking the limit for $r_2 \to +\infty$

$$M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} c$$

$$\xrightarrow[r_2 \to +\infty]{} M(r_1)$$

i.e. $\quad \varphi(r) = \max\limits_{\partial B_r(0)} u \leq \varphi(r_1)$ for every $r_1 > 0$

Taking the limit for $r_1 \to 0^+$ we get

$$\max_{\partial B_r(0)} u \leq \lim_{r_1 \to 0^+} \varphi(r_1) = u(0) \quad \forall r > 0$$

(u is C^2, then continuous)

that is 0 is a (global) maximum point.

Then u must be constant (by the maximum principle).

EXERCICE The analogous does not hold in dimension 3

(and in fact in every dimension $n \geq 4$)

We show this fact ($n = 3$)

Laplacian in polar coordinates ($n = 2$)

$$u(x,y) \qquad F(\rho,\theta) = (\rho \cos\theta, \rho \operatorname{sen}\theta)$$
$$\tilde{u}(\rho,\theta) := u \circ F(\rho,\theta)$$

Δu becomes

$$\tilde{u}_{\rho\rho}(\rho,\theta) + \frac{1}{\rho}\tilde{u}_\rho(\rho,\theta) + \frac{1}{\rho^2}\tilde{u}_{\theta\theta}(\rho,\theta) \quad , \quad \text{i.e.}$$

$u_{xx}(\rho\cos\theta, \rho \operatorname{sen}\theta) + u_{yy}(\rho\cos\theta, \rho\operatorname{sen}\theta) =$

$$= \tilde{u}_{\rho\rho}(\rho,\theta) + \frac{1}{\rho}\tilde{u}_\rho(\rho,\theta) + \frac{1}{\rho^2}\tilde{u}_{\theta\theta}(\rho,\theta)$$

in $\mathbb{R}^3$, for $u = u(x_1, x_2, x_3)$ and radial, we have

Δu becomes $\quad \tilde{u}_{\rho\rho} + \frac{2}{\rho}\tilde{u}_\rho \qquad \left(\tilde{u} = \tilde{u}(\rho,\theta,\phi)\right)$

Consider

$$u(x) = \begin{cases} e^{-\frac{1}{|x|}} & |x| \neq 0 \\ 0 & |x| = 0 \end{cases}$$

$-\Delta u$ in radial (or spherical) coordinates (since u is radial the derivatives w.r.t. the angular variables are zero) is

$$-\tilde{u}_{\rho\rho}(\rho) - \frac{2}{\rho}\tilde{u}_\rho$$

$$\tilde{u}_\rho = \frac{\partial}{\partial\rho} e^{-\frac{1}{\rho}} = \frac{d}{d\rho} e^{-\frac{1}{\rho}} = e^{-\frac{1}{\rho}}\frac{1}{\rho^2} \qquad \rho \neq 0$$

$\tilde{u}_\rho(0) = 0$

$$\tilde{u}_{\rho\rho}(\rho) = \begin{cases} e^{-1/\rho}\frac{1}{\rho^4} - 2e^{-1/\rho}\frac{1}{\rho^3} \\ 0 \quad \text{if } \rho = 0 \end{cases}$$

then $-\tilde{u}_{\rho\rho} - \frac{2}{\rho}\tilde{u}_\rho$ is

$$e^{-\frac{1}{\rho}}\left(-\frac{1}{\rho^4} + \frac{2}{\rho^3} - \frac{2}{\rho}\frac{1}{\rho^2}\right) = -\frac{e^{-\frac{1}{\rho}}}{\rho^4}$$

if $\rho \neq 0$

otherwise is 0

Then $-\Delta u \leq 0$ and u is sub-harmonic, is bounded $(0 \leq u \leq 1)$, but not constant.

Notice that the analogous in $\mathbb{R}^2$ is not sub-harmonic:

$-\Delta u$ would be:

1 instead of 2

$$-\left(\tilde{u}_{\rho\rho}(\rho,\theta) + \frac{1}{\rho}\tilde{u}_\rho(\rho,\theta)\right) =$$

$$= e^{-\frac{1}{\rho}}\left(-\frac{1}{\rho^4} + \frac{2}{\rho^3} - \frac{1}{\rho}\frac{1}{\rho^2}\right) =$$

$$= \frac{e^{-1/\rho}}{\rho^3}\left(1 - \frac{1}{\rho}\right)$$

GRADIENT IN POLAR COORDINATES

$u(x,y) \longrightarrow$ polar coordinates

$$F(\rho,\theta) = (\rho\cos\theta, \rho \operatorname{sen}\theta)$$

$\tilde{u} := u \circ F$ and then $u = \tilde{u} \circ F^{-1}$

$$\tilde{u}(\rho,\theta) = u(\rho\cos\theta, \rho\operatorname{sen}\theta)$$

Then
$$\tilde{u}_\rho = u_x \cos\theta + u_y \operatorname{sen}\theta$$
$$\tilde{u}_\theta = u_x(-\rho\operatorname{sen}\theta) + u_y\, \rho\cos\theta$$

and
$$\nabla_{\rho,\theta}\tilde{u}(\rho,\theta) = \underbrace{\begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\rho\operatorname{sen}\theta & \rho\cos\theta \end{pmatrix}}_{= A^{-1}} \cdot \nabla_{x,y} u\,(F(\rho,\theta))$$

Observe that $A^{-1} = (DF)^t$

Then $\nabla_{x,y} u\,(F(\rho,\theta))$ is $A \cdot \nabla_{\rho,\theta}\tilde{u}(\rho,\theta)$

where
$$A = \begin{pmatrix} \cos\theta & -\frac{1}{\rho}\operatorname{sen}\theta \\ \operatorname{sen}\theta & \frac{1}{\rho}\cos\theta \end{pmatrix}$$

That is

$\nabla_{x,y}\, u\,(F(\rho,\theta))$ becomes

$$\begin{pmatrix} \cos\theta & -\frac{1}{\rho}\,\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \frac{1}{\rho}\cos\theta \end{pmatrix} \cdot \begin{pmatrix} \tilde{u}_\rho\,(\rho,\theta) \\ \tilde{u}_\theta\,(\rho,\theta) \end{pmatrix} =$$

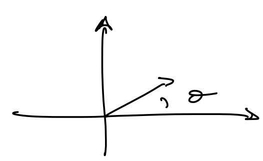
$$= \left(\cos\theta\ \tilde{u}_\rho - \frac{1}{\rho}\,\mathrm{sen}\,\theta\ \tilde{u}_\theta\ ,\ \mathrm{sen}\,\theta\ \tilde{u}_\rho + \frac{1}{\rho}\cos\theta\ \tilde{u}_\theta \right)$$

$$\nabla_{x,y}\, u(x,y) = u_x\,(1,0) + u_y\,(0,1)$$

$$\nabla_{\rho\theta}\, \tilde{u}\,(x,y) = \tilde{u}_\rho\, e_1 + \frac{1}{\rho}\tilde{u}_\theta\, e_2 \qquad e_1 = (\cos\theta,\ \mathrm{sen}\,\theta)$$

$$e_2 = (-\mathrm{sen}\,\theta,\ \cos\theta)$$

<u>Remark</u> e_1 radial



e_2 tangential