A - STRONG AND WEAK TOPOLOGY

EXERCISE A.1 Consider the sequence of functions

$$f_n : [0, 2\pi] \longrightarrow \mathbb{R}, \qquad f_n(x) = \operatorname{sen} nx$$

$f_n \in L^p(0, 2\pi) \quad \forall\, p \in [1, +\infty]$

Consider $p = 1$. Study the convergence of $\{f_n\}_n$ in the weak and the strong topology of $L^1(0, 2\pi)$.

EXERCISE A.2 Consider the sequence $(\alpha \geq 0)$

$$f_n(x) = \begin{cases} n^\alpha \operatorname{sen} nx & x \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right] \\ 0 & x \notin \left[-\frac{\pi}{n}, \frac{\pi}{n}\right] \end{cases}$$

Study weak and strong convergence in $L^2(-\pi, \pi)$.

EXERCISE A.3 As above, but with

$$f_n(x) = \begin{cases} n^\alpha \operatorname{sen} nx & x \in \left[0, \frac{\pi}{n}\right] \\ 0 & x \notin \left[0, \frac{\pi}{n}\right] \end{cases}$$

EXERCISE 1.4

Study pointwise, weak and strong convergence in $L^p(\mathbb{R})$, $p \in [1, +\infty]$, of the following sequences of functions $f_h : \mathbb{R} \to \mathbb{R}$:

- $f_h(x) = \begin{cases} \operatorname{sen}(hx) & x \in [0, 2\pi] \\ 0 & \text{otherwise} \end{cases}$

- $f_h(x) = \begin{cases} e^{-hx} \operatorname{sen}(hx) & x \in [0, 2\pi] \\ 0 & \text{otherwise} \end{cases}$

- $f_h(x) = \begin{cases} \operatorname{sen}(hx) & x \in [0, \frac{2\pi}{h}] \\ 0 & \text{otherwise} \end{cases}$

- $f_h(x) = \min \{ |hx - 1|, 1 \}$

EXERCISE 1.5

Find a sequence $\{f_h\}$ such that $\int_{\mathbb{R}} f_h \, dx = 1$,

$f_h \longrightarrow 0$ a.e. and

$$\int_{\mathbb{R}} f_h(x) \varphi(x) \, dx \xrightarrow[h \to +\infty]{} \varphi(0) \qquad \forall \varphi \in C_c^0(\mathbb{R})$$

EXERCISE A.6

Find a sequence $\{f_h\}$ such that $\int_{\mathbb{R}} f_h \, dx = 1$,

$f_h \longrightarrow 0$ a.e. and

$$\int_{\mathbb{R}} f_h(x)\, \varphi(x)\, dx \xrightarrow[h \to +\infty]{} 0 \qquad \forall \varphi \in C^0_c(\mathbb{R})$$

EXERCISE A.7

Prove that for $f \in L^\infty(\Omega)$ and $\{\rho_n\}_n$ a sequence of mollifiers (or an approximate identity) in general does not hold

$$\rho_n * f \longrightarrow f \quad \text{in } L^\infty(\Omega).$$

EXERCISE 1.8 Consider the space

$$X = \{ \gamma : [0,1] \to \mathbb{R}^2 \mid \gamma \ C^1\text{-piecewise}, \ \gamma(0) = (0,0) \}$$

and for $\gamma \in X$ let us introduce

$$\|\gamma\| = \int_0^1 |\dot{\gamma}(t)| \, dt$$

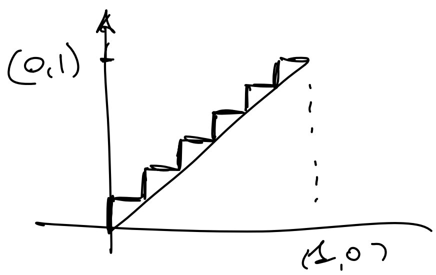
(a) Verify that $\|\cdot\|$ is a norm

(b) What happens if we drop condition $\gamma(0) = (0,0)$?

(c) Consider the sequence

$$\gamma_n(t) = \begin{cases} (0,t) & t \in [0, 1/2^n] \\ (t,0) + \left(0, \frac{1}{2^n}\right) - \left(\frac{1}{2^n}, 0\right) & t \in \left(1/2^n, \frac{2}{2^n}\right] \\ (0,t) + \left(\frac{1}{2^n}, \frac{1}{2^n}\right) - \left(0, \frac{2}{2^n}\right) & t \in \left(\frac{2}{2^n}, \frac{3}{2^n}\right] \\ (t,0) + \left(\frac{1}{2^n}, \frac{2}{2^n}\right) - \left(\frac{3}{2^n}, 0\right) & t \in \left(\frac{3}{2^n}, \frac{4}{2^n}\right] \\ \vdots & \\ (t,0) + \left(\frac{2^n-1}{2^n}, 1\right) - \left(\frac{2^n-1}{2^n}, 0\right) & t \in \left(\frac{2^n-1}{2^n}, 1\right] \end{cases}$$

In the following picture one element of the sequence



As n goes to $+\infty$ the sequence of γ_n is converging pointwise to

$\gamma(t) = (t, t) \qquad t \in [0, 1]$

(b.1) What is the dual space of X ?

(b.2) is $\{\gamma_n\}_n$ strongly converging to γ ?

(b.3) is $\{\gamma_n\}_n$ weakly " to γ ?

EXERCISE B.1 Prove that the function $(q>1)$

$$v(x) = \frac{1}{x\,(|\log x| + 1)^q} \in L^1(0,1)$$

but $v \notin L^p(0,1)$ for $p>1$.

Given the function

$$u(x) := \int_0^x \frac{1}{t\,|\log t|^q}\,dt$$

prove that there are not $c>0$, $\alpha \in (0,1)$ such that

$$|u(x) - u(y)| \leq c\,|x-y|^\alpha$$

EXERCISE 0.1 Given $u \in L^\infty(\Omega)$, $|\Omega| < +\infty$, show that the function

$$m(p) = \left(\int_\Omega |u|^p \, dx \right)^{1/p} \qquad m : [1, +\infty] \to [0, +\infty)$$

is increasing.

EXERCISE 0.2 Say why the norm of the supremum

$\left(\text{i.e. } \|u\| := \sup_{x \in \Omega} |u(x)| \text{ or } \operatorname{ess\,sup}_{x \in \Omega} |u(x)| \right)$ is

called $\|\cdot\|_\infty$ (the question is: why infinity ?)

(Hint: start from $\mathbb{R}^2$ and show that

$$\|(x,y)\|_\infty = \max\{|x|, |y|\})$$

EXERCISE 0.3 Prove, using the Poincaré inequality, that for $p \in [1, n)$

$$\left(\int_{B_r(x_0)} |u|^q \, dx \right)^{1/q} \leq c\, r \left(\int_{B_r(x_0)} |Du|^p \, dx \right)^{1/p}$$

$\forall\, u \in W_0^{1,p}(B_r(x_0))$ or u with null mean value and for every $q \in [p, p^*]$, and then for every $q \in [1, p^*]$.