1 - STRONG AND WEAK TOPOLOGY

1.1

Consider the sequence of functions

$$f_n : [0, 2\pi] \longrightarrow \mathbb{R}, \qquad f_n(x) = \operatorname{sen} nx$$

$f_n \in L^p(0, 2\pi) \quad \forall\, p \in [1, +\infty]$

Consider $p = 1$. Then

$$\int_0^{2\pi} |f_n(x)|\, dx = \int_0^{2\pi} |\operatorname{sen} nx|\, dx =$$

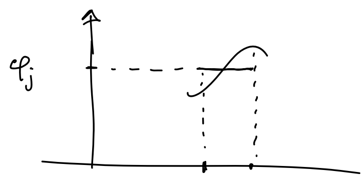
$$= \frac{1}{n} \int_0^{2\pi n} |\operatorname{sen} y|\, dy = \int_0^{2\pi} |\operatorname{sen} y|\, dy = 2 \int_0^{\pi} \operatorname{sen} y\, dy = 4$$

Then $\|f_n\|_{L^1} \leq 4$. Consider $\varphi \in C^0([0, 2\pi])$.

$$\int_0^{2\pi} f_n(x)\, \varphi(x)\, dx = \sum_{m=1}^{n} \int_{2\pi \frac{m-1}{n}}^{2\pi \frac{m}{n}} \operatorname{sen}(nx)\, \varphi(x)\, dx$$

For each m between 1 and n consider the mean value of φ

$$\varphi_j := \frac{n}{2\pi} \int_{2\pi \frac{m-1}{n}}^{2\pi \frac{m}{n}} \varphi(x)\, dx$$



By the continuity of φ for every $\varepsilon > 0$ we can find $H \in \mathbb{N}$ such that if $h \geq H$ then

$$|\varphi(x) - \varphi_j| < \varepsilon \quad \forall\, x \in \left[2\pi \frac{n-1}{h}, 2\pi \frac{n}{h}\right]$$

Then

$$\int_{2\pi\frac{n-1}{h}}^{2\pi\frac{n}{h}} \operatorname{sen}(hx)\, \varphi(x)\, dx = \int_{2\pi\frac{n-1}{h}}^{2\pi\frac{n}{h}} \operatorname{sen} hx \left(\varphi_j + \varphi(x) - \varphi_j\right) dx$$

$$= \underbrace{\int_{2\pi\frac{n-1}{h}}^{2\pi\frac{n}{h}} \operatorname{sen} hx\, \varphi_j\, dx}_{\substack{\shortparallel \\ 0}} + \int_{2\pi\frac{n-1}{h}}^{2\pi\frac{n}{h}} \operatorname{sen} hx \left(\varphi(x) - \varphi_j\right) dx$$

$$\left| \int_0^{2\pi} \operatorname{sen} hx\, \varphi(x)\, dx \right| \leq \sum_{n=1}^{h} \left| \int_{2\pi\frac{n-1}{h}}^{2\pi\frac{n}{h}} \operatorname{sen} hx \left(\varphi(x) - \varphi_j\right) dx \right|$$

$$\leq \varepsilon \int_0^{2\pi} |\operatorname{sen} hx|\, dx = 4\varepsilon .$$

If $\varphi \in L^\infty(0,2\pi)$ one can approximate φ by a step function as done above and come to the same conclusion.

If one looks $\{f_n\}_n$ as a sequence in L^p for $p \in (1,+\infty)$ the same argument can be done for $\varphi \in L^{p'}(0,2\pi)$.

Then $\{f_n\}_n$ weakly converges to $f \equiv 0$ in $L^p(0,2\pi)$ for every $p \in [1,+\infty)$.

Does $\{f_n\}_n$ strongly converge?

No! $\int_0^{2\pi} |f_n|\, dx = 4$ for every n

Then $\|f_n\|_{L^1} \not\xrightarrow[n\to+\infty]{} 0$.

1.2 $f_n(x) = \begin{cases} n^\alpha \operatorname{sen} nx & x \in [-\pi/n, \pi/n] \\ 0 & x \notin [-\pi/n, \pi/n] \end{cases}$

$$\int_{-\pi/n}^{\pi/n} (n^\alpha \operatorname{sen} nx)^2\, dx = n^{2\alpha} \int_{-\pi}^{\pi} \operatorname{sen}^2 y\, dy\, \frac{1}{n} = \pi\, n^{2\alpha-1}$$

Then
$$\left(\int_{-\pi}^{\pi} f_n^2(x)\,dx \right)^{1/2} = \sqrt{\pi}\; n^{\alpha - 1/2}$$

We conclude that for $\alpha > \frac{1}{2}$ $\quad \|f_n\|_{L^2} \longrightarrow +\infty$ and $\{f_n\}_n$ cannot converge.

For $\alpha = \frac{1}{2}$ $\{f_n\}_n$ is bounded, but $\{\|f_n\|_n\}_n$ is not converging to 0. Anyway the mean value

$$\int_{-\pi}^{\pi} f_n = \int_{-\pi/n}^{\pi/n} \operatorname{sen} nx\, dx = 0$$

and, as in the previous exercise,

$$f_n \longrightarrow 0 \quad \text{in } L^2\text{-weak}$$

(and not strong!)

For $\alpha \in [0, 1/2)$ $\quad \|f_n\|_{L^2} \longrightarrow 0$ and then

$$f_n \longrightarrow 0 \quad \text{in } L^2 \quad (L^2\text{-strong})$$

A.3 The only difference is that for $\alpha = 1/2$

$\int_{-\pi}^{\pi} f_n\,dx$ is not 0! But $\int_{-\pi}^{\pi} f_n(x)\,\varphi(x)\,dx \longrightarrow 0$
$\forall \varphi \in L^2$

1.4

- The first sequence does not converge strongly, but only weakly

$$\left(\int_{\mathbb{R}} f_n(x)\,\varphi(x)\,dx \xrightarrow[n]{} 0 \quad \forall \varphi \in L^{p'}(\mathbb{R}) \text{ if } p \in [1,\infty) \quad \forall \varphi \in L^1(\mathbb{R}) \text{ if } p = +\infty \right)$$

It strongly in L^p $\forall p$ and pointwise converges to zero in $L^p(A)$ $\forall A \in \mathbb{R} \setminus [0, 2\pi]$

- The second one pointwise converges to zero. It also strongly converges to zero in $L^p(\mathbb{R})$ $\forall p \in (1,\infty)$ and $p = +\infty$ – Let's compute the maximum and the minimum of f_n. We get that

$$|f_n(x)| \leq e^{-n\frac{3}{4}\pi}\,\frac{\sqrt{2}}{2} \xrightarrow[n\to\infty]{} 0$$

Then $f_n \xrightarrow[n]{} 0$ also in $L^\infty(\mathbb{R})$

- The third one converges pointwise, weakly, strongly to 0 in each L^p.
- In this case the pointwise limit is $f \equiv 1$, and it is also the strong and the weak limit.

1.5

$$f_n(x) = \begin{cases} 2n & x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases}$$

1.6

$$f_n(x) = \begin{cases} 1 & x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}$$

1.7 Take, for instance, $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

1.8 The dual space is that of the differential forms (or of the vector fields).

If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field, F continuous, we can associate to F a linear form φ_F to F in this way

$$\langle \varphi_F , \gamma \rangle := \int_0^1 \left(F(\gamma(t)) , \gamma'(t) \right) dt$$

$(\cdot , \cdot)$ scalar product in $\mathbb{R}^2$

In the same way given a differential form $\omega = \omega_1(x,y)\,dx + \omega_2(x,y)\,dy$ we can consider φ_ω defined by

$$\langle \varphi_\omega , \gamma \rangle := \int_0^1 \left[\omega_1(\gamma(t))\, \dot\gamma_1(t) + \omega_2(\gamma(t))\, \dot\gamma_2(t) \right] dt$$

$F_1\,dx + F_2\,dy$ is a differential form

These are all linear and continuous forms on X.

4.8

First of all notice that $\|\cdot\|$ is a norm just because $\gamma(0) = (0,0)$. Indeed

$\|\gamma\| = 0 \Rightarrow$ the lenght of γ is 0 and γ is a constant curve, but since $\gamma(0) = (0,0)$ then $\gamma(t) = (0,0)$ $\forall t$

The other points defining a norm are easy to be verified.

Then if $\gamma(0) \neq (0,0)$ we cannot conclude that

$\int_0^1 |\dot\gamma(s)|\, ds = 0 \Rightarrow \gamma \equiv (0,0)$ (but only $\gamma \equiv (x_0, y_0)$).

Notice that one could also consider the space

$$Y = \{ \gamma : [0,1] \longrightarrow \mathbb{R}^2 \mid \gamma \ C^1\text{-piecewise} \},$$

which contains X, with the norm

$$\|\gamma\|_Y := \max_{t \in [0,1]} |\gamma(t)| + \int_0^1 |\dot\gamma(t)|\, dt$$

Verify that $\|\gamma\|_Y$ is a norm and that $\|\cdot\|_Y$ and $\|\cdot\|_X = \|\cdot\|$ are equivalent on X

(i.e. there are two positive constants c_1, c_2 s.t.

$$c_1 \|\gamma\|_X \leq \|\gamma\|_Y \leq c_2 \|\gamma\|_X \quad).$$

Now we state we can identify X' with the set of curves

$$\eta : [0,1] \longrightarrow L^\infty(0,1) \times L^\infty(0,1)$$
$$t \mapsto (\eta_1(t), \eta_2(t))$$

Knowing that $(L^1(\Omega))' = L^\infty(\Omega)$, observe that

$$\gamma' \in L^1(0,1) \times L^1(0,1)$$

The map

$$L_\eta(\gamma) = \int_0^1 (\eta_1(t)\,\dot{\gamma}_1(t) + \eta_2(t)\,\dot{\gamma}_2(t))\,dt$$

is a linear and continuous map on X.

Linearity is obvious. Continuity comes from the Hölder inequality

$$|L_\eta(\gamma)| \leq \|\eta\|_\infty \int_0^1 |\dot{\gamma}(t)|\,dt$$

Now let's come to the sequence $\{\gamma_n\}$.

This sequence cannot converge strongly to γ, i.e.

$$\lim_n \|\gamma_n - \gamma\|_X \quad \text{cannot be } 0.$$

Observe that, in general, by

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x - y\| ,$$

one would have $\lim_n \|\gamma_n\| = \|\gamma\|$ if

$\lim_n \|\gamma_n - \gamma\|_X$ were zero.

Observe that

$$\|\gamma_n\|_X = \int_0^1 |\dot{\gamma}_n(t)|\, dt = 2 \quad \text{for every } n \in \mathbb{N}$$

<u>but</u>

$$\|\gamma\|_X = \sqrt{2}$$

As a consequence $\{\gamma_n\}$ cannot converge to γ

w.r.t. the strong topology otherwise we would have

$$\|\gamma_n\| \xrightarrow[n \to +\infty]{} \|\gamma\| .$$

On the other side, since

$$\dot{\gamma}_n(t) = \begin{cases} (0,1) & \text{in } \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right] \quad k \text{ odd} \\ (1,0) & \text{in } \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right] \quad k \text{ even} \end{cases}$$

that is

$$(\gamma_n)_1'(t) = \begin{cases} 1 & \text{in } \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right] \quad k \text{ even} \\ 0 & \text{in } \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right] \quad k \text{ odd} \end{cases}$$

We have that $(\gamma_n)_1' \longrightarrow 1/2$ in the sense that

$$\int_0^1 \eta_1(t)\,(\gamma_n)_1'(t)\,dt \longrightarrow \frac{1}{2}\int_0^1 \eta_1(t)\,dt$$

$$\forall\ \eta_1 \in L^\infty(0,1)$$

and $$\int_0^1 \eta_2(t)\,(\gamma_n)_2'(t)\,dt \longrightarrow \frac{1}{2}\int_0^1 \eta_2(t)\,dt$$

$$\forall\ \eta_2 \in L^\infty(0,1)$$

B.1 Let's see that $\displaystyle\int_0^1 \frac{1}{x(|\log x|+1)^q}\,dx < +\infty$

(One could also consider $\dfrac{1}{x\,|\log x|^q}$ in $(0,a)$, $a<1$)

Consider the change of variable $x = \frac{1}{y}$ and get

$$\int_0^1 \frac{1}{x(|\log x|+1)^q}\,dt = \int_{+\infty}^1 y\,\frac{1}{(|\log \frac{1}{y}|+1)^q}\cdot\left(-\frac{1}{y^2}\right)dy$$

$$= \int_1^{+\infty} \frac{1}{(|\log y|+1)^q}\,\frac{1}{y}\,dy$$

Now consider $f(y) = y\,(|\log y|+1)^q$. One gets

$$f'(y) = (|\log y|+1)^q + q\,y\,(|\log y|+1)^{q-1}\,\frac{\log y}{|\log y|}\,\frac{1}{y}$$

$$= (|\log y|+1)^{q-1}\left[\,|\log y|+1-q\,\right]$$

$\Rightarrow f' > 0$ in some interval $(0,a)$

(Notice that the same holds for

$f_A(y) = y\,(|\log y|+A)^q$ for every $A \geq 0$)

Then $\frac{1}{f}$ is decreasing, at least definetively.

Then we can use the integral criterion for numerical series and study

$$\sum \frac{1}{n(\log n + 1)^q}$$

and then, again because this is decreasing, the Cauchy condensation test to conclude.

Therefore $v \in L^1(0,1)$. To see that $v \notin L^p$ if $p>1$ it is sufficient to integrate (as before $x = \frac{1}{y}$)

$$\int_0^1 \frac{1}{x^p(|\log x|+1)^{qp}} dt = \int_1^{+\infty} \frac{1}{(|\log y|+1)^q} \frac{1}{y^{2-p}} dy = +\infty$$

Now to see that

$$u(x) := \int_0^x \frac{1}{t\,|\log t|^q} dt$$

is not α-Hölder continuous for any $\alpha \in (0,1]$ (we omitted to consider "+1" just for the sake of simplicity) we proceed by contradiction.

Suppose there are $c>0$ and $\alpha \in (0,1]$ s.t.

$$|u(x) - u(y)| \leq c\,|x-y|^\alpha$$

In particular for $y = 0$ we would get

$$u(x) = \int_0^x \frac{dt}{t\,|\log t|^q} \le c\, x^\alpha$$

As already seen above the function $x \mapsto \dfrac{1}{x\,|\log x|^q}$ is decreasing, at least in $(0, a)$ for some $a > 0$.

Then

$$\int_0^x \frac{dt}{t\,|\log t|^q} > x \frac{1}{x\,|\log x|^q} = \frac{1}{|\log x|^q}$$

and we would get

$$\circledast \qquad \frac{1}{x^\alpha\,|\log x|^q} \le c \qquad \text{for every } x \in (0, a).$$

But taking the limit for $x \to 0^+$ we get

$$\lim_{x \to 0^+} \frac{1}{x^\alpha\,|\log x|^q} = +\infty$$

and so $\circledast$ is impossible.

C.1 It is sufficient to verify that $m(p) \leq m(q)$ for $p \leq q$.

C.2 In $\mathbb{R}^2$ consider $\|(x,y)\|_p = \left(|x|^p + |y|^p\right)^{1/p}$.

Suppose $|x| = \max\{|x|, |y|\}$ (otherwise ...)

Then $\|(x,y)\|_p = |x| \left(1 + \left(\frac{|y|}{|x|}\right)^p\right)^{1/p}$

and since

$$\lim_{p \to +\infty} \left(1 + \left(\frac{|y|}{|x|}\right)^p\right)^{1/p} = 1$$

one gets that

$$\lim_{p \to +\infty} \|(x,y)\|_p = \|(x,y)\|_\infty .$$

In $\mathbb{R}^n$ the proof is the same.

For $f \in L^\infty(\mathbb{R})$ one can find (this is a theorem)

a sequence

$$S_n(x) = \sum_{i=1}^{N} c_i \chi_{\omega_i}(x) \qquad c_i \in \mathbb{R}, \quad \omega_i \subseteq \mathbb{R}$$

strongly converging to f in $L^p(\mathbb{R})$, $p \in [1, +\infty]$.

One can argue as before on $\{S_k\}_{k\in\mathbb{N}}$ and then take the limit.

C.2 First, if u is a generic L^1 function

$u - \fint_\Omega u\,dx$ has its avarage equal to zero.

Then we have that for $r=1$ there is $c>0$ s.t.

$$\circledast \qquad \left(\fint_{B_1(0)} |u|^p dx\right)^{1/p} \leq c\left(\fint_{B_1(0)} |Du|^p dx\right)^{1/p} \qquad (x_0=0)$$

for $u \in W_0^{1,p}(B_1(0))$ or $u \in W^{1,p}(B_1(0))$ with $\int u = 0$.

Now

$$\fint_{B_r(0)} |u(x)|^p dx = \frac{1}{r^n |B_1(0)|} \int_{B_1(0)} |u(ry)|^p r^n dy =$$

$$= \fint_{B_1(0)} |\tilde{u}(y)|^p dy \qquad \text{where } \tilde{u}(y) := u(ry)$$

In the same way

$$\fint_{B_r(0)} |Du(x)|^p dx = \fint_{B_1(0)} |Du(ry)|^p dy .$$

Now notice that

$$D_i \tilde{u}(y) = \frac{\partial}{\partial y_i}\left(u(ry)\right) = \frac{\partial u}{\partial y_i}(ry) \quad r$$

therefore $\quad D\tilde{u}(y) = r\, Du(ry) \quad$ and then

$$\fint_{B_r(0)} |Du(x)|^p dx = \frac{1}{r^p} \fint_{B_1(0)} |D\tilde{u}(y)|^p dy$$

Since for $\tilde{u}$ (*) holds we have

$$\left(\fint_{B_1(0)} |\tilde{u}(y)|^p dy\right)^{1/p} \leq c \left(\fint_{B_1(0)} |D\tilde{u}(y)|^p dy\right)^{1/p} =$$

$$\left(\fint_{B_r(0)} |u(x)|^p dx\right)^{1/p} = c\, r \left(\frac{1}{r^p} \fint_{B_1(0)} |D\tilde{u}(y)|^p dy\right)^{1/p}$$

$$= c\, r \left(\fint_{B_r(0)} |Du(x)|^p dx\right)^{1/p}.$$

If $x_0 \neq 0$ it is sufficient to translate.

To prove the result for p^* it is sufficient to use

$$\left(\int_{B_1(0)} |u|^{p^*} dx\right)^{1/p^*} \leq c \left(\int_{B_1(0)} |u|^p dx + \int_{B_1(0)} |Du|^p dx\right)^{1/p}$$

To reach every $q \in [1, p^*]$ use EXERCISE C.1.