L^p SPACES

Topological space (X,τ) is a topological space if X is a set and τ a collection of subsets of subsets of X satisfying :

i) $\phi \in \tau$, $X \in \tau$

ii) $V_1, \dots V_n \in \tau \Rightarrow \bigcap_{i=1}^{n} V_i \in \tau$

iii) if $\{V_\alpha\}_{\alpha \in A}$ is an arbitrary collection of members of $\tau \Rightarrow \bigcup_{\alpha \in A} V_\alpha \in \tau$

COMMENTS Giving a topology τ on a set X means to decide who are the open sets of X.

There are infinite many ways to give a topology on a set X.

Examples : $X = \mathbb{R}$

$\tau = \{\phi, \mathbb{R}\}$ (trivial topology)

τ = all subsets of $\mathbb{R}$ (discrete topology)

τ contains only the subsets $(a, +\infty)$, $a \in \mathbb{R}$

" " " " " $(-\infty, b)$, $b \in \mathbb{R}$

" " " " " $(-a, a)$ $a \in \mathbb{R}$

" " " " " $(x_0 - a, x_0 + a)$ for some $x_0 \in \mathbb{R}$

REMARK $V_n = (-\frac{1}{n}, \frac{1}{n})$ open but $\bigcap_{n=1}^{+\infty} V_n = \{0\}$ is closed

σ-algebra A collection $\mathcal{M}$ of subsets of X is said to be a σ-algebra if it satisfies:

i) $X \in \mathcal{M}$

ii) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$ (A^c the complement of A relative to X)

iii) if $A_n \in \mathcal{M}$, $n \in \mathbb{N}$ $\Rightarrow \bigcup_{n=0}^{+\infty} A_n \in \mathcal{M}$

Comments

(a) Since $\phi = X^c \Rightarrow \phi \in \mathcal{M}$

(b) if we consider $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$, $A_j = \phi \ \forall j \geq k$

$\Rightarrow A_1 \cup A_2 \cup \ldots \cup A_k \in \mathcal{M}$

(c) Since $\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c \right)^c$ then

$\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M} \Rightarrow \bigcap_{n=1}^{+\infty} A_n \in \mathcal{M}$

(While this is not true for a topology).

(d) Since $A \setminus B = B^c \cap A$ then

$A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$.

Measurable set The members of $\mathcal{M}$, σ-algebra on X, are said measurable and X is said measurable space.

If X measurable space
Y topological space
$f : X \to Y$ is said to be measurable if

$f^{-1}(V)$ is measurable for every V open set in Y

σ-algebra of Borel. There exists the smallest σ-algebra containing τ, where τ is a topology on X.
This σ-algebra is said the Borel σ-algebra of (X, τ).

Take in $\mathbb{R}^n$ the topology τ generated by balls

$$B_r(x_0) = \{ x \in \mathbb{R} \mid |x - x_0| < r \}$$

or by rectangles

$$Q_r(x_0) = \{ x \in \mathbb{R}^n \mid |x - x_0|_\infty \leq r \}$$

for $x_0 \in \mathbb{R}^n$ and $r > 0$.

EXERCISE Show that these two ways generates the same topology.

Consider the Borel σ-algebra containing τ and call it $\mathcal{B}$.

The Lebesgue measure is (or it can be) introduced by a theorem that states the existence of a measure m (which turns out to be the Lebesgue measure, that we will denote by $\mathcal{L}^n$, but often also with $|\cdot|$) that is defined on the σ-algebra $\mathcal{B}$.

For this part see, e.g.,
RUDIN Real and Complex Analysis

Once introduced the Lebesgue measure one can ask:
- is every subset of $\mathbb{R}^k$ Lebesgue-measurable?
- is every Lebesgue measurable set a Borel set?

No! The σ-algebra (of measurable sets) is strictly greater than $\mathcal{B}$, the σ-algebra of Borel sets. Then there are measurable sets that are not Borelian.
Moreover it is possible to find sets that are not measurable.

From now on we consider the Lebesgue measure, $X = \Omega$ open subset of $\mathbb{R}^n$ and by measurable set we mean Lebesgue measurable set.

Examples (of measurable functions)

Consider $\mathbb{R}$ with the usual topology and $\mathcal{M}$ the σ-algebra of Lebesgue measurable sets.

- $f : \mathbb{R} \to \mathbb{R}$ continuous is measurable

V open $\Rightarrow$ $f^{-1}(V)$ is open $\Rightarrow$ $f^{-1}(V) \in \mathcal{B}$ and then $f^{-1}(V) \in \mathcal{M}$

- fix $x_0 \in \mathbb{R}$. Then the set $\{x_0\}$ is measurable

($\{x_0\} = (\mathbb{R} \setminus \{x_0\})^c = ((-\infty, x_0) \cup (x_0, +\infty))^c$ and $(-\infty, x_0) \cup (x_0, +\infty)$ is open and then measurable)

If we consider $\mathbb{Q}$ (which is countable) we can write

$$\mathbb{Q} = \bigcup_{n=0}^{+\infty} \{q_n\} \quad \text{where}$$

$Q : \mathbb{N} \longrightarrow \mathbb{Q}$ $\quad Q(n) = q_n$ is bijective

Then $\mathbb{Q}$ turn out to be a measurable set.

Recall: the Lebesgue measure of $\mathbb{Q}$ is zero!

- Consider the Dirichlet function

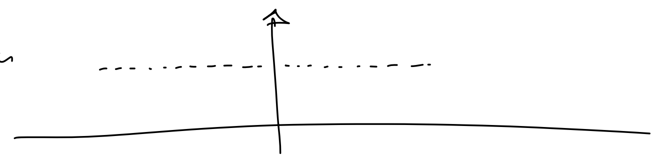
$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Since $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}^c$ are measurable one has that

$$D^{-1}\left(\left(\tfrac{1}{2}, \tfrac{3}{2}\right)\right) = \mathbb{Q} \quad \text{is measurable}$$

$$D^{-1}\left(\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)\right) = \mathbb{R} \setminus \mathbb{Q} \quad " \quad "$$

Then the function d is measurable.



For an example of a not measurable set look for VITALI set.

NORMED, BANACH and HILBERT SPACES

A norm $\|\cdot\|$ is a function defined in X and valued in $\mathbb{R}$, X vectorial space, satisfying

i) $\|x\| \geq 0 \quad \forall x \in X$ and $\|x\| = 0 \Rightarrow x = 0$

ii) $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall x \in X, \ \forall \lambda \in \mathbb{R}$

iii) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

A scalar product on a vectorial space X is a bilinear form defined in $X \times X$ and valued in $\mathbb{R}$ satisfying

i) $(x, x) \geq 0$ and $(x, x) = 0 \Leftrightarrow x = 0$

ii) $(x, y) = (y, x)$

By definition (it is bilinear) it satisfies

iii) $(x+y, z) = (x, z) + (y, z) \quad \forall\ x, y, z \in X$

iv) $(\lambda x, y) = \lambda (x, y) \quad \forall\ x, y \in X$
$\forall\ \lambda \in \mathbb{R}$

Given a scalar product in $X \times X$ the following inequality holds (Cauchy-Schwarz inequality)

$$|(u, v)| \leq (u, u)^{1/2} (v, v)^{1/2} \quad \forall\ u, v \in X.$$

proof: it is equivalent to show that

$$(u, v)^2 \leq (u, u)(v, v)$$

By iii)-iv) if $v = 0$ we have $(u, v) = 0$ and (by i)) $(v, v) = 0$ and then the inequality is true. Suppose then $(v, v) \neq 0$.

Consider $\lambda \in \mathbb{R}$. We have

$$0 \leq (u + \lambda v, u + \lambda v) = (u, u) + 2\lambda (u, v) + \lambda^2 (v, v)$$

This is a second order polynomial in λ with $(v, v) > 0$.
Minimizing $p(\lambda) = (v, v)\lambda^2 + 2\lambda (u, v) + (u, u)$
we get

$$2\lambda (v, v) + 2(u, v) = 0$$

$$\Rightarrow \quad \lambda = -\frac{(u,v)}{(v,v)} \qquad \left((v,v) \neq 0\right)$$

Then in particular

$$(v,v)\frac{(u,v)^2}{(v,v)^2} - 2\frac{(u,v)}{(v,v)}(u,v) + (u,u) \geq 0$$

i.e. $$(u,v)^2 \leq (u,u)(v,v) .$$ //

REMARK - EXERCISE

A scal product induces a norm simply defining

$$\|x\| = (x,x)^{1/2}$$

Def A Banach space $(X, \|\cdot\|_X)$ is a normed vectorial space that is complete with respect to the topology induced by $\|\cdot\|_X$.

Def A Hilbert space $(H, (\cdot,\cdot))$ is a Banach space where the norm is induced by the scalar product $(\cdot,\cdot)$.

Def We define $L^p(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, $p \in (0, +\infty)$ the space

$$\{ f : \Omega \to \mathbb{R} \mid f \text{ measurable and } \|f\|_p < +\infty \}$$

where

$$\|f\|_p^p := \int_\Omega |f|^p \, dx$$

We define $L^\infty(\Omega)$ the set

$$\{ f : \Omega \to \mathbb{R} \mid f \text{ measurable for which there is } c > 0 \text{ s.t. } |f| \le c \text{ a.e. in } \Omega \}$$

and define

$$\|f\|_{L^\infty(\Omega)} := \inf \{ c \geq 0 \mid |f| \le c \text{ a.e. in } \Omega \}$$

$$|f| \le c \ \text{a.e. in } \Omega \iff |f(x)| \le c \ \text{for a.e. } x \in \Omega$$

a.e. = almost everywhere; a.e. = almost every

Meaning of "almost everywhere"

A property $P(x)$ with $x \in \Omega$, $\Omega \subseteq \mathbb{R}^n$, is true almost everywhere in Ω (a.e. in Ω)

if there is a set $E \subseteq \Omega$, $|E| = 0$,

for which

$$P(x) \text{ is true } \forall x \in \Omega \setminus E$$

REMARK Honestly L^p spaces are equivalence classes of functions. One should consider

$$\mathcal{L}^p(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \text{ measurable}, \; \|f\|_p < +\infty \right\},$$

the equivalence relation $\sim$ among functions in $\mathcal{L}^p$

$$f \sim g \quad \text{if} \quad f(x) = g(x) \quad \text{for almost every } x \in \Omega$$

i.e. $|D_{f,g}| = \mathcal{L}^n(D_{f,g}) = 0$ where

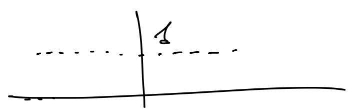
$\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$ and

$$D_{f,g} := \{ x \in \Omega \mid f(x) \neq g(x) \}$$

and finally define $L^p(\Omega)$ as $\dfrac{\mathcal{L}^p(\Omega)}{\sim}$.

Then two functions that differ only in a set of measure zero are in the same class, and in fact we identify them with the same function.

Example: D is in the same class of the null function!
(the Dirichlet function)



$$\int_{\mathbb{R}} D(x)\,dx = 0$$

Given $p \in [1, +\infty]$ we denote by p' the conjugate exponent of p, i.e. the number such that

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{if} \quad 1 < p < +\infty$$

$p' = +\infty$ if $p = 1$

$p' = 1$ if $p = +\infty$

Theorem Given $f, g : \Omega \to \mathbb{R}$ measurable functions and $1 \leq p \leq +\infty$ we have

$$\int_\Omega |fg| \, dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)} \qquad \text{(Hölder)}$$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \qquad \text{(Minkowski)}$$

REMARK $\|\cdot\|_p$ with $p \in [1, +\infty]$ is a norm (see Minkowski's inequality)

REMARK $(f, g)_2 = \int_\Omega f g \, dx$ is a scalar product in $L^2(\Omega)$

Theorem $\left(L^p(\Omega), \|\cdot\|_p\right)$ is a Banach space for each $p \in [1, +\infty]$.

$\left(L^2(\Omega), (\cdot,\cdot)_2\right)$ is a Hilbert space.

EX Why $p \geq 1$?

Which is the set $\{ (x,y) \in \mathbb{R}^2 \mid |x|^p + |y|^p \leq 1 \}$, $p \in (0,1)$?

Is it convex ?

EX Draw the image of $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$$\gamma(t) = (\cos^3 t, \ \sin^3 t)$$

Do $\gamma_1(t)$ and $\gamma_2(t)$ satisfy an implicit equation ?

Theorem Given a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq L^p(\Omega)$, $f \in L^p(\Omega)$, such that $\| f_n - f \|_{L^p(\Omega)} \xrightarrow[n \to +\infty]{} 0$, $p \in [1, +\infty]$, then there is a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ such that

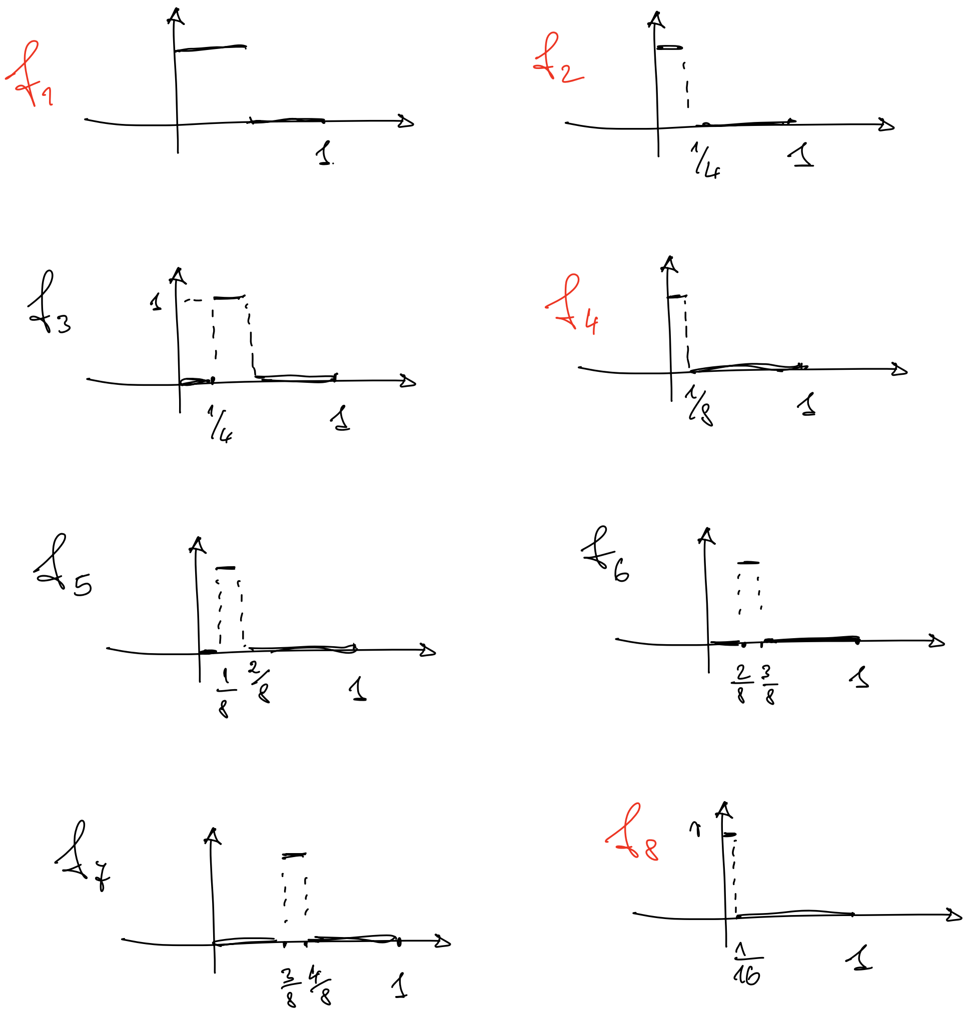
$$\lim_{k \to +\infty} f_{n_k}(x) = f(x) \quad \text{for a.e. } x \in \Omega.$$

Example Consider the sequence defined as follows:

$$f_1(x) = \begin{cases} 1 & x \in [0, 1/2] \\ 0 & x \in (1/2, 1] \end{cases}$$

$$f_2(x) = \begin{cases} 1 & x \in [0, 1/4] \\ 0 & x \in (\frac{1}{4}, 1] \end{cases}$$

$$f_3(x) = \begin{cases} 1 & x \in [1/4, 1/2] \\ 0 & x \in [0,1] \setminus [1/4, 1/2] \end{cases}$$



Clearly $f_n \longrightarrow 0$ in L^p $\forall\, p \in [1,+\infty)$

In the points $\frac{k}{2^{n+1}}$ $\quad k, n \in \mathbb{N}$, $\quad k \leq 2^n$

$f_n(x)$ does not converge, but clearly the subsequence

$f_1, f_2, f_4, f_8, f_{16}, \dots f_{2^n}$ converges to $f = 0$

(those marked in red) $\quad \forall\, x \in (0,1]$, i.e.

a.e. in $[0,1]$.

SOME RECALLS By $C_c^k(\Omega)$ we denote the subset of $C^k(\Omega)$, $k \in \mathbb{N} \cup \{+\infty\}$, Ω open set of $\mathbb{R}^n$, of those functions that have compact support in Ω.

If $k=0$ usually one writes $C_c(\Omega)$ instead of $C_c^0(\Omega)$

A step function in Ω is (! a finite sum)

$$s(x) = \sum_{k=1}^{N} c_k \chi_{A_k}$$

where $c_k \in \mathbb{R}$, $\bigcup_{k=1}^{N} A_k = \Omega$ where

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \quad (x \in \Omega \setminus E) \end{cases}$$

We will denote by $S(\Omega)$ the set of step functions defined in Ω.

Theorem Consider Ω an open subset of $\mathbb{R}^n$. Then $C_c^k(\Omega)$ and $C^k(\Omega) \cap L^p(\Omega)$ are dense in $L^p(\Omega)$ $\forall\, p \in [1, +\infty)$ and $k \in \mathbb{N} \cup \{\infty\}$.

$S(\Omega)$ is dense in $L^p(\Omega)$ $\forall\, p \in [1, +\infty]$.

EX Why $C_c^k(\Omega)$ and $C^k(\Omega) \cap L^\infty(\Omega)$ are not dense in $L^\infty(\Omega)$?

DUAL SPACE OF L^p

Given a Banach space $(X, \|\cdot\|)$ we define the dual space X' as the set of linear and continuous maps from X to $\mathbb{R}$, i.e.

$$L : X \to \mathbb{R} \quad \text{linear and continuous}$$

i.e. $\lim_{n \to +\infty} L(x_n) = L(x)$ if $\{x_n\}_n \subseteq X$ is converging to $x \in X$.

Equivalently one could say that (this has to be proven)

$$X' = \{ L : X \to \mathbb{R} \mid L \text{ linear and bounded} \}$$

L is bounded if

$$|L(x)| \leq c \|x\|_X \quad \forall x \in X.$$

We can endow X' with a norm. We define

$$\boxed{\|\varphi\|_{X'} = \sup_{\|x\|_X \leq 1} \langle \varphi, x \rangle_{X' \times X}} \qquad \left(\langle \varphi, x \rangle = \varphi(x) \right)$$

One can verify that this is a norm on the space X'.
X' endowed with this norm is a Banach space
(if X is a Banach space)

One can also prove (this is a theorem and not a definition)

Theorem Given $x \in X$ one has that

$$\|x\| = \sup_{\|\varphi\|_{X'} \leq 1} \langle \varphi, x \rangle = \max_{\|\varphi\|_{X'} \leq 1} \langle \varphi, x \rangle .$$

EXAMPLE Given $X = \mathbb{R}^n$ with $\|\cdot\|_X = |\cdot|$ we can consider X'. We know that we can identify X' with $\mathbb{R}^n$ and if $\varphi \in X'$ we have that there is $\xi \in \mathbb{R}^n$ such that $\langle \varphi, x \rangle = \langle \varphi_\xi, x \rangle = (\xi, x)_{\mathbb{R}^n} =$

$$= \xi_1 x_1 + \dots + \xi_n x_n$$

Given $\varphi \in X'$, $\varphi = \varphi_\xi$, we define

$$\|\varphi_\xi\|_{X'} = \sup_{|x| \leq 1} \langle \varphi_\xi, x \rangle = \sup_{|x| \leq 1} (\xi, x)$$

$$(\xi, x) \leq |\xi| |x| \Rightarrow \|\varphi_\xi\|_{X'} \leq |\xi|$$

On the other side taking

$x_\xi = \frac{\xi}{|\xi|}$ we have

$$\langle \varphi_\xi, x_\xi \rangle = (\xi, x_\xi) = \frac{1}{|\xi|}(\xi, \xi) = \frac{1}{|\xi|}|\xi|^2 = |\xi|$$

Example Consider the vectorial space made by sequences in $\mathbb{R}$, denote by x an element $\{x_n\}_{n\in\mathbb{N}}$ and we denote this space by $\mathbb{R}^{\mathbb{N}}$.

If we consider $x, y \in \mathbb{R}^{\mathbb{N}}$, $\lambda \in \mathbb{R}$, we simply define

$x+y$ as the sequence $\{x_n+y_n\}_{n\in\mathbb{N}}$ and

λx " " " $\{\lambda x_n\}_{n\in\mathbb{N}}$.

There are many subspaces of $\mathbb{R}^{\mathbb{N}}$: for instance, consider, for $p \in [1, +\infty)$, the space $\ell^p(\mathbb{R})$, the set of sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=0}^{+\infty} |x_n|^p$ converges endowed with the norm

$$\|x\|_{\ell^p} = \left(\sum |x_n|^p\right)^{1/p}.$$

For $p=\infty$ ℓ^∞ denotes the set of bounded sequences: if $x = \{x_n\}_{n \in \mathbb{N}}$ then

$$\|x\|_{\ell^\infty} = \max_{n \in \mathbb{N}} |x_n| .$$

We can consider subspaces of these ℓ^p.
For instance the space $(c_{00}, \|\cdot\|_\infty)$ where

$$c_{00} = \left\{ x = \{x_n\}_n \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \begin{array}{l} \text{a finite number of } \{x_n\}_{n \in \mathbb{N}} \\ \text{are different from } 0 \end{array} \right\}$$

For instance $x = (x_0, x_1, x_2, 0, 0, x_5, x_6, \dots x_k, \overbrace{0, 0, \dots}^{\text{always } 0})$

belongs to c_{00}. This is a vectorial space, since

$x + y \in c_{00}$ $\quad x + y = (x_0 + y_0, x_1 + y_1, x_2, y_3$

$y = y_0, y_1, 0, y_3, y_4, y_5, \dots y_m, 0, \longrightarrow)$ always 0

x as above

Consider the elements $x^N = (0, \dots 0, 1, 0, \dots 0) \in c_{00}$
$\uparrow$ $(N+1)$-th position

and $L : c_{00} \longrightarrow \mathbb{R}$ defined as follows

$$L x^N = N$$

So that $L(x) = \sum_{k=1}^{+\infty} k\, x_k$ for $x \in c_{00}$.

Notice that in fact this is not a series, being indeed a finite sum. L is linear and valued in $\mathbb{R}$.

But notice that L is not bounded. Indeed

$$|Lx| \leq N \|x\|_\infty$$ if the last term of the sequence defining x is exactly the $(N+1)$-th term

Notice that L is not even continuous.
In fact if we consider a sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ of sequences, $x^{(k)} = \left(x_0^{(k)}, x_1^{(k)}, \dots x_k^{(k)}, 0, \dots\right)$,
converging to $0 = (0, 0, 0, \dots)$
the convergence of $Lx^{(k)}$ to 0 is not guaranteed.
Indeed if we consider

$$x^{(k)} = (0, 0, 0, \dots 0, \tfrac{1}{k}, 0, \dots)$$

$(k+1)$-th position

we have that

$$Lx^{(k)} = 1 \quad \text{for every } k \in \mathbb{N}^*$$

even if $\|x^{(k)}\|_\infty \xrightarrow[k\to+\infty]{} 0$.

See the following result to understand the connection between boundedness and continuity.

Theorem Given X, Y Banach spaces and $L : X \to Y$ linear. Then L is continuous iff L is bounded, i.e. there exists a constant $c > 0$ such that

$$\|Lx\|_Y \leq c \|x\|_X \quad \forall \; x \in X.$$

Without proof

Theorem (Riesz) Consider $p \in [1, \infty)$ and $\varphi \in (L^p)'$. Then there is $v \in L^{p'}$, unique, where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$ if $p > 1$, $p' = \infty$ if $p = 1$, such that

$$\langle \varphi, f \rangle_{(L^p)' \times L^p} = \int v f \, dx \qquad \forall f \in L^p.$$

Moreover $\|\varphi\|_{(L^p(\Omega))'} = \|v\|_{L^{p'}(\Omega)}$.

without proof

REMARK It is clear that if $v \in L^{p'}$ by Hölder's inequality

$$L^p \ni f \longmapsto \int v f \, dx$$

is a linear and continuous form for every $p \in [1, +\infty]$. Indeed

$$\left| \int v f \, dx \right| \leq \|v\|_{L^{p'}} \|f\|_{L^p}$$

What is to prove is that each linear form can be represented by an element of $L^{p'}$. This is true for every $p \in [1, +\infty)$, but not for $p = \infty$. Indeed

$$(L^\infty)' \underset{\neq}{\supseteq} L^1$$

TOWARDS WEAK TOPOLOGY

Now we see how to construct a topology in $\mathbb{R}^2$ in two ways, in one of them looking at $\mathbb{R}^2$ as a product space looking at it as $\mathbb{R}\times\mathbb{R}$.
The first way is to consider as basis the sets

$$B_r(x_0) = \{ x \in \mathbb{R}^2 \mid |x - x_0| < r \} .$$

The second is the following.

How to construct a topology on $\mathbb{R}^2$ starting from a topology on $\mathbb{R}$?
Suppose to have the topology τ on $\mathbb{R}$ generated by the balls $B_r(x) = (x-r, x+r)$, $x \in \mathbb{R}$, $r > 0$.
The idea is to consider the cheapest topology that makes the projections π_1 and π_2 continuous

$$\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R} \qquad \begin{aligned} \pi_1(x,y) &= (x,0) \\ \pi_2(x,y) &= (0,y) \end{aligned}$$

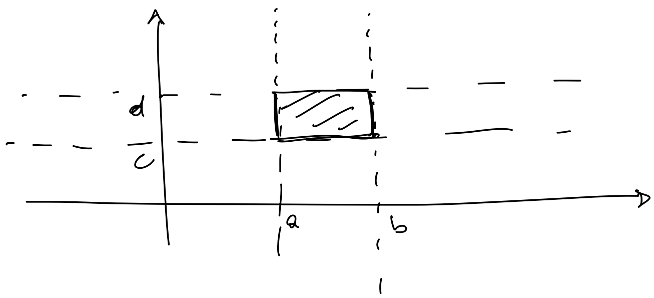
(in real $\pi_i : \mathbb{R}^2 \longrightarrow \mathbb{R}_i$ where $\mathbb{R}_1 = \mathbb{R}\times\{0\}$, $\mathbb{R}_2 = \{0\}\times\mathbb{R}$)

π_1 continuous means $\pi_1^{-1}(V)$ open in $\mathbb{R}^2$ where V is open in $\mathbb{R}$

Since $\pi_{u_1}^{-1}((a,b)) = (a,b) \times \mathbb{R}$ and

$\pi_{u_2}^{-1}((c,d)) = \mathbb{R} \times (c,d)$

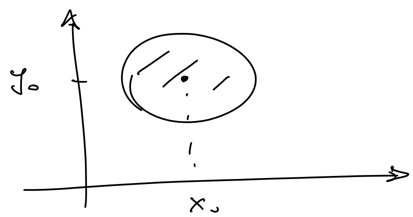
it is convenient to consider the coarsest topology which contains $(a,b) \times \mathbb{R}$, $\mathbb{R} \times (c,d)$ and finite intersections of sets of this type, i.e.



$(a,b) \times (c,d)$

Then we have two ways to construct the usual topology in $\mathbb{R}^2$, one is to consider as a basis the sets $B_r(x_0,y_0) = \{ (x,y) \in \mathbb{R}^2 \mid |(x,y) - (x_0,y_0)| < r \}$

and the other that has as a basis rectangles $(a,b) \times (c,d)$.



(These two bases generate the same topology!)

In $\mathbb{R}^3$ this way to construct the topology gives parallelepipeds obtained by the intersections of sets as

$(a,b) \times \mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times (c,d) \times \mathbb{R}$, $\mathbb{R} \times \mathbb{R} \times (e,f)$

Consider now a vectorial space of infinite dimension X, let's think to $\mathbb{R}^{\mathbb{N}} := \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$, the space of sequences in $\mathbb{R}$.

Then $x \in \mathbb{R}^{\mathbb{N}} \iff x = (x_1, x_2, x_3, \dots)$

If we have to consider sets of the type

$$\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \times (a,b) \times \mathbb{R} \times \dots \qquad (1)$$

and to generate the topology we have to consider only finite intersections of sets of this type, then

$$\mathbb{R} \times \dots \times \mathbb{R} \times (a_1,b_1) \times \mathbb{R} \times \dots \times \mathbb{R} \times (a_2,b_2) \times \mathbb{R} \times \dots \times (a_n,b_n) \times \mathbb{R} \times \dots$$

If we have a normed space, i.e. a space E endowed with a norm $\|\cdot\|_E$, we can construct a topology using the balls

$$B_r(x_0) = \{ x \in E \mid \|x - x_0\|_E < r \}$$

The topology generated by these balls is <u>different</u> from the topology generated by (finite intersection of) of sets like those in (1).

Given a Banach space X we have now two (not only two!) ways to construct a topology:

i) the basis is given by balls $B_r(x)$, $x \in X$, $r>0$
(called strong topology)

ii) the other one is that we are going to define
(and will be called weak topology)

Def Given a Banach space $(X, \|\cdot\|_X)$ we define the weak topology $\sigma(X, X')$ on X the coarsest topology that makes all $\varphi \in X'$ continuous (the coarsest topology among all topology that makes all $\varphi \in X'$ continuous).

REMARK If $\varphi \in X'$, φ is continuous with respect to the strong topology. Indeed if $x_n \to x$ w.r.t. the strong topology we have

$$|\langle \varphi, x_n \rangle - \langle \varphi, x \rangle| \leq \|\varphi\|_{X'} \|x_n - x\|_X \xrightarrow[n\to\infty]{} 0$$

Then the weak topology $\sigma(X, X')$ is coarser than the strong topology.

Theorem Consider a sequence $\{x_n\}_n \subseteq X$. Then

i) $x_n \to x$ in $\sigma(X, X') \iff \langle \varphi, x_n \rangle \to \langle \varphi, x \rangle$ $\forall \varphi \in X'$

ii) $x_n \to x$ strongly in $X \Rightarrow x_n \to x$ weakly in X

iii) $x_n \to x$ weakly in $X \Rightarrow \{x_n\}_n$ is bounded and $\|x\| \leq \liminf \|x_n\|$

iv) if $x_n \to x$ weakly in X

and $\varphi_n \to \varphi$ strongly in X'

$$\Rightarrow \quad \langle \varphi_n, x_n \rangle_{X' \times X} \longrightarrow \langle \varphi, x \rangle_{X' \times X}$$

proof : i) we do not prove point i) .

ii) this is obvious, since $\sigma(X, X')$ is coarser than the strong topology and anyway

$$| (f, x_n) - (f, x) | \leq \|f\|_{X'} \|x_n - x\|_X \xrightarrow[n \to +\infty]{} 0$$

iii) we do not see the proof of the fact that $\{x_n\}_n$ is bounded. For the second part :

$$| \langle f, x_n \rangle | \leq \|f\|_{X'} \|x_n\|_X$$

and taking the limit as $n \to +\infty$

$$|\langle f, x \rangle| \leq \|f\| \liminf_{n \to +\infty} \|x_n\|$$

iv) $$| \langle f_n, x_n \rangle - \langle f, x \rangle | =$$

$$= | \langle f_n, x_n \rangle - \langle f, x_n \rangle + (f, x_n) - (f, x) |$$

$$\leq | (f_n - f, x_n) | + | (f, x_n) - (f, x) |$$

$$\leq \underbrace{\|f_n - f\|}_{\to 0} \underbrace{\|x_n\|}_{\text{bounded}} + \underbrace{| \langle f, x_n \rangle - \langle f, x \rangle |}_{\to 0} \quad //$$

An open neighbourhood V of x_0 the weak topology is a set like

$$V = \{ x \in X \mid \langle f_i, x - x_0 \rangle < r , \quad i = 1, \dots n \}$$

(i.e. for a finite number of elements in X').

Then V always contains a line if X has infinite dimension (in fact, infinite lines, each one orthogonal to the others).

This has as a consequence the following facts:

- the set

$$S = \{ x \in X \mid \|x\| = 1 \} \quad \text{is not closed}$$

if X has infinite dimension

and $$\overline{S}^{\sigma(X,X')} = B_1(0) = \{ x \in X \mid \|x\| \leq 1 \}$$

- the set $B = \{ x \in X \mid \|x\| < 1 \}$ is not open

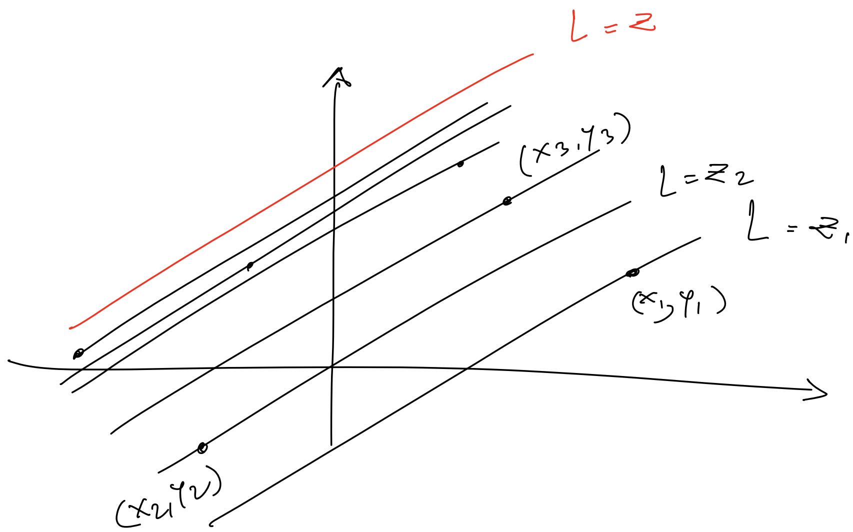
if X has infinite dimension

If we consider a linear function L on $\mathbb{R}^2$ and valued in $\mathbb{R}$ and we consider a sequence $\{(x_n, y_n)\}_n$ such that

$$L(x_n, y_n) \longrightarrow L(x_0, y_0)$$

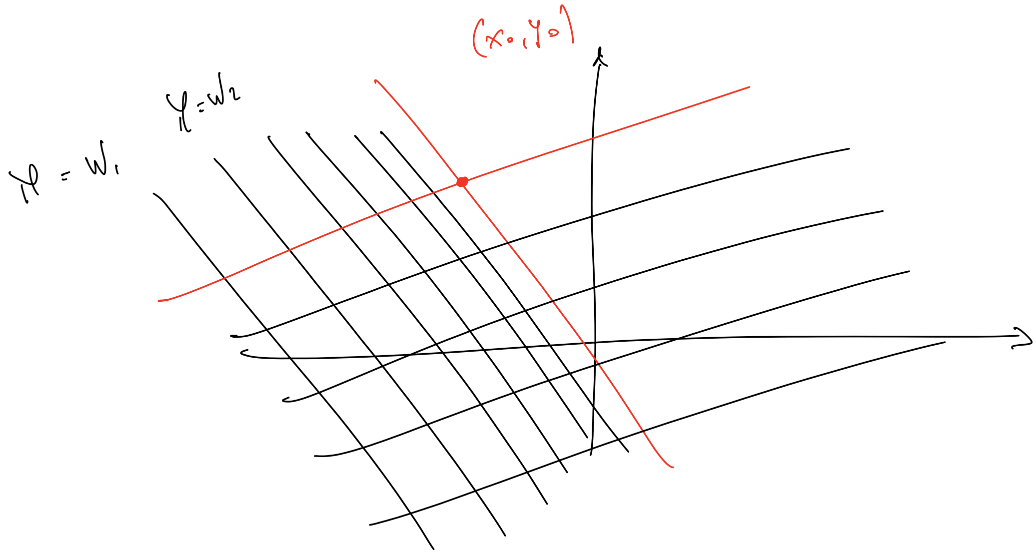
this means that the value $L(x_n, y_n) = z_n \in \mathbb{R}$ is converging to $z = L(x_0, y_0)$.

The linear map is constant on parallel lines and this means that (x_n, y_n) belong to the line where $L = z_n$



If we have another linear map Υ that is constant on lines not parallel to those where L is constant and we have

$$\Upsilon(x_n, y_n) = w_n \xrightarrow[n \to +\infty]{} w = \Upsilon(x, y)$$



We get that $(x_n, y_n) \longrightarrow (x_0, y_0)$ in $\mathbb{R}^2$.

But in infinite dimension having

$$\langle \varphi, x_n \rangle_{X' \times X} \longrightarrow \langle \varphi, x \rangle_{X' \times X}$$

even if for every $\varphi \in X'$ does not guarantee the strong convergence of $\{x_n\}_n$ to x.

In one case this is true. Before we see a definition

Def A Banach space $(X, \|\cdot\|)$ is said to be uniformly convex if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that

$$x, y \in X, \quad \|x\| \leq 1, \; \|y\| \leq 1, \quad \|x - y\| > \varepsilon$$

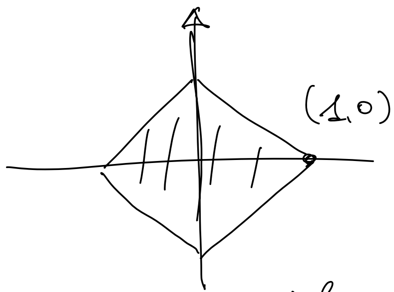
$$\Rightarrow \quad \left\| \frac{x+y}{2} \right\| < 1 - \delta$$

EXAMPLE $(\mathbb{R}^2, \|\cdot\|_p)$

$$\|(x,y)\|_p = \left(|x|^p + |y|^p\right)^{1/p}$$

$$\|(x,y)\|_\infty = \max\{|x|, |y|\}$$

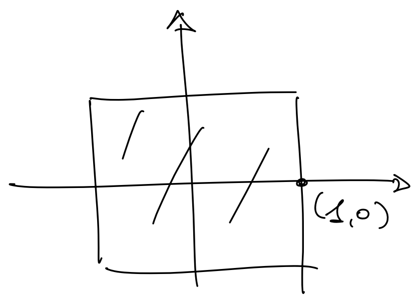
$B_1((0,0))$ in $(\mathbb{R}^2, \|\cdot\|_1)$

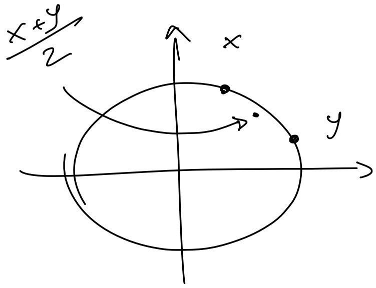


this is not uniformly convex

$(\mathbb{R}^2, \|\cdot\|_\infty)$

$B_1((0,0))$ in $(\mathbb{R}^2, \|\cdot\|_\infty)$



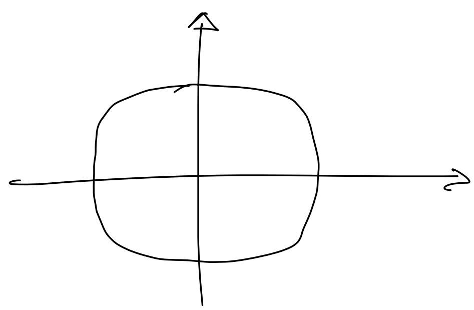


$B_1((0,0))$ in $(\mathbb{R}^2, \|\cdot\|_2)$

$B_1((0,0))$ in

$(\mathbb{R}^2, \|\cdot\|_p)$

$p > 2$



Theorem Let $(X, \|\cdot\|)$ be uniformly convex.

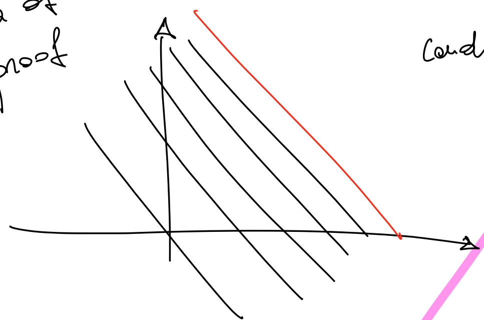
If $\{x_n\}_n$ is a sequence such that

i) $x_n \to x$ in $\sigma(X, X')$ and

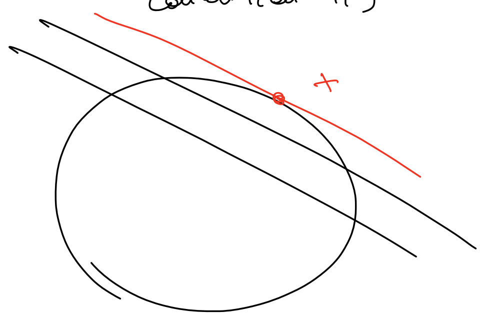
ii) $\limsup \|x_n\| \leq \|x\|$

Then $x_n \to x$ strongly in X.

idea of the proof



condition i)



condition ii)

EXAMPLES $L^1(\Omega)$ and $L^\infty(\Omega)$ are not uniformly convex

$L^p(\Omega)$, $1 < p < +\infty$, are uniformly convex

$$\left(L^p(\Omega)\right)' = L^{p'}(\Omega) \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

$p \in [1, +\infty)$

$$\left(L^\infty(\Omega)\right)' \supsetneq L^1(\Omega)$$

More precisely $\quad (L^p(\Omega))' \simeq L^{p'}(\Omega)$ for $p \in [1,+\infty)$

i.e. for every $\varphi \in (L^p)'$ $\exists!\ f \in L^{p'}(\Omega)$

s.t. $\quad \langle \varphi, u \rangle_{(L^p(\Omega))' \times L^p(\Omega)} = \int_\Omega f u \, dx \quad \forall u \in L^p(\Omega)$.

Moreover

$$\|\varphi\|_{(L^p(\Omega))'} = \|f\|_{L^{p'}(\Omega)} .$$

THEOREM (Weak and weak-* compactness)

If $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$, $p \in [1,+\infty]$

there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $f \in L^p(\Omega)$

such that

$$\int_\Omega f_{n_k}(x)\, g(x)\, dx \longrightarrow \int_\Omega f(x)\, g(x)\, dx$$

$$\forall g \in L^{p'}(\Omega) \quad \text{if } p \in [1,+\infty)$$

$$\forall g \in L^1(\Omega) \quad \text{if } p = +\infty$$

We will be more interested in the case $p = 2$.

REMARK Then, while $\{x \in \mathbb{R}^n \mid |x| \leq c\}$ is compact in $\mathbb{R}^n$ and a bounded sequence $\{x_n\}_n \subseteq \mathbb{R}^n$ admits a convergent subsequence, in a Banach space (of infinite dimension) one has to specify the topology.

If $X = L^p(\Omega)$, $1 \leq p < +\infty$, $\{x \in X \mid \|x\| \leq c\}$ is compact with respect to $\sigma(X, X')$ and a sequence $\{x_n\}_n \subseteq X$ such that $\|x_n\| \leq c$ admits a converging subsequence, i.e. there are $x \in L^p(\Omega)$ and $\{x_{n_k}\}_k$ s.t.

$$x_{n_k} \longrightarrow x \quad \text{in } L^p(\Omega) - \text{weak}.$$

SEPARABLE SPACES

A topological space X is said separable if there exist a subset D of X that is countable and dense in X.

EXAMPLES $L^p(\Omega)$ for $p \in [1, +\infty)$ is separable.