# SOBOLEV SPACES (dim $n = 1$)

We introduce now a notion of derivative that is "weaker" with respect to the classical one.

Consider first $n = 1$ and $I$ interval.

Consider for the moment $u \in C^1([a,b])$.

Then we have

$$\int_a^b u(x)\,\varphi'(x)\,dx = -\int_a^b u'(x)\,\varphi(x)\,dx$$

$$\forall \varphi \in C^1_c((a,b))$$

Suppose now to know that

$$\int_a^b u\,\varphi'\,dx = -\int_a^b v\,\varphi\,dx$$

for some $v \in C^0([a,b])$ and every $\varphi \in C^1_c((a,b))$

Then $\displaystyle\int_a^b u\,\varphi'\,dx = -\int_a^b u'\varphi\,dx$ and then

$$\int_a^b (u' - v)\,\varphi\,dx = 0 \quad \forall \varphi \in C^1_c((a,b))$$

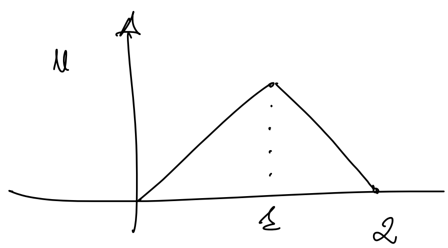
by which $v = u'$. In this spirit for $u \in L^p(I)$ we say that $v \in L^p(I)$ is a (the) weak derivative of $u$ if

$$\int_I u \varphi' \, dx = - \int_I v \varphi \, dx \quad \forall \varphi \in C_c^1((a,b)) \qquad (1)$$

EXERCISE Observe (= prove) that if "a" weak derivative exists, this is unique, and so it is "the" weak derivative.

EXAMPLE Consider the function defined in $[0,2]$

$$u(x) = \begin{cases} x & x \in [0,1] \\ -x+2 & x \in (1,2] \end{cases}$$




The function $v(x) = \begin{cases} 1 & x \in (0,1) \\ -1 & x \in (1,2) \end{cases}$

is the weak derivative of $u$.

Consider $u(x) = \sqrt{x}$, $x \in [0,1]$

Then $v(x) = \dfrac{1}{2\sqrt{x}}$ is the weak derivative in $L^1(0,1)$

REMARK Notice that for a continuous function the two statements

$$u \in C^0([a,b]) \quad \text{and} \quad u \in C^0((a,b))$$

have different meanings, for an $L^p$ function one writes $L^p(a,b)$ where $a$ and $b$ are the extremes of the interval without distinguishing between $(a,b)$, $(a,b]$, $[a,b)$, $[a,b]$ since the set $\{a,b\}$ has measure zero.

From now on $I$ will denote an interval in $\mathbb{R}$.

Def We define $W^{1,p}(I)$ the set

$$W^{1,p}(I) = \left\{ u \in L^p(I) \;\middle|\; \begin{array}{l} u \text{ admits a weak} \\ \text{derivative } v \in L^p(I) \\ (v \text{ satisfies } (1)) \end{array} \right\}$$

REMARK The weak derivative of $u$ is a distributional derivative which in particular is a function in $L^p(I)$.

One can iterate the argument and define $W^{m,p}(I)$ for $m \geq 2$ by induction:

$$W^{m,p}(I) = \{ u \in W^{m-1,p}(I) \mid u' \in W^{m-1,p}(I) \}$$

**Theorem** $W^{m,p}(I)$ is a Banach space $(p \in [1, +\infty])$ with the norm

$$\|u\|_{W^{m,p}(I)} := \|u\|_{L^p} + \|u'\|_{L^p} + \dots + \|u^{(m)}\|_{L^p}$$

or the equivalent one

$$\|u\|_{W^{m,p}(I)} := \left( \int |u|^p dx + \int |u'|^p dx + \dots + \int |u^{(m)}|^p dx \right)^{1/p}$$

**Theorem** $C^1(I) \cap L^p(\Omega)$ is dense in $W^{1,p}(I)$

$1 \leq p < +\infty$

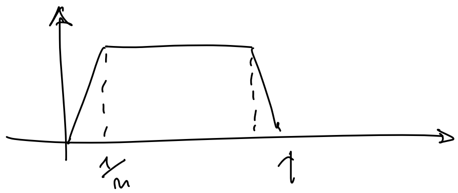
$C^m(I) \cap L^p(\Omega)$ " " in $W^{m,p}(I)$

$C^1(\bar{I})$ " " in $W^{1,p}(I)$

**REMARK** $C^1_c(I)$ is not dense in $W^{1,p}(I)$ (while is dense in $L^p(I)$).

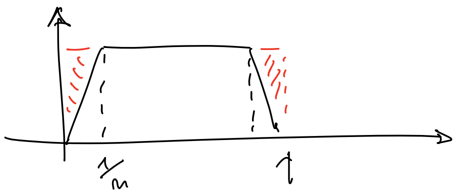
As example consider the function $u \equiv 1$ in $[0,1]$

If we consider the sequence (even if not $C^1$)




$$u_n(x) = \begin{cases} nx & x \in [0, 1/n] \\ 1 & x \in [1/n, 1 - 1/n] \\ -nx + n & x \in (1 - 1/n, 1] \end{cases}$$

Then $\| u_n - u \|_{L^p(0,1)} \longrightarrow 0 \quad \big( p \in [1, +\infty) \big)$




for $p = 1$ $\| u_n - u \|_{L^1}$ coincides with the area of the two red triangles

If we consider $\| u_n - u \|_{W^{1,1}(I)}$ we have to consider

$$\int_0^1 | u_n' - u' | \, dx = \int_0^{1/n} n \, dx + \int_{1-1/n}^1 n \, dx = 2$$

**Def** By $W_0^{1,p}(I)$ we denote the closure of $C_c^1(I)$ in $W^{1,p}(I)$ $\big( \text{for } p \in [1, +\infty) \big)$

**Lemma 1** Consider $u \in L^1_{loc}(I)$ such that

$$\int_I u \varphi \, dx = 0 \quad \forall \varphi \in C_c^1(I)$$

Then $u = 0$. (see the other part of the course)

**Lemma 2** Consider $u \in L^1_{loc}(I)$ such that

$$\int_I u \varphi' \, dx = 0 \quad \forall \varphi \in C^1_c(I)$$

Then there is a constant $c$ such that

$$u = c \quad \text{in } I.$$

proof: we prove first the second point. Fix a function $\psi$ such that $\int_I \psi = 1$. For every $w \in C_c(I)$ $\exists \varphi \in C^1_c(I)$ such that

$$\varphi' = w - \left(\int_I w\right) \psi .$$

Indeed the function $h = w - (\int_I w)\psi \in C_c(I)$ and moreover $\int_I h = 0$. Then the function

$$\varphi(x) := \int_a^x h(t)\, dt \quad \text{belongs to } C^1_c(I)$$

where $I = (a,b)$ ($a$ possibly $-\infty$).

Then

$$\int_I u \left[ w - \left(\int_I w\right) \psi \right] dx = 0 \quad \forall w \in C_c(I)$$

that is

$$0 = \int_I u(x)\, w(x)\, dx - \int_I u(y) \left( \int_I w(t)\, dt \right) \psi(y)\, dy$$

$$= \int_I u(x)\, w(x)\, dx - \int_I w(t) \left( \int_I u(y)\, \psi(y)\, dy \right) dt$$

$$= \int_I \left[ u - \int_I u \psi \right] w \qquad \forall\, w \in C_c(I)$$

Then $\quad u(x) - \int_I u \psi\, dt = 0 \quad$ for a.e. $x$

i.e. $u$ is constant

**Theorem A** (·) The space $W^{1,p}(I) \subseteq C^0(\bar{I})$ $\forall p \in [1, \infty]$ and the injection is continuous.

(··) the injection $W^{1,p}(I) \subseteq C^0(\bar{I})$ is compact for $p \in (1, +\infty]$

(···) the injection $W^{1,1}(I) \subseteq L^q(I)$ is compact for $q \in [1, +\infty)$

**COMMENT** $j : X \to Y$ is compact if given a bounded set $B \subseteq X$ the set $j(B)$ is relatively compact in $Y$. For instance, consider the sequence

$$u_h(x) = \begin{cases} 0 & x \in [-1,0] \\ hx & x \in (0, 1/h) \\ 1 & x \in [1/h, 1] \end{cases}, \quad h \in \mathbb{N}^*.$$

$\{u_h\}_h$ is bounded in $W^{1,1}(-1,1)$, each $u_h$ is continuous, but

$$u_h(x) \longrightarrow u(x) \quad \forall x \in [-1,1] \qquad u(x) = \begin{cases} 0 & x \in [-1,0] \\ 1 & x \in (0,1] \end{cases}$$

that is not continuous !

To prove in (o) that the injection is continuous means that there is $c>0$ such that

$$\|u\|_{L^\infty(I)} \leq c \, \|u\|_{W^{1,p}(I)} \qquad \left( \|\cdot\|_{C^0(\bar{I})} = \|\cdot\|_{L^\infty(I)} \right)$$

Once proved that $u$ is continuous one will have that

$$|u(x) - u(y)| \leq \int_y^x |u'(t)| \, dt \leq \left( \int_y^x |u'(t)|^p dt \right)^{1/p} |x-y|^{1/p'}$$

This condition if $p>1$ (if $p' < +\infty$) helps to get a condition that helps to get the compactness in $C^0(\bar{I})$ and this condition is lacking if $p=1$.

In the following we will prove only that a function $u \in W^{1,p}(I)$ is continuous, without proving the continuity of the injection.

REMARK A function $u \in W^{1,p}(I)$ is in fact an equivalence class and we know that we could have $u_1, u_2$ different functions, but

$$u_1 = u_2 \quad \text{a.e. in } I,$$

which belongs to the same class, represented by $u$.

The previous theorem says that in the class

$$[u] = \{ v : I \to \mathbb{R} \mid v = u \text{ a.e. in } I \}$$

there is a special function $\tilde{u}$, which is continuous in $I$.

Before proving the theorem we see a preliminary result and an auxiliary theorem.

Theorem (Beppo Levi's monotone convergence).

Let $\{f_n\}_n$ a sequence of measurable functions defined in $\Omega \subseteq \mathbb{R}^n$ and suppose that

•) $g \le f_1 \le f_2 \le f_3 \le \dots$ in $\Omega$ for some $g \in L^1(\Omega)$

••) $f_k(x) \to f(x)$ for a.e. $x \in \Omega$.

Then $f$ is measurable and

$$\lim_{k \to +\infty} \int_\Omega f_k \, dx = \int_\Omega f \, dx .$$

Without proof

REMARK Abroad this theorem is often referred to as Lebesgue's monotone converge theorem.

EXAMPLE We do not see the proof, but we see an example to show that assumptions are sharp.

First notice that assumption ••) is free: by monotonicity $\lim_{k \to +\infty} f_k(x)$ exists for a.e. $x \in \Omega$.

Consider

$$f_k(x) = -\frac{1}{k}\frac{1}{x} \qquad x \in (0,1)$$

$$\int_0^1 f_k(x)\, dx = -\frac{1}{k}\int_0^1 \frac{1}{x}\, dx = -\infty$$

but

$$\lim_{k \to +\infty} f_k(x) = 0 \qquad \forall\, x \in (0,1)$$

Theorem (Lebesgue's dominated convergence).

Let $\{f_k\}_k$ a sequence of measurable functions defined in $\Omega \subseteq \mathbb{R}^n$ and suppose that

•) $f_k(x) \to f(x)$ for a.e. $x \in \Omega$,

$\circ\circ$) $|f_k| \leq g \quad \forall k$ for some $g \in L^1(\Omega)$.

Then
$$\lim_{k\to\infty} \int_\Omega |f_k - f| \, dx = 0$$
and in particular
$$\lim_{k\to\infty} \int_\Omega f_k \, dx = \int_\Omega f \, dx$$

EXAMPLE As before, consider
$$f_k(x) = \begin{cases} 1/x & x \in (1/k, 1) \\ 0 & x \in (0, 1/k] \end{cases}$$

Then $f_k \to f$, $f(x) = 1/x$ with $x \in (0,1)$

$\int_0^1 f_k = \log k < +\infty$ while $\int_0^1 f \, dx = +\infty$

and therefore $\int_0^1 |f_k - f| \, dx = +\infty$

Notice that there is not a $g \in L^1(0,1)$ such that
$$|f_k| \leq g \quad \forall k.$$

**Lemma 3** Consider $g \in L^1_{loc}(I)$, $x_0 \in I$ and consider

$$v(x) = \int_{x_0}^{x} g(t)\, dt, \qquad x \in I.$$

Then $v \in C^0(I)$ and

$$\int_I v \varphi' \, dx = - \int_I g \varphi \qquad \forall \varphi \in C^1_c(I)$$

proof: suppose $I = (a,b)$. Let's see first that $v$ is continuous. Consider

$$\chi_k(x) := \chi_{\{g > k\}}(x) \qquad \text{where} \qquad \chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$$

Define $g_k := \chi_k g$ and observe that

$$|(1 - \chi_k) g| \leq |g| \quad \text{and} \quad (1-\chi_k) g \longrightarrow g \quad \text{a.e. in } I$$

because if $g \in L^1$ then $g$ is finite a.e.

Then

$$\int_I \chi_k g \, dx \xrightarrow[k \to \infty]{} 0 .$$

Fix $\varepsilon > 0$. We can find $\bar{k} \in \mathbb{N}$ such that

$$\left| \int_{\{g > k\}} g \, dt \right| < \varepsilon/2 \qquad \forall k \geq \bar{k}$$

Then, once fixed $x_0 \in I$, consider another

$x \in I$. We can write

$$\int_{x_0}^{x} |g(t)|\,dt = \int_{x_0}^{x} |\chi_{E_{\bar h}}\, g(t)|\,dt + \int_{x_0}^{x} |(1-\chi_{E_{\bar h}})\, g(t)|\,dt$$

$$< \varepsilon/2 + |x-x_0|\,\bar h .$$

since $|(1-\chi_{E_{\bar h}})\, g| \le \bar h$.

Then, choosing $|x-x_0|\,\bar h < \varepsilon/2$, i.e.

$$\delta < \frac{1}{\bar h}\,\frac{\varepsilon}{2}$$

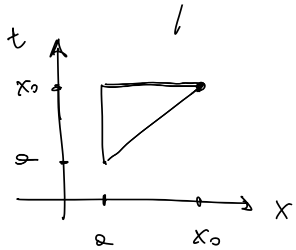
one gets that if

$$|x-x_0| < \delta \Rightarrow \left| \int_{x_0}^{x} g(t)\,dt \right| < \varepsilon$$

that is $\lim_{x \to x_0} \int_{x_0}^{x} g(t)\,dt = 0$.

Second part: $\int_I v\varphi'\,dx = \int_I \left( \int_{x_0}^{x} g(t)\,dt \right) \varphi'(x)\,dx =$

$$= -\int_{a}^{x_0} \int_{x}^{x_0} g(t)\,\varphi'(x)\,dt\,dx + \int_{x_0} \int_{x_0}^{x} g(t)\,\varphi'(x)\,dt\,dx .$$




By Fubini's theorem

$$\int_I v\,\varphi'\,dx = -\int_a^{x_0} g(t)\int_a^t \varphi'(x)\,dx\,dt + \int_{x_0}^b g(t)\int_t^b \varphi'(x)\,dx\,dt$$

$$= -\int_a^{x_0} g(t)\,\varphi(t)\,dt - \int_{x_0}^b g(t)\,\varphi(t)\,dt = -\int_I g\,\varphi\,dt$$

**EX** Prove that $v \in C^0(I)$ assuming $g \in L^p(I)$, $p > 1$.

<u>proof of the theorem</u>: given $u \in W^{1,p}(I)$, let's fix $x_0 \in I$ and consider

$$\bar u(x) = \int_{x_0}^x u'(t)\,dt$$

By the previous lemma

$$\int_I \bar u\,\varphi'\,dx = -\int_I u'\,\varphi\,dx \qquad \forall\, \varphi \in C^1_c(I)$$

Then

$$\int_I (u - \bar u)\,\varphi'\,dx = 0 \qquad \forall\, \varphi \in C^1_c(I)$$

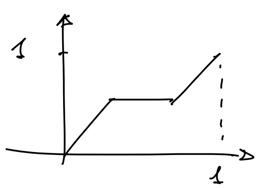
by which $u = \bar{u}$ a.e. in $I$.

By the theorem we deduce that if $u \in W^{1,p}(I)$, not only $u \in C(\bar{I})$, but it also holds

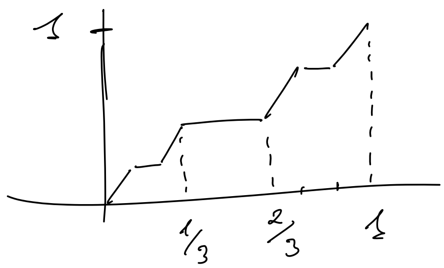
$$\boxed{u(x) - u(y) = \int_y^x u'(t)\,dt} \qquad (2)$$

Formula (2) it is not obvious at all. There are some continuous functions for which (2) does not hold even if they admit a derivative defined almost everywhere.

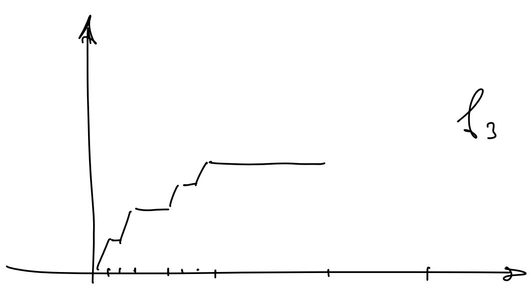
EXAMPLE Consider the sequence of continuous functions




$f_1$




$f_2$



$f_3$

The function $f_n : [0,1] \to [0,1]$ is continuous, increasing, $f_n(0) = 0$, $f_n(1) = 1$ and $f_n'(x) = 0$ for $x \notin C_n$, where

$$C_1 = [0,1] \setminus (1/3, 2/3)$$
$$C_2 = \left([0, 1/3] \setminus (1/9, 2/3)\right) \cup \left([2/3, 1] \setminus (7/9, 8/9)\right)$$
$$\vdots$$

$C_n$ is obtained removing the third central part from each interval (connected part) defining $C_{n-1}$.

It is possible to prove that $\{f_n\}$ uniformy converges to a continuous function $f : [0,1] \to [0,1]$, increasing, $f(0) = 0$, $f(1) = 1$, where $f'(x) = 0$ for every $x \in [0,1] \setminus C$ where

$$C = \bigcap_{n=1}^{+\infty} C_n \, , \qquad \text{but } |C| = 0 .$$

The set $C$ is called Cantor's set

Just for curiosity $C$ is a closed set (since it is intersection of closed sets), has the power of the continuum, does not contain any interval, its measure is 0 and, if one gives a suitable definition of

"dimension", can prove that $C$ has a dimension that turns out to be neither 1 nor 0, but an intermediate number.
Even if this procedure leads to construct a set of null measure, the same procedure can lead to construct sets of positive measure (removing less than one third each time).

The limit function $f$ is called Cantor function

(scala di Cantor in italiano)

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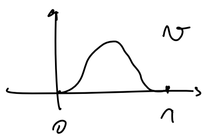
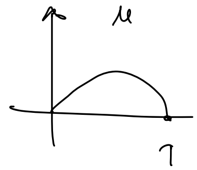
As a consequence of the previous theorem (Theorem A) we get

Theorem If $u \in W_0^{1,p}(I)$, $I$ is $(a,b)$, $(a,b]$, $[a,b)$, $[a,b]$ then $u(a) = u(b) = 0$.

COMMENT If $v \in C_c^1((0,1))$ then there is a compact $K$ of $(0,1)$ such that $v = 0$ in $(0,1) \setminus K$




while $u \in W_0^{1,p}(0,1)$ may be different from 0 in $(0,1)$ and 0 only in 0 and 1




REMARK If $I = \mathbb{R}$ $W_0^{1,p}(I) = W^{1,p}(I)$.

**Def** A function $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be Lipschitz continuous if there is a constant $c > 0$ such that

$$|u(x) - u(y)| \leq c\,|x-y| \qquad \forall\ x,y \in \Omega;$$

is said to be Hölder continuous of exponent $\alpha$ or $\alpha$-Hölder continuous if $\alpha \in (0,1)$ and there is $c > 0$ such that

$$|u(x) - u(y)| \leq c\,|x-y|^{\alpha} \qquad \forall\ x,y \in \Omega.$$

**EX** What happens if $\alpha > 1$? Classify all functions for which there is $c > 0$ such that

$$|u(x) - u(y)| \leq c\,|x-y|^{\alpha} \qquad \forall\ x,y \in \Omega$$

with $\alpha > 1$.

Other consequences of Theorem A are the following.

**Theorem** If $u \in W^{1,p}(I)$ then

$u$ is Lipschitz continuous if $p = \infty$

$u$ is $\alpha$-Hölder continuous if $p \in (1, +\infty)$

where $\alpha = \frac{1}{p'} = 1 - \frac{1}{p}$.

**COMMENT** If $p = 1$ $u$ is continuous, but a priori not Hölder continuous (see EXERCISES).

proof: from (2) we derive

$$|u(x) - u(y)| = \left| \int_y^x u'(t)\, dt \right| \leq \int_y^x |u'(t)|\, dt$$

By Hölder's inequality we get

$$|u(x) - u(y)| \leq \left( \int_y^x |u'(t)|^p\, dy \right)^{1/p} |x-y|^{1/p'} \quad \text{if } p \in (1, +\infty),$$

$$|u(x) - u(y)| \leq \|u'\|_\infty \, |x-y| \quad \text{if } p = \infty.$$

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Theorem B (Poincaré inequality)

If $u \in W_0^{1,p}(I)$, $1 \le p < \infty$, $I$ bounded,

then $(I = (a,b))$

$$\|u\|_{L^p(I)} \le (b-a)\,\|u'\|_{L^p(I)}.$$

proof: suppose $I = (a,b)$, $u \in C^0([a,b])$ and $u(a) = 0$. Then

$$u(x) - u(a) = \int_a^x u'(t)\,dt \qquad \text{by which if } p = 1$$

$$|u(x)| \le \int_a^b |u'(t)|\,dt \Rightarrow \int_a^b |u| \le (b-a)\int_a^b |u'| \,;$$

if $p \in (1,+\infty)$

$$|u(x)| \le \int_a^b |u'(t)|\,dt \le \left(\int_a^b |u'(t)|^p\,dt\right)^{1/p} (b-a)^{1/p'}$$

$$\int_a^b |u(x)|^p\,dx \le \int_a^b \left(\int_a^b |u'(t)|^p\,dt \quad (b-a)^{p/p'}\right) dx$$

$$\Rightarrow \quad \|u\|_{L^p(I)} \le (b-a)\,\|u'\|_{L^p(I)}. \qquad //$$

REMARK Notice that in fact also for $u \in W^{1,\infty}(a,b)$ with $u(a) = u(b) = 0$ one has

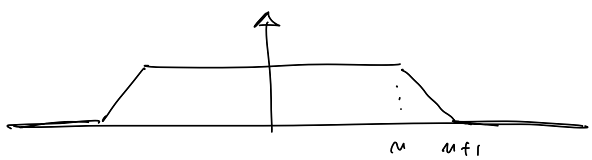
$$|u(x)| \le \int_a^x |u'(t)|\,dt \implies |u(x)| \le (b-a)\,\|u'\|_{L^\infty(I)}.$$

<u>REMARK</u> Notice that if $I$ is unbounded the previous result cannot be true, i.e. it is not possible to find a constant $c>0$ such that

$$\|u\|_{L^p(I)} \le c\,\|u'\|_{L^p(I)} \qquad \text{for } p \in [1,+\infty).$$

For instance, consider the sequence

$$u_n(x) = \begin{cases} 1 & \text{if } x \in [-n,n] \\ x+n+1 & \text{if } x \in (-n-1,-n) \\ -x+n+1 & \text{if } x \in (n,n+1) \\ 0 & \text{if } x \notin (-n-1,n+1) \end{cases}$$




Clearly

$$\int_{\mathbb{R}} |u_n'(t)|^p\,dt = 2 \qquad \forall n \in \mathbb{N},\ \forall p \in [1,\infty)$$

but

$$\int_{\mathbb{R}} |u_n|^p \xrightarrow[n\to+\infty]{} +\infty.$$

**Def** Given a vectorial space $X$ and two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $X$ we say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there are two constants $c_1, c_2 > 0$ such that

$$\|x\|_1 \le c_1 \|x\|_2 \quad \text{and} \quad \|x\|_2 \le c_2 \|x\|_1 \quad \forall x \in X.$$

By the Poincaré inequality the quantity $\|u'\|_{L^p(a,b)}$ turns out to be a norm equivalent to $\|u\|_{W^{1,p}(a,b)}$ in the space $W_0^{1,p}(a,b)$.

# DUAL SPACE OF $W_0^{1,p}$

The dual space of $W_0^{1,p}(I)$, $I$ interval, $p \in [1,\infty)$, is denoted by $W^{-1,p'}(I)$.

If $p=2$, $\left(W_0^{1,2}(I)\right)' = \left(H_0^1(I)\right)' = H^{-1}(I)$.

REMARK We observe an important thing, only for $p=2$.

We have already seen that the dual space of $L^p(I)$ is isomorphic to $L^{p'}(I)$ $(1 < p < \infty)$ and in particular $(L^2(I))'$ is isomorphic to $L^2(I)$.

Call $\mathcal{R} : L^2(I) \to (L^2(I))'$ the Riesz isomorphism which is linear and continuous and bijective.

Moreover we have that the immersion

$$i : H_0^1(I) \longrightarrow L^2(I), \qquad i(u) = u$$

is linear and continuous and the immersion

$$j : (L^2(I))' \longrightarrow H^{-1}(I), \qquad j(u) = u$$

is linear and continuous. Indeed, given $f \in (L^2(I))'$, $j(f)$ turns out to be a linear and continuous form on $H_0^1(I)$ acting as follows:

$$\langle j(f), v \rangle_{H^{-1} \times H^1_0} := \int r^{-1}(f)\, v \, dx \qquad \forall v \in H^1_0(I)$$

where $r^{-1}(f) = u$ for some (unique) $u \in L^2$.

Then

$$|\langle j(f), v \rangle| \leq \|u\|_{L^2} \|v\|_{L^2} \leq \|u\|_{L^2} \|v\|_{H^1}$$

and then $\|j(f)\|_{H^{-1}} \leq \|u\|_{L^2}$, or

$$\| j \circ r^{-1}(u) \|_{H^{-1}} \leq \|u\|_{L^2}$$

! As the map $r$, one can prove that there is a linear, continuous and bijective map

$$R : H^1_0(I) \longrightarrow H^{-1}(I)$$

but usually the space $H^{-1}(I)$ is not identified with $H^1_0(I)$, even if this is possible.
This is because one prefers to identify $(L^2)'$ with $L^2$ itself. Now we try to explain.

Since $H^1_0 \subset L^2$ then $(L^2)' \subset H^{-1} = (H^1_0)'$

we have the three maps

$$i : H_0^1 \to L^2 \qquad \text{injective}$$
$$r : L^2 \to (L^2)' \qquad \text{bijective}$$
$$j : (L^2)' \to H^{-1} \qquad \text{injective}$$

If we identify, by $r$, $(L^2)'$ with $L^2$ and think of $i$ and $j$ as identity we can write

$$H_0^1 \subset L^2 \subset H^{-1}$$

**Theorem** Consider $p \in [1, +\infty)$ and $F \in W^{-1,p'}(I)$. Then there are $f_0, f_1 \in L^{p'}(I)$ (! not unique) such that

$$\langle F, u \rangle_{W^{-1,p'} \times W_0^{1,p}} = \int f_0 u + \int f_1 u' \quad \forall u \in W_0^{1,p}$$

Moreover $$\| F \|_{W^{-1,p'}} = \max \{ \| f_0 \|_{L^p}, \| f_1 \|_{L^p} \}$$

without proof

The non-uniqueness of $f_0$ and $f_1$ is easily seen if we observe that if we consider

$$h_0 \in W^{1,p}, \quad h_0' = g_0$$

we can write

$$\int f_0 u + \int f_1 u' = \int f_0 u + \int f_1 u' + \int g_0 u - \int g_0 u =$$

$$= \int f_0 u + \int f_1 u' + \int g_0 u - \int h_0' u =$$

since $u \in W_0^{1,p}(I)$

$$= \int f_0 u + \int f_1 u' + \int g_0 u + \int h_0 u' =$$

$$= \int (f_0 + g_0) u + \int (f_1 + h_0) u'$$

$$= \int \tilde{f}_0 u + \int \tilde{f}_1 u'$$

with $\tilde{f}_0 = f_0 + g_0$, $\tilde{f}_1 = f_1 + h_0$

If $I$ is bounded, for instance $I = (a,b)$, we can consider

$$h_0(x) = -\int_a^x f_0(t)\, dt \qquad (h_0' = -f_0)$$

and get that $F$ can be represented as (taking $\tilde{f}_0 = 0$)

$$\langle F, u \rangle = \int f_1 u' .$$

This is coherent with the fact that $W_0^{1,p}(a,b)$ can be endowed with the norm ($a, b \in \mathbb{R}$)

$$\| u' \|_{L^p}$$