SOBOLEV SPACES (dim $n = 1$)

We introduce now a notion of derivative that is "weaker" with respect to the classical one.

Consider first $n = 1$ and I interval.

Consider for the moment $u \in C^1([a,b])$.

Then we have

$$\int_a^b u(x)\,\varphi'(x)\,dx = -\int_a^b u'(x)\,\varphi(x)\,dx$$

$$\forall \varphi \in C_c^1((a,b))$$

Suppose now to know that

$$\int_a^b u\,\varphi'\,dx = -\int_a^b v\,\varphi\,dx$$

for some $v \in C^0([a,b])$ and every $\varphi \in C_c^1((a,b))$

Then $\displaystyle\int_a^b u\,\varphi'\,dx = -\int_a^b u'\varphi\,dx$ and then

$$\int_a^b (u' - v)\,\varphi\,dx = 0 \quad \forall \varphi \in C_c^1((a,b))$$

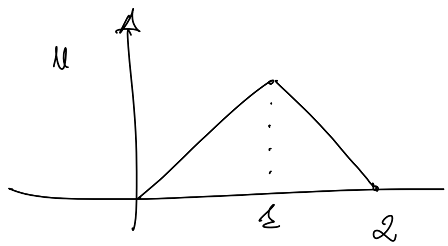
by which $v = u'$. In this spirit for $u \in L^p(I)$ we say that $v \in L^p(I)$ is a (the) weak derivative of u if

$$\int_I u \varphi' \, dx = - \int_I v \varphi \, dx \quad \forall \varphi \in C_c^1((a,b)) \qquad (1)$$

EXERCISE Observe (= prove) that if "a" weak derivative exists, this is unique, and so it is "the" weak derivative.

EXAMPLE Consider the function defined in $[0,2]$

$$u(x) = \begin{cases} x & x \in [0,1] \\ -x+2 & x \in (1,2] \end{cases}$$



The function $v(x) = \begin{cases} 1 & x \in (0,1) \\ -1 & x \in (1,2) \end{cases}$

is the weak derivative of u.

Consider $u(x) = \sqrt{x}$, $x \in [0,1]$

Then $v(x) = \dfrac{1}{2\sqrt{x}}$ is the weak derivative in $L^1(0,1)$

REMARK Notice that for a continuous function the two statements

$$u \in C^0([a,b]) \quad \text{and} \quad u \in C^0((a,b))$$

have different meanings, for an L^p function one writes $L^p(a,b)$ where a and b are the extremes of the interval without distinguishing between (a,b), $(a,b]$, $[a,b)$, $[a,b]$ since the set $\{a,b\}$ has measure zero.

From now on I will denote an interval in $\mathbb{R}$.

Def We define $W^{1,p}(I)$ the set

$$W^{1,p}(I) = \left\{ u \in L^p(I) \;\middle|\; \begin{array}{l} u \text{ admits a weak} \\ \text{derivative } v \in L^p(I) \end{array} \right\}$$

$(v \text{ satisfies } (1))$

REMARK The weak derivative of u is a distributional derivative which in particular is a function in $L^p(I)$.

Theorem A The space $W^{1,p}(I) \subseteq C^0(\bar{I})$ $\forall p \in [1,\infty]$