SOBOLEV SPACES IN HIGHER DIMENSION $(n \geq 2)$

Def The space $W^{1,p}(\Omega)$, $\boxed{\Omega \overset{\circ}{\subseteq} \mathbb{R}^n}$ $(n \geq 1)$ is defined as

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \exists\, v_1, \dots v_n \in L^p(\Omega) \text{ s.t.} \\ \displaystyle\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega v_i\, \varphi\, dx \\ \forall \varphi \in C^1_c(\Omega),\ i = 1, \dots n \end{array} \right\}$$

Usually when $p = 2$ often one writes

$$H^1(\Omega) \quad \text{instead of} \quad W^{1,2}(\Omega).$$

The function v_i is called weak derivative of u (w.r.t. x_i) and denoted by

$$D_i u \quad \text{or} \quad D_{x_i} u$$

By Du one denotes the weak gradient

$$Du = (D_1 u, \dots D_n u)$$

(sometimes also the classical notations $\partial_i u$, $\partial_{x_i} u$, $\frac{\partial u}{\partial x_i}$ and ∇u are used, unless one wants to stress the difference between classical (strong) and weak derivatives)

The space $W^{1,p}(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{W^{1,p}} := \left(\int_\Omega \left[|u|^p + |Du|^p \right] dx \right)^{1/p}$$

for $p \in [1, +\infty)$

$$\|u\|_{W^{1,\infty}} := \|u\|_\infty + \|Du\|_\infty$$

$H^1(\Omega)$ is a Hilbert space with the scalar product

$$(u,v)_{H^1} := \int_\Omega \left(u v + (Du, Dv) \right) dx$$

where for two vectors of $\mathbb{R}^n$

$$(\xi, \eta) := \sum_{i=1}^n \xi_i \eta_i$$

Moreover $W^{1,p}(\Omega)$ is separable for $p \in [1, +\infty)$

IMPORTANT THING

If a function u belongs to $L^p(\Omega) \cap Lip_{loc}(\Omega)$ admits a classical derivative $\frac{\partial u}{\partial x_i}$ defined almost everywhere in Ω. If $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$,

then this is also the weak derivative $D_i u$.

! This does not mean that an arbitrary function u for which $\frac{\partial u}{\partial x_i}$ exists almost everywhere has the weak derivative $D_i u$.

Example: the Cantor's function f ($f'(x) = 0$ a.e.)

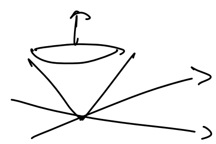
$$\int_0^1 f \varphi' \, dx \text{ cannot be equal to } - \int_0^1 f' \varphi \, dx = 0 \quad \forall \varphi$$

The function u has to be locally Lipschitz continuous (and f is not)

EXAMPLES of function in $W^{1,p}$

(•) Consider the set $B_1(0,0) = \{ (x,y) \in \mathbb{R}^2 \mid |(x,y)| < 1 \}$ and some functions defined in this domain.

The first example is the function



$$v(x,y) = \sqrt{x^2+y^2}$$

$v \in L^\infty(B_1(0,0))$ and then in every $L^p(B_1(0,0))$ $\forall p \in [1, \infty]$

$$D_x v = \frac{x}{\sqrt{x^2+y^2}} \quad , \quad D_y v = \frac{y}{\sqrt{x^2+y^2}}$$

$$|Dv| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = 1$$

then $v \in W^{1,p}(B_1(0,0)) \quad \forall\, p \in [1, \infty]$

since $|Dv|^p = \left(|D_x v|^2 + |D_y v|^2\right)^{p/2}$

and then if $|Dv| \in L^p$, also $D_x v$ and $D_y v \in L^p$.

(••) $\quad w(x,y) = \dfrac{1}{(x^2+y^2)^{\alpha/2}} \qquad \alpha > 0$

$$\int_{B_1(0,0)} |w|^p \, dx\,dy = \int_{B_1(0,0)} \left(\frac{1}{(x^2+y^2)^{1/2}}\right)^{\alpha p} dx\,dy =$$

$$= \int_0^{2\pi} d\theta \int_0^1 \frac{1}{\rho^{\alpha p}}\, \rho \, d\rho < +\infty \quad \text{iff} \quad \alpha p - 1 < 1$$

i.e. $p < 2/\alpha$

(then α has to be less than 2)

$$\frac{\partial w}{\partial x} = -\alpha \frac{x}{(x^2+y^2)^{\frac{\alpha}{2}+1}}$$

$$\frac{\partial w}{\partial y} = -\alpha \frac{y}{(x^2+y^2)^{\frac{\alpha}{2}+1}}$$

$$|Dw| = \alpha \left(\frac{x^2}{(x^2+y^2)^{\alpha+2}} + \frac{y^2}{(x^2+y^2)^{\alpha+2}}\right)^{1/2} = \alpha \frac{1}{(x^2+y^2)^{\frac{\alpha+1}{2}}}$$

$\Rightarrow |Dw| \in L^p(B_1(0,0))$ iff $(\alpha+1)p - 1 < 1$,

that is $p < \frac{2}{\alpha+1}$ (then α has to be less than 1)

(∘∘∘) Consider the function u of two variables

that expressed in polar coordinates is

$$\tilde{u}(\rho,\theta) = \operatorname{sen}\theta, \quad \theta\in[0,2\pi], \quad \rho\in(0,1]$$

To express $\tilde{u}$ in Cartesian coordinates we use

$$\left(\rho(x,y) = \sqrt{x^2+y^2}\right)$$

$$\theta(x,y) = \begin{cases} \operatorname{arctg}\frac{y}{x} & \text{for } x>0,\ y\geq 0 \\ \pi/2 & \text{for } x=0,\ y>0 \\ \operatorname{arctg}\frac{y}{x}+\pi & \text{for } x<0 \\ \frac{3}{2}\pi & \text{for } x=0,\ y<0 \\ \operatorname{arctg}\frac{y}{x}+2\pi & \text{for } x>0,\ y<0. \end{cases}$$

$$u(x,y) := \operatorname{sen}(\theta(x,y))$$

Clearly $u \in L^p(B_1(0,0))$ $\forall\, p\in[1,+\infty]$

$$D_x u = \cos(\theta(x,y)) \, \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) =$$

$$= -\cos(\theta(x,y)) \, \frac{y}{x^2+y^2}$$

$$D_y u = \cos(\theta(x,y)) \, \frac{1}{1+\left(\frac{y}{x}\right)^2} \, \frac{1}{x} =$$

$$= \cos(\theta(x,y)) \, \frac{x}{x^2+y^2}$$

Then $$|D_x u|^2 + |D_y u|^2 = \left(\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} \right) \cos^2(\theta(x,y))$$

$$= \frac{\cos^2(\theta(x,y))}{x^2+y^2}$$

and $$|Du|^p \leq \frac{1}{(x^2+y^2)^{p/2}}$$

Jacobian

$$\int_{B_1(0,0)} |Du|^p \, dx\,dy \leq \int_0^{2\pi} d\theta \int_0^1 \frac{1}{\rho^p} \rho \, d\rho = 2\pi \int_0^1 \frac{1}{\rho^{p-1}} \, d\rho$$

and this is finite if $p < 2$.

Conclusion: the function u belongs to

$$W^{1,p}(B_1(0,0)) \quad \forall \, p \in [1,2)$$

and is (not defined and) not continuous in $(0,0)$.

Summarizing:

(•) v is continuous, but not differentiable (everywhere)

(••) w and its derivatives are not defined in a point where w and $D_i w$ go to ∞

(•••) u is bounded, not defined in $(0,0)$ and even if we define u in $(0,0)$ the extension is not continuous!

but nevertheless they are $W^{1,p}(B_1(0,0))$ for some p.

<u>Theorem</u> Given $u \in L^p(\Omega)$ there is a sequence $\{u_n\}_n \subseteq C_c^\infty(\mathbb{R}^n)$ such that

$1 \leq p < +\infty$

$$u_n\big|_\Omega \longrightarrow u \quad \text{in } L^p(\Omega).$$

Given $u \in W^{1,p}(\Omega)$ there is a sequence $\{u_n\}_n \subseteq C_c^\infty(\mathbb{R}^n)$ such that

$$u_n\big|_\Omega \longrightarrow u \quad \text{in } L^p(\Omega),$$

$$Du_n\big|_\omega \longrightarrow Du\big|_\omega \quad \text{in } L^p(\omega) \ \forall\, \omega \subset\subset \Omega.$$

<u>idea of the proof</u>: we extend $u : \Omega \to \mathbb{R}$ in this way

$$\bar{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

given $\rho \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp} \rho = B_1(0)$,

$\int_{\mathbb{R}^n} \rho(x)\,dx = 1$, define

$$\rho_h(x) := \frac{1}{h^n}\,\rho\left(\frac{x}{h}\right)$$

[REMARK $\int \rho_h\,dx = 1$, $\operatorname{supp} \rho_h = B_{\frac{1}{h}}(0)$,

$\rho_h \to \delta_0$ in the distributional sense]

You know that $\rho_h * \bar{u}\big|_\Omega \longrightarrow u$ in $L^p(\Omega)$

where

$$f * g(x) = \int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy = \int_{\mathbb{R}^n} g(y)\,f(x-y)\,dy$$

Verify !

and $\rho_h * \bar{u} \in C_c^\infty(\mathbb{R}^n)$.

In the proof, that we do not see, it is used:

suppose f admits a derivative with respect to x_i

then $f * g$ " " " " " " "

(f differentiable in the classical or in the weak sense)

and

$$\frac{\partial}{\partial x_i}(f*g)(x) = \frac{\partial f}{\partial x_i} * g\ (x)$$

Then one can prove that

$$\frac{\partial}{\partial x_i}(\rho_\varepsilon * \bar{u}) = \rho_\varepsilon * \overline{\frac{\partial u}{\partial x_i}} \qquad (3)$$

and not

$$\frac{\partial}{\partial x_i}(\rho_\varepsilon * \bar{u}) = \rho_\varepsilon * \frac{\partial \bar{u}}{\partial x_i} \qquad (\text{why?})$$

Then, using the result for L^p function and (3), one conclude that (just because of (3)!)

that
$$\left(\frac{\partial}{\partial x_i} \rho_\varepsilon * \bar{u}\right)\Big|_\omega = \frac{\partial \rho_\varepsilon}{\partial x_i} * \bar{u}\Big|_\omega \longrightarrow u\Big|_\omega$$

in $L^p(\omega)$ and not $L^p(\Omega)$!

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Theorem Given $u \in W^{1,p}(\Omega)$ there is a sequence $\{u_n\}_n \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$ such that

$p \in [1,+\infty)$

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\Omega).$$

Note that we do not assert $\{u_n\}_n \subseteq C^\infty(\overline{\Omega})$.

[see Evans-Gariepy] (Ω is an open set)

Theorem Suppose Ω open, bounded, $\partial\Omega$ Lipschitz

Given $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$,

there is a sequence

$\{u_n\}_n \subseteq W^{1,p}(\Omega) \cap C^\infty(\overline{\Omega})$ such that

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\Omega).$$

[see Evans-Gariepy]

One can also consider a sequence $\{u_n\} \subseteq C^\infty_c(\mathbb{R}^n)$

such that $u_n|_\Omega \to u$ in $W^{1,p}(\Omega)$

[see Brezis, Corollary IX.8]

SPACES $W^{k,p}$

As in dimension 1 we have:

Def $$W^{k,p}(\Omega) = \left\{ u \in W^{k-1}(\Omega) \;\middle|\; D_i u \in W^{k-1,p}(\Omega) \right\}$$

$k \geq 1$, $p \in [1, \infty]$

Usually one denotes by $H^k(\Omega)$ the space $W^{k,2}(\Omega)$.

for $p \in (1, +\infty) \rightarrow$ By $W^{1,p}_0(\Omega)$ we denote the closure of $C^1_c(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$.

In the following we summarise many results that can be proved in many steps and whose prove we do not see.
We confine to two cases, even if other types of sets can be considered.

Theorem (Sobolev inequalities)

Let Ω be $\mathbb{R}^n$ or a bounded open set with Lipschitz boundary. Then if

i) $1 \le p < n$

$$W^{1,p}(\Omega) \subset L^q(\Omega) \qquad \forall\, q \in [p, p^*]$$

with continuous injection and where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \qquad \text{i.e.} \quad p^* = \frac{pn}{n-p} \;;$$

ii) $p = n$

$$W^{1,n}(\Omega) \subset L^q(\Omega) \qquad \forall\, q \in [n, +\infty)$$

with continuous injection;

iii) $p > n$, possibly ∞,

$$W^{1,p}(\Omega) \subset L^\infty(\Omega)$$

with continuous injection. Moreover

there is a costant c (depending only on p, n, Ω) such that

$$|u(x) - u(y)| \leq c \, \|Du\|_{L^p(\Omega)} \, |x-y|^\alpha$$

with $\alpha = 1 - \frac{n}{p}$.

(for $p = \infty$ $\alpha = 1$).

REMARK Continuous injection means that

$j : W^{1,p} \to L^q$ is continuous, i.e.

$$\|u\|_{L^q} = \|j(u)\|_{L^q} \leq c \, \|u\|_{W^{1,p}} .$$

Theorem Let Ω bounded with $\partial\Omega$ Lipschitz. Then

$u \in W^{1,\infty}(\Omega)$ if and only if $f \in \mathrm{Lip}(\bar{\Omega})$.

without proof

EXAMPLE Here Ω cannot be unbounded.

For instance, $u(x) = x$ $(x \in \mathbb{R})$ is Lipschitz continuous, $u' \in L^\infty$, but

$$u \notin L^\infty(\mathbb{R}) \; !$$

Corollary Let Ω be $\mathbb{R}^n$ or a bounded open set with Lipschitz boundary.

Let $m \geq 1$ and $1 \leq p \leq \infty$. Then if

i) $\frac{1}{p} - \frac{m}{n} > 0 \Rightarrow W^{m,p} \hookrightarrow L^q$

for $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$

ii) $\frac{1}{p} - \frac{m}{n} = 0 \Rightarrow W^{m,p} \subset L^q \quad \forall q \in [p, +\infty)$

iii) $\frac{1}{p} - \frac{m}{n} < 0 \Rightarrow W^{m,p} \subset L^\infty$

with continuous injections. Moreover in the last case, i.e. if $m > \frac{n}{p}$, we have that

$$u \in C^{m - [\frac{n}{p}] - 1, \alpha}(\overline{\Omega}) \quad \text{where}$$

$$\alpha = \begin{cases} [\frac{n}{p}] - \frac{n}{p} + 1 & \text{if } \frac{n}{p} \notin \mathbb{N} \\ \text{any positive number} < 1 & \text{if } \frac{n}{p} \in \mathbb{N} \end{cases}$$

and there is a constant $c = c(n, p, \alpha, \Omega)$

Def $\|u\|_{C^{k,\alpha}(\Omega)} =$

$$= \sum_{j=0}^{k} \|D^j u\|_\infty + \sup_{\substack{i=1,\dots n\\ x,y\in\Omega}} \frac{|D_i^k u(x) - D_i^k u(y)|}{|x-y|^\alpha}$$

For $k=0$

$$\|u\|_{C^{0,\alpha}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

EXAMPLE Being in $L^q(\Omega)$ $\forall q \in [1,+\infty)$ or $(p,+\infty)$ does not imply to belong to L^∞!

Consider, for instance,

$$x \in \mathbb{R}^2, \quad u(x) = \begin{cases} \log|\log|x|| & \text{if } |x| < \frac{1}{e} \\ 0 & \text{if } |x| \geq \frac{1}{e} \end{cases}$$

EX Verify that $u \in H^1(\mathbb{R}^2)$

Since $u \in H^1(\mathbb{R}^2)$ (after doing the exercise!)

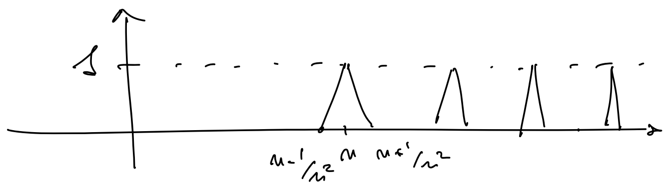
then $\left(\frac{1}{p} - \frac{m}{n} = \frac{1}{2} - \frac{1}{2} = 0\right)$ $u \in L^q$ $\forall q \in [2,+\infty)$,

but u is clearly unbounded.

EXAMPLE A function $u \in L^p(\mathbb{R}^n)$ not necessarily satisfies

$$\lim_{|x| \to +\infty} u(x) = 0 .$$

For instance, consider the function



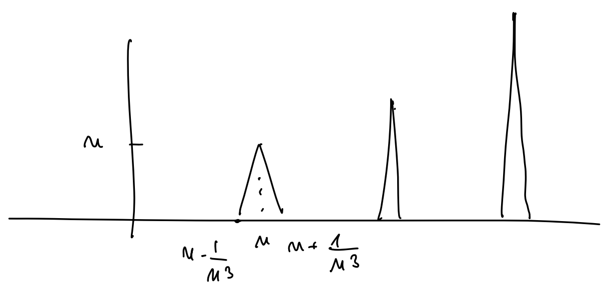
The area of one triangle is $\frac{1}{n^2}$, then

$$\int_0^{+\infty} u(x)\,dx = \sum \frac{1}{n^2} < +\infty$$

but $\lim_{x \to +\infty} u(x)$ is not 0.

One can also consider

$$u(n) \to +\infty$$



If $u \in W^{1,p}(\mathbb{R}^n)$ with $p > n$ we have $W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ with continuous injection.

Since we can find a sequence $\{u_n\}_n \in C^1_c(\mathbb{R}^n)$ such that

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\mathbb{R}^n)$$

and $\|u_n - u\|_{L^\infty} \leq C \|u_n - u\|_{W^{1,p}}$

we derive that $u_n \to u$ uniformly in $\mathbb{R}^n$

and being u uniform limit of functions compactly supported then $\lim_{|x| \to +\infty} u(x) = 0$.

EXAMPLE Why p^*? The Sobolev injection when $1 \leq p < n$ can be motivated as follows.

Suppose $u \in W^{1,p}(\mathbb{R}^n)$. If an inequality like

$$\circledast \qquad \|u\|_{L^q(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \text{for some } q$$

holds for u, taking $\{u_\lambda\}_{\lambda>0}$, $u_\lambda(x) = u(\lambda x)$, we have

$$\int_{\mathbb{R}^n} |u_\lambda(x)|^q \, dx \overset{\lambda x = y}{=} \int_{\mathbb{R}^n} |u(y)|^q \frac{1}{\lambda^n} \, dy$$

and $(D_i u_\lambda(x) = \lambda D_i u(\lambda x))$

$$\int_{\mathbb{R}^n} |Du_\lambda(x)|^p \, dx = \int_{\mathbb{R}^n} |\lambda Du(\lambda x)|^p \, dx = \lambda^{p-n} \int_{\mathbb{R}^n} |Du(y)|^p \, dy$$

If $\circledast$ has to hold for some q

$$\frac{1}{\lambda^{n/q}} \|u\|_{L^q} \leq C \lambda^{1-\frac{n}{p}} \|Du\|_{L^p}$$

$\Leftrightarrow \quad \|u\|_{L^q} \leq c \; \lambda^{1+\frac{n}{q}-\frac{n}{p}} \|Du\|_{L^p}$

and the exponent $1+\frac{n}{q}-\frac{n}{q}$ has to be non-negative, otherwise (letting $\lambda \to +\infty$) the inequality could not hold. Then

$$1+\frac{n}{q}-\frac{n}{p} \geq 0 \quad \Rightarrow \quad q \leq \frac{pn}{n-p} = p^*.$$

COMPACTNESS The injections seen above, besides being continuous, are also compact provided Ω is bounded. The following result holds.

Theorem (Rellich - Kondrachov)

Consider Ω bounded, $\partial\Omega$ Lipschitz. If

i) $p < n \quad \Rightarrow \quad W^{1,p}(\Omega) \subset L^q(\Omega) \quad \forall q \in [1, p^*)$

ii) $p = n \quad \Rightarrow \quad W^{1,p}(\Omega) \subset L^q(\Omega) \quad \forall q \in [1, +\infty)$

iii) $p > n \quad \Rightarrow \quad W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$

with compact injections.

Without proof

REMARK The result holds true also for $p = +\infty$, but in this case is known as Ascoli - Arzelà theorem.

REMARK Notice that the spaces $L^p(\Omega)$ are boxed into each other if Ω is bounded (or with finite measure):

$$L^\infty(\Omega) \subset L^p(\Omega) \subset L^q(\Omega) \subset L^1(\Omega)$$

$$\forall\, p,q \in (1,+\infty)\,,\quad p > q$$

Indeed, for $q \in [1,+\infty)$ and $p > q$

$$\int_\Omega |u|^q dx \leq \left(\int_\Omega |u|^p dx \right)^{q/p} |\Omega|^{\frac{p-q}{p}}$$

using Hölder's inequality taking the power p/q of $|u|^q$ and $\frac{p}{p-q}$ of 1 $\left(\frac{q}{p} + \frac{p-q}{p} = 1 \right)$

Taking the power $\frac{1}{q}$ we get

$$\|u\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p}} \|u\|_{L^p(\Omega)} .$$

Letting $p \to +\infty$, or directly, we also get

$$\|u\|_{L^q(\Omega)} \leq |\Omega|^{1/q} \|u\|_{L^\infty(\Omega)} .$$

REMARK Observe that in the Sobolev inequalities we have, for instance when $1 \leq p < n$, that $W^{1,p}(\Omega)$ continuously embeds in $L^q(\Omega)$ for every $q \in [p, p^*]$. But Ω can be unbounded, for instance $\mathbb{R}^n$. In that case $W^{1,p}(\mathbb{R}^n)$ with $p > 1$ does not immerse in $L^q(\mathbb{R}^n)$ with $q < p$. For instance the function

$$u : \mathbb{R} \to \mathbb{R}, \qquad u(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ \frac{1}{x} & \text{if } |x| \geq 1 \end{cases}$$

belongs to $W^{1,p}(\mathbb{R})$ $\forall p > 1$, but $u \notin L^1(\mathbb{R})$.

In the statement of the theorem of Rellich - Kondrachov we are saying two things: the first one (that could be said before) is that if Ω is bounded with $\partial\Omega$ Lipschitz the injection $W^{1,p} \hookrightarrow L^q$ does not hold only for $q \in [p, p^*]$ but for $q \in [1, p^*]$.

The second one is that the injection is compact, but not for p^*!, in $L^q(\Omega)$ for $q \in [1, p^*)$.

This means that if we have a sequence

$\{u_h\}_{h\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ $\quad$ (Ω bdd, $\partial\Omega$ lip)

and $\quad \|u_n\|_{W^{1,p}(\Omega)} \leq c$, $\quad c \in \mathbb{R}$,

there exist a subsequence $\{u_{h_j}\}_{j\in\mathbb{N}}$ and $u \in L^q(\Omega)$ $\forall\, q \in [1, p^*)$ such that

$$u_{h_j} \longrightarrow u \quad L^q(\Omega) \qquad (\text{strongly!})$$

REMARK IN THE REMARK $\quad$ In fact we have more!

The function u belongs to $W^{1,p}(\Omega) \cap L^q(\Omega)$.

To see that $u \in W^{1,p}(\Omega)$ we can observe that:

$$\|u_h\|_{L^p} + \|D_1 u_h\|_{L^p} + \dots + \|D_n u_h\|_{L^p} \leq c$$

Then, by the weak compactness seen in the previous chapter, we have that there is a subsequence $\{u_{h_j}^{(0)}\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty} u_{h_j}^{(0)} = u^{(0)} \quad \text{weakly in } L^p(\Omega)$$

and n other subsequences $\{u_{h_j}^{(1)}\}, \dots \{u_{h_j}^{(n)}\}$ such that

$$\lim_{j\to+\infty} u_{h_j}^{(k)} = v^{(k)} \quad \text{weakly in } L^p(\Omega)$$

for each $k \in \{1, \dots n\}$

First observe that we can select a same subsequence for all these limits:
indeed, since $\{u_{k_j}^{(0)}\}_{j\in\mathbb{N}}$ is bounded, we can select a subsequence from $\{u_{k_j}^{(0)}\}_{j\in\mathbb{N}}$ and this subsequence will converge to $u^{(0)}$ in L^p.
Then we can suppose to have selected one subsequence $\{u_{k_j}\}_j$ such that

$$\begin{aligned}
\lim_j u_{k_j} &= u \\
\lim_{j\to+\infty} D_1 u_{k_j} &= v^{(1)} \\
&\vdots \\
\lim_{j\to\infty} D_n u_{k_j} &= v^{(n)}
\end{aligned}
\quad \left|\quad \begin{array}{l} \text{weakly} \\ \text{in } L^p(\Omega) \\ \text{for some} \\ u, v^{(1)}, \dots v^{(n)} \\ \text{in } L^p(\Omega) \end{array}\right.$$

Then for every $\varphi \in C_c^\infty(\Omega)$ $\left(D_k\varphi \in C_c^\infty\right)$ we have

$$\int_\Omega u_{k_j} D_k\varphi \, dx \xrightarrow[j\to\infty]{} \int u \, D_k\varphi \, dx$$

$$\left(\left(\quad - \int D_k u_{k_j} \varphi \, dx \xrightarrow[j\to\infty]{} -\int v^{(k)} \varphi \, dx\right.\right.$$

Then

$$\int u \, D_k \varphi \, dx = - \int v^{(k)} \varphi \, dx \qquad \forall \varphi \in C_c^\infty(\Omega)$$

and then $\qquad v^{(k)} = D_k u \qquad$ for each k

(*) Then, if $\{u_h\}_h$ bounded in $W^{1,p}(\Omega)$,

Ω bounded and $\partial\Omega$ Lipschitz,

$\{u_h\}$ converges, up to a subsequence,

strongly in $L^q(\Omega) \quad \forall q \in [1, p^*)$

and weakly in $W^{1,p}(\Omega)$

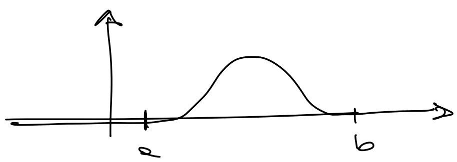
to a function $u \in W^{1,p}(\Omega) \cap L^q(\Omega)$.

Clearly analogous conclusions hold for $p \geq n$.

We conclude this remark with a counterexample.

If Ω is unbounded the Rellich compactness result clearly may be false.

Consider u like that



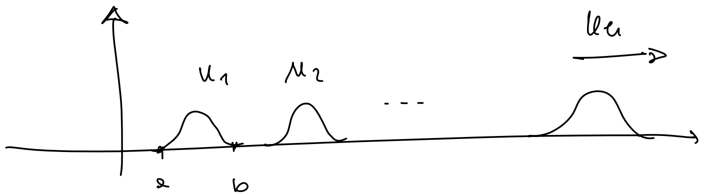
$\left(u \in C_c^\infty(\mathbb{R}) \quad \text{supp}(u) \subseteq (a,b) \right)$

and define

$$u_h(x) := u(x-h).$$

Then $\|u_h\|_{W^{1,p}} = \|u\|_{W^{1,p}} \quad \forall h \in \mathbb{N}$

but $\{u_h\}$ does not converge!



EXAMPLE We show now an example to see how

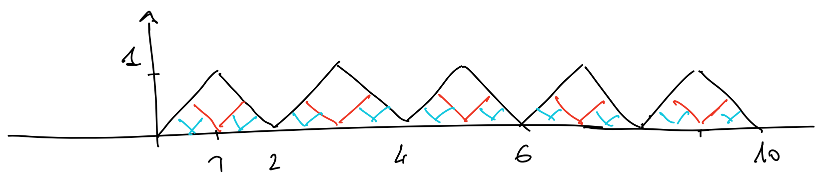
⊛ is true and the derivatives do not converge strongly, but only weakly!

Let u_1

u_2

u_3

⋮

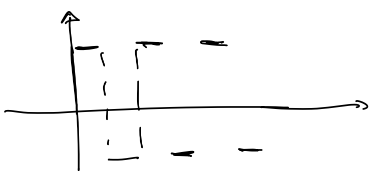


in such a way that $|u_h'(x)| = 1$ for a.e. $x \in \mathbb{R}$

We have that (is not necessary to extract a subseq.)

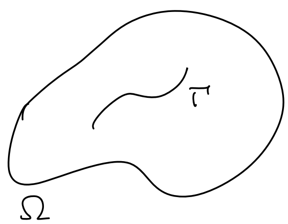
$$u_h \longrightarrow 0 \quad \text{strongly in } L^p(0,10) \ \forall p \in [1,\infty]$$

but $u_h' \longrightarrow 0$ only weakly

u_h' 

BOUNDARY VALUES FOR SOBOLEV FUNCTIONS

We have seen that a function in a Sobolev space is defined almost everywhere. So, if we have a function $u : \Omega \longrightarrow \mathbb{R}$, is it possible to trace the value of u in $\Gamma \subseteq \overline{\Omega}$ subset of codimension 1? What does this mean?



In particular, is it possible to consider the value of u in $\partial\Omega$?

Def Given $\Omega \overset{\circ}{\subseteq} \mathbb{R}^n$ we define $W_0^{1,p}(\Omega)$, $1 \leq p < +\infty$, the closure of $C_c^1(\Omega)$ (or $C_c^k(\Omega)$, or $C_c^\infty(\Omega)$) with respect to the topology induced by the norm $\|\cdot\|_{W^{1,p}(\Omega)}$.
If $p = 2$ we often write $H_0^1(\Omega)$ instead of $W_0^{1,2}(\Omega)$

The space $W_0^{1,p}(\Omega)$ endowed with the norm $\|\cdot\|_{W^{1,p}}$ is a Banach space, $H_0^1(\Omega)$ endowed with the scalar product $(\cdot,\cdot)$ of $H^1(\Omega)$ is a Hilbert space.

REMARK $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$, while in general

$$W_0^{1,p}(\Omega) \subsetneq W^{1,p}(\Omega)$$

REMARK $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, as well as $C_c^k(\Omega)$ for every $k \geq 1$.

Theorem (Poincaré inequality)

Consider $p \in [1, +\infty)$, Ω open and bounded. Then there exists a positive constant c, depending only on Ω and p, such that

$$\|u\|_{L^p(\Omega)} \leq c \, \|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W_0^{1,p}(\Omega).$$

If moreover Ω is connected, $p \in [1, +\infty]$, there is $c > 0$ as above such that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq c \, \|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W^{1,p}(\Omega)$$

where $$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx = \fint_\Omega u \, dx .$$

REMARK If Ω bounded the quantity $\left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}$ turns out to be a norm on $W_0^{1,p}(\Omega)$.

Moreover $(u,v) := \int_\Omega (\nabla u, \nabla v)\, dx$ is a scalar product on $H^1_0(\Omega)$.

EX Show that the inequality (for some $p \geq 1$)

$$\|u\|_{L^p(\Omega)} \leq c\, \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}_0(\Omega)$$

cannot hold if Ω is unbounded.

Till now we have seen how to consider the value 0 at the boundary of a set, at least for $p < +\infty$. But this way is almost a trick and does not completely solve the problem.

Theorem Consider Ω open, bounded with Lipschitz boundary. Consider $p \in [1, +\infty)$.

There exists a bounded linear operator

$$T : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega; \mathcal{H}^{n-1})$$

such that

$$Tu = u \text{ in } \partial\Omega \qquad \forall u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$$

Moreover

$$\int_\Omega u \operatorname{div} \phi \, dx = - \int_\Omega (Du, \phi)\, dx + \int_{\partial\Omega} (\phi, \nu)\, Tu \, d\mathcal{H}^{n-1}$$

for every $u \in W^{1,p}(\Omega)$ and $\phi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, being ν the unit outer normal to $\partial\Omega$ (which is defined only almost everywhere in $\partial\Omega$)

[see Evans - Gariepy, pag 133]

One can also prove that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$

$$\lim_{r\to 0} \fint_{B_r(x)\cap\Omega} |u(y) - Tu(x)|\, dy = 0$$

and then

$$\boxed{Tu(x) = \lim_{r\to 0} \fint_{B_r(x)\cap\Omega} u(y)\, dy}$$

In this sense we can give a meaning to the value of u in $\partial\Omega$ (in an analogous way one can "trace" u in $\Gamma \subseteq \overline{\Omega}$, Γ of codimension 1)

Def Tu is called **trace** of u and is defined up to sets of $\mathcal{H}^{n-1} \llcorner \partial\Omega$ measure zero.

THE DUAL SPACE OF $W_0^{1,p}$

It is denoted by $W^{-1,p'}$ (p' s.t. $\frac{1}{p}+\frac{1}{p'}=1$ for $p\in(1,\infty)$, $p'=\infty$ if $p=1$)

$p\in[1,+\infty)$

If $p=2$ we denote

$$(H_0^1(\Omega))' =: H^{-1}(\Omega)$$

and one has

$$H_0^1(\Omega)\subseteq L^2(\Omega)\subseteq H^{-1}(\Omega)$$

with dense and continuous injections.

Notice that if Ω is bounded one also has

$$W_0^{1,p}(\Omega)\subseteq L^2(\Omega)\subseteq W^{-1,p'}(\Omega) \qquad \text{for } p\leq 2$$

and in fact for p such that $p^*=2$, i.e. $p\geq\frac{2n}{n+2}$

Observe that $\frac{2n}{n+2}<2$ for $n\geq 3$.

Theorem $F\in W^{-1,p'}(\Omega) \Rightarrow \exists\, f_0, f_1, \ldots f_n \in L^{p'}(\Omega)$ s.t.

$$\langle F, v\rangle_{W^{-1,p'}\times W_0^{1,p}} = \int_\Omega f_0 v + \sum_{j=1}^{n}\int_\Omega f_j D_j v$$

$\forall v\in W_0^{1,p}(\Omega)$

We conclude with another results that connect the notion of trace with $W^{1,p}_0$.

Theorem Assume Ω bounded with Lipschitz boundary and consider $u \in W^{1,p}(\Omega)$, $p \in [1,+\infty)$. Then

$$u \in W^{1,p}_0(\Omega) \iff Tu = 0 \text{ in } \partial\Omega.$$

Without proof

[see EVANS, pag 259]

Question: how can be represented an element of $(W^{1,p}_0(\Omega))' = W^{-1,p'}$?

We focus our attention to the case $p=2$.

Observe that given $u \in H^1_0(\Omega)$, $D_i u \in L^2$, and therefore the vector $(u, D_1 u, \dots D_n u)$ may be seen as an element of $H^{-1}(\Omega)$ in the sense that, given $v \in H^1_0$,

$$\langle F_u, v \rangle_{H^{-1} \times H^1_0} := \int_\Omega u v \, dx + \sum_{j=1}^n \int_\Omega D_j u \, D_j v \, dx$$

is a linear and continuous map from $H^1_0(\Omega) \to \mathbb{R}$.

Indeed one can prove that for every $F \in H^{-1}(\Omega)$ there are $f_0, f_1, \dots f_n \in L^2(\Omega)$ (not unique) such that

$$\langle F, u \rangle_{H^{-1} \times H^1_0} = \int_\Omega f_0 u \, dx + \int_\Omega f_1 D_1 u + \dots + \int f_n D_n u \, dx \quad (\bullet)$$

Observe that, since

$$H^1_0(\Omega) \subset H^1(\Omega) \subset L^2(\Omega)$$

We have

$$\left(L^2(\Omega)\right)' \subseteq \left(H^1(\Omega)\right)' \subset \left(H^1_0(\Omega)\right)' = H^{-1}(\Omega)$$

and therefore also each element of $(H^1(\Omega))'$ can be represented as in $(\bullet)$

GENERAL FACT

Consider $p \in (1,+\infty)$ and $X = L^p(\Omega)$

$X = W^{1,p}(\Omega)$ (or $W^{k,p}(\Omega)$, $k \in \mathbb{N}$).

If $\{u_n\} \subseteq X$ and

i) $u_n \rightharpoonup u$ weakly in X

ii) $\|u_n\|_X \longrightarrow \|u\|_X$

then $u_n \to u$ strongly in X.

proof: we prove it only for $p=2$, i.e. when X is a Hilbert space.

i) $u_n \rightharpoonup u$ means $(u_n, v)_X \longrightarrow (u,v)_X \quad \forall v \in X$

where $(\cdot,\cdot)_X$ denotes the scalar product in X.

ii) $\|u_n\|_X \to \|u\|_X$ implies that

$$\|u_n\|_X^2 = (u_n, u_n)_X \longrightarrow (u,u)_X = \|u\|_X^2$$

Then $\|u_n - u\|_X^2 = (u_n - u, u_n - u)_X =$

$$= \|u_n\|_X^2 + \|u\|_X^2 - 2(u_n, u)_X$$

Taking the limit we have that

$$(u_n, u)_X \longrightarrow (u, u)_X = \|u\|_X^2 \quad \text{by i)}$$

$$\|u_n\|_X^2 \longrightarrow \|u\|_X^2 \quad \text{by ii)}$$

and then $\lim_n \|u_n - u\|_X^2 = 0$. //