VARIATIONAL FORMULATION OF SOME PROBLEMS

We start from a toy problem, the most simple.
Suppose you want to solve

$$(P) \qquad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

for some function f, Ω bounded.
We already know what a classic solution (even if in general does not exists) is:
provided that $f \in C^0(\Omega)$ a solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is a function satisfying (P)
If you multiply the equation in (P) by $v \in C^1_c(\Omega)$ assuming $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ we get

$$\int_\Omega -\Delta u \, v \, dx = \int_\Omega f v \, dx$$

and

$$\int_\Omega -\Delta u \, v \, dx = \int_\Omega (Du, Dv) \, dx + \underbrace{\int_{\partial\Omega} (Du, \nu) v \, d\mathcal{H}^{n-1}}_{= 0}$$

for $v \in C^1_c(\Omega)$

Consider $\boxed{f \in L^2(\Omega)}$

A weak solution of (P) is a function $u \in H^1_0(\Omega)$ s.t.

$$\int_\Omega (Du, Dv)\, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H^1_0(\Omega) \qquad (1)$$

In (1) there are both the two informations: that u is the solution and that $u = 0$ in $\partial\Omega$.

! Every classic solution is also a weak solution.

Indeed if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and $u = 0$ in $\partial\Omega$ (Ω bounded) then $u \in H^1_0(\Omega)$ and satisfies (1) for every $v \in C^1_c(\Omega)$. By density it satisfies (1) also $\forall\, v \in H^1_0(\Omega)$.

! Every weak solution $u \in H^1_0(\Omega) \cap C^2(\Omega) \cap C^0(\bar{\Omega})$ is also a classic solution if $f \in L^2(\Omega)$.

Indeed, being $u \in C^0(\bar{\Omega})$ and $u \in H^1_0(\Omega)$ then $u = 0$ in $\partial\Omega$.

Moreover, being $u \in C^2(\Omega)$, one gets

$$-\int_\Omega \Delta u \, v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H^1_0(\Omega)$$

and, in particular, $\forall\, v \in C^1_c(\Omega)$. Then

$$-\Delta u = f \quad \text{a.e. in } \Omega.$$

But being $\Delta u \in C^0(\Omega)$ we get that

$$-\Delta u(x) = f(x) \quad \forall\, x \in \Omega \quad \left(\text{and } f \in C^0(\Omega)\right)$$

REMARK For the equation $-\Delta u = f$ one looks for a solution in $H^1_0(\Omega) = W^{1,2}_0(\Omega)$ because eq. (1) is to hold in particular for $v = u$ and then

$$\int_\Omega |Du|^2 dx = \int_\Omega f u \, dx$$

The left hand term makes sense if $u \in W^{1,2}$!

One could study also equations like

$$-\Delta_p u = -\operatorname{div}\left(|Du|^{p-2} Du\right) = f \qquad \begin{pmatrix} -\Delta_p \text{ is called} \\ p\text{-Laplacian} \end{pmatrix}$$

Multiplying by $v \in C^1_c(\Omega)$ one gets $\quad 1 < p < +\infty$

$$\int_\Omega |Du|^{p-2} (Du, Dv)\, dx = \int_\Omega f v \, dx$$

and taking $v = u$ one would get $\int |Du|^p dx$.

In this case the right space where to look for a solution would be $W^{1,p}$!

Anyway we will confine to study linear problems in H^1.

But why looking for a weak solution in a Sobolev space? First, we have seen (or we will see) that C^2 is not the right space where to look for a solution of (P). Indeed, even if $f \in C^0(\Omega)$, not always $u \in C^2(\Omega)$ exists such that $-\Delta u = f$.

Another, apparently trivial, reason is that looking for a solution in a wider space gives more chances to find a solution (it is easier to prove existence theorems).

Finally, one can solve equation with data f that could not admitted in the classical case (f may be L^2 and also $H^{-1}(\Omega)$!)

Let's see a couple of examples in dimension 1.

1) Consider $f : (-1,1) \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if } x \in (-1,0) \\ 1 & \text{if } x \in (0,1) \end{cases}$$

! f is not continuous

We want to solve in $H^1_0(-1,1)$

$$\begin{cases} -u'' = f & \text{in } (-1,1) \\ u(-1) = u(1) = 0 \end{cases}$$

We start from the equation: u has to satisfy

in $(-1,0)$ $\quad u'' = 0 \Rightarrow u' = a \Rightarrow u(x) = ax + b$

in $(0,1)$ $\quad u'' = -1 \Rightarrow u'(x) = -x + c \Rightarrow u(x) = -\frac{x^2}{2} + cx + d$

Now: if $u \in H^1 \Rightarrow u$ is continuous (also in $x = 0$), then we have the three conditions

$u(-1) = 0$
$u(1) = 0$
u cont. in 0

$$\begin{cases} -a + b = 0 \\ -\frac{1}{2} + c + d = 0 \\ b = d \end{cases}$$

by which

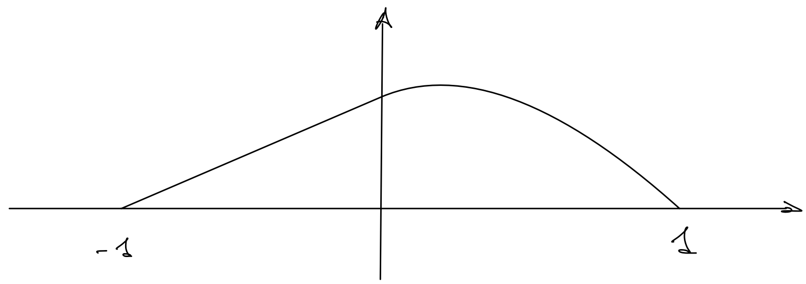
$$\begin{cases} b = a \\ d = a \\ c = \frac{1}{2} - a \end{cases} \Rightarrow u(x) = \begin{cases} ax + a & x \in (-1,0) \\ -\frac{x^2}{2} + \left(\frac{1}{2} - a\right)x + a & x \in (0,1) \end{cases}$$

Notice that $u'(x) = \begin{cases} a & x \in (-1,0) \\ -x + \frac{1}{2} - a & x \in (0,1) \end{cases}$

If u'' has a discontinuity of the first type, a jump discontinuity, in 0 then u' is to be continuous. Then $a = \frac{1}{2} - a \Rightarrow a = \frac{1}{4}$.

Finally we get that

$$u(x) = \begin{cases} \frac{1}{4}x + \frac{1}{4} & \text{for } x \in (-1,0) \\ -\frac{x^2}{2} + \frac{1}{4}x + \frac{1}{4} & \text{for } x \in (0,1) \end{cases}$$



This is the solution. Notice that $f \in L^2(-1,1)$ and $u \in C^0([-1,1])$, $u \in C^1([-1,1])$, but $u \notin C^2([-1,1])$.

Ex Verify that u is a weak solution

2) We know that an element in $H^{-1}(a,b)$ may be represented as

$$H^1_0(a,b) \ni v \mapsto \int_a^b f_1 v' \, dx \qquad (f_0 = 0)$$

Notice that $\delta \in H^{-1}(\mathbb{R})$ or $H^{-1}(-1,1)$, where

$$\langle \delta, v \rangle_{H^{-1}(-1,1) \times H^1_0(-1,1)} = v(0) \qquad (v \text{ is continuous !}$$

Who is the function f_1 associated to δ ?

It is simply

$$f_1(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases}$$

or $f_1 + c$, $c \in \mathbb{R}$. Indeed

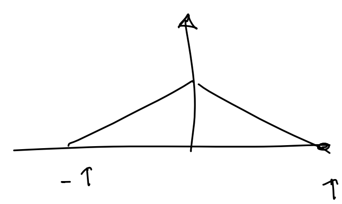
$$\int_{-1}^{1} f_1 v' \, dx = \int_{-1}^{0} v'(x) dx = v(0) - v(-1) = v(0)$$

If we want to solve

$$\begin{cases} -u'' = \delta & \text{in } (-1,1) \\ u(-1) = u(1) = 0 \end{cases}$$

We find that a solution (in fact, <u>the</u> solution)

$$u(x) = \begin{cases} \frac{1}{2}(x+1) & x \in [-1,0] \\ -\frac{1}{2}(x-1) & x \in [0,1] \end{cases}$$



and $u \notin C^2$ and $u \notin C^1$.

3) Given $a(x) = \begin{cases} \alpha & x \in [0, 1/2) \\ \beta & x \in [1/2, 1] \end{cases}$ we want to solve

$\alpha, \beta > 0$

$$\begin{cases} -\dfrac{d}{dx}\left(a \dfrac{du}{dx}\right) = 0 \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

If u is a solution we have that $a u' = c$ for some $c \in \mathbb{R}$

In $(0, 1/2)$ we have

$$\alpha u'(x) = c \Rightarrow u'(x) = \frac{c}{\alpha} \Rightarrow u(x) = \frac{c}{\alpha} x + k_1, \quad k_1 \in \mathbb{R}$$

In $(1/2, 1)$ we have

$$\beta u'(x) = c \Rightarrow u'(x) = \frac{c}{\beta} \Rightarrow u(x) = \frac{c}{\beta} x + k_2, \quad k_2 \in \mathbb{R}$$

Then

$$u(x) = \begin{cases} \dfrac{c}{\alpha} x + k_1 & \text{in } (0, 1/2) \\ \dfrac{c}{\beta} x + k_2 & \text{in } (1/2, 1) \end{cases}$$

for some constants c, k_1, k_2

Now we impose the boundary conditions:

$$u(0) = 0 \Rightarrow k_1 = 0$$

$$u(1) = 1 \Rightarrow \frac{c}{\beta} + k_2 = 1 \Rightarrow k_2 = 1 - \frac{c}{\beta}$$

Since we look for a solution in $H^1(0,1) \subseteq C^0([0,1])$ we impose continuity in $\frac{1}{2}$, i.e.

$$\frac{c}{\alpha}\frac{1}{2} + k_1 = \frac{c}{\beta}\frac{1}{2} + k_2$$

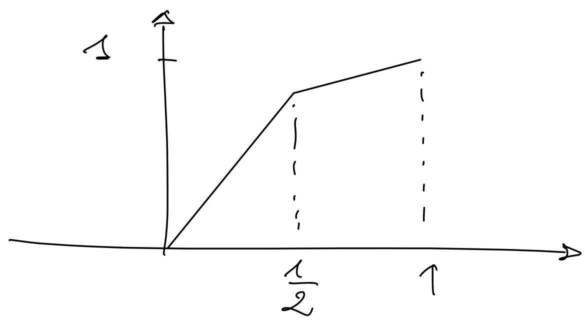
that is $$\frac{c}{2\alpha} = \frac{1}{2}\frac{c}{\beta} + 1 - \frac{c}{\beta} = 1 - \frac{c}{2\beta}$$

$$\Rightarrow \quad \frac{c}{2\alpha} + \frac{c}{2\beta} = 1 \quad \Rightarrow \quad c = \frac{2\alpha\beta}{\alpha+\beta}$$

Then the solution is

$$u(x) = \begin{cases} \dfrac{2\beta}{\alpha+\beta}\, x & x \in [0, 1/2) \\[2ex] \dfrac{2\alpha}{\alpha+\beta}\, x + \dfrac{\beta-\alpha}{\alpha+\beta} & x \in [1/2, 1] \end{cases}$$

Suppose $\alpha < \beta$. Then the graph of u is



indeed u' is greater in $(0, 1/2)$ than in $(1/2, 1)$

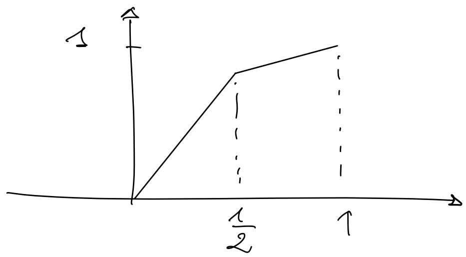
The solution u is not neither C^2 nor C^1.

REMARK The constant $\frac{2\alpha\beta}{\alpha+\beta}$ is called harmonic mean of α and β.

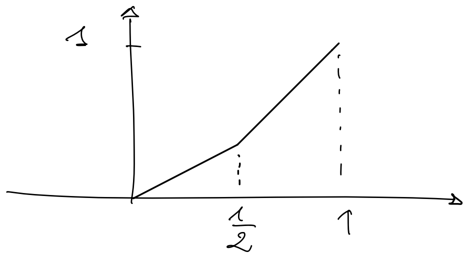
Observe that if $\alpha = \beta$ we get $u(x) = x$.

If $\alpha \neq \beta$ u is "more increasing" where a takes its minimum, i.e.

$\alpha < \beta$



$\alpha > \beta$



We will understand the reason of this behaviour later.

Some recalls and preliminary results on Hilbert spaces

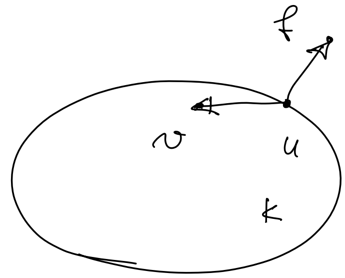
Theorem Consider H a Hilbert space, $K \subseteq H$ a convex and closed subset of H. Then for every $f \in H$ there exist a unique $u \in K$ such that

$$\|f - u\|_H = \min_{v \in K} \|f - v\|_H =: \mathrm{dist}(f, K)$$

Moreover u is characterized by the property

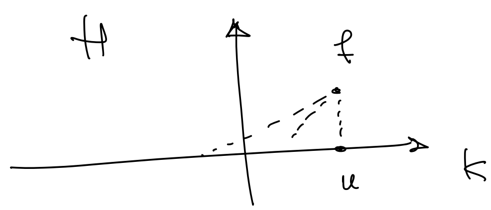
$$\begin{cases} u \in K \\ (f-u,\, v-u) \leq 0 \quad \forall\, v \in K. \end{cases}$$

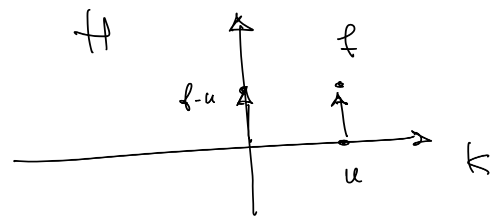
already seen (GUIOTTO)



REMARK In particular if K is a closed subspace the characterization of u is simply

$$\begin{cases} u \in K \\ (f-u,\, v) = 0 \quad \forall\, v \in K. \end{cases} \qquad (2)$$



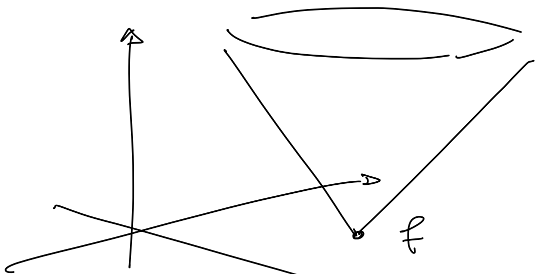


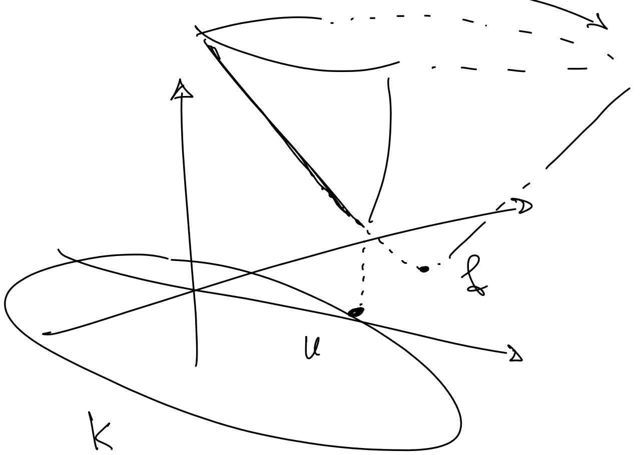
and moreover u satisfies

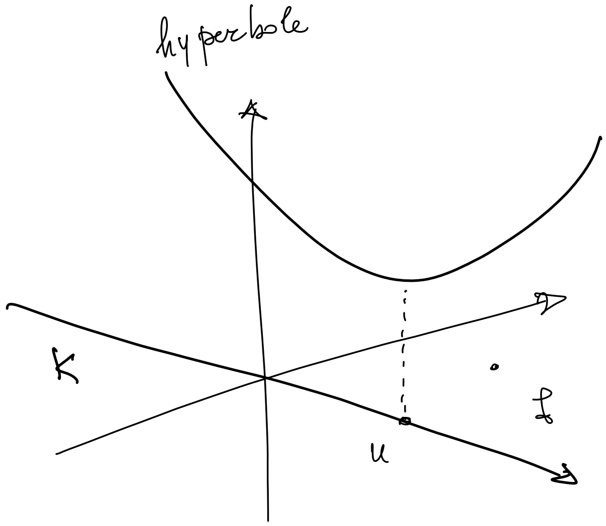
$$\|f-u\|_H = \min_{v \in K} \|f-v\|_H \qquad (3)$$

EX May a subspace of a Hilbert space not be closed? Answer: yes. Show an example.

The graph of the function $g(v) = \|f - v\|_{H}$ is a cone







Notice that minimise g is equivalent to minimise $\varphi(v) = (g(v))^2 = \|f - v\|_{H}^2$.

Theorem (Riesz-Frechet) Consider H a Hilbert space.

For every $\varphi \in H'$ there is $f \in H$ such that

$$\langle \varphi, v \rangle_{H' \times H} = (f, v)_H \quad \forall\, v \in H.$$

Moreover $\|f\|_H = \|\varphi\|_{H'}$.

Without proof

Examples $H = L^2$, $H = H^1_0$, $H = H^1$ already seen

Now we see, and prove, an important result.

Before we recall some notions.

Def A map $a : H \times H \longrightarrow \mathbb{R}$ is a bilinear form if

$\forall u \in H$ $H \ni v \mapsto a(u,v)$ is linear (in v),

$\forall v \in H$ $H \ni u \mapsto a(u,v)$ is linear (in u);

is symmetric if $a(u,v) = a(v,u)$ $\forall u,v \in H$;

is continuous if $\exists$ $M > 0$ s.t.

$$|a(u,v)| \leq M \|u\|_H \|v\|_H ;$$

is coercive if there is $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|_H^2 \quad \forall u \in H.$$

EXAMPLE Given a matrix $A = (a_{ij})_{i,j=1}^n$ defined in $\mathbb{R}^n$

the map $a : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as

$$a(x,y) := (A \cdot x, y) = \sum_{i,j=1}^n a_{ij} x_j y_i$$

is a bilinear form.

It is symmetric iff the matrix A is symmetric

Notice that the quadratic form

$$x \mapsto a(x,x) = (Ax, x)$$

can be always associated to a symmetric matrix.
Indeed in the sum there are the two addends

$$a_{ij}\, x_i x_j \qquad \text{and} \qquad a_{ji}\, x_i x_j \,.$$

Summing we have that

$$a_{ij}\, x_i x_j + a_{ji}\, x_i x_j = 2\, \frac{a_{ij} + a_{ji}}{2}\, x_i x_j$$

and then substituting the matrix A with the symmetric matrix $\tilde{A}$ defined by

$$\tilde{a}_{ij} := \frac{a_{ij} + a_{ji}}{2}$$

We have that the quadratic form $x \mapsto a(x,x)$ is represented by the symmetric matrix $\tilde{A}$.
Then we have infinite matrices defining a quadratic form, but only one of them is symmetric.
Now suppose A is symmetric (otherwise we modify it).
Then its eigenvalues are real. Then

$$a(x,x) = \big(Ax, x\big) \geq \lambda |x|^2$$

where λ is the minimum eigenvalue

If $\lambda > 0$ then a is coercive.

Orthogonal subspace   Consider a Hilbert space H.

Given $\mathcal{V}$ a vectorial subspace of H we denote by

$$\mathcal{V}^\perp = \{ h \in H \mid (h, u)_H = 0 \quad \forall u \in \mathcal{V} \}$$

and $\mathcal{V}^\perp$ is called the space orthogonal to $\mathcal{V}$.

Observe that $\mathcal{V}^\perp$ is closed, but in general a subspace of a Hilbert space is not closed.

For instance, $H^1(\Omega)$ is a subspace of $L^2(\Omega)$, but it is dense in $L^2(\Omega)$ and so it cannot be closed. Then $(H^1(\Omega))^\perp = \{0\}$ ($\perp$ in L^2).

$\mathcal{V}$ closed in H means: if $\{u_n\}_n$ is a converging sequence, $\{u_n\}_n \subseteq \mathcal{V}$, $u_n \to u$, then $u \in \mathcal{V}$.

In $\mathbb{R}^2$ if $\mathcal{V} = \{ (x,y) \in \mathbb{R}^2 \mid x = y \}$ then

$$\mathcal{V}^\perp = \{ (x,y) \in \mathbb{R}^2 \mid x = -y \}$$

If $H = L^2(\Omega)$ and $\mathcal{V} = \{ u \in L^2(\Omega) \mid u \equiv 0 \text{ in } \omega \subset \Omega \}$

then $\mathcal{V}^\perp = \{ u \in L^2(\Omega) \mid u \equiv 0 \text{ in } \Omega \setminus \omega = \omega^c \}$

If $H = L^2(-1,1)$ and $\mathcal{V} = \{ \text{constant function} \}$

then $\mathcal{V}^\perp = \{ u \in L^2(-1,1) \mid \int_{-1}^{1} u \, dx = 0 \}$.

Theorem (Lax-Milgram) Consider a Hilbert space H and a bilinear form $a : H \times H \longrightarrow \mathbb{R}$ for which there are two positive constants α, β such that

$$a(u,v) \le \beta \|u\| \|v\| \qquad (a \text{ bounded})$$

$$a(u,u) \ge \alpha \|u\|^2 \qquad (a \text{ coercive})$$

Then for every $\varphi \in H'$ there exists a unique $u \in H$ such that

$$a(u,v) = \langle \varphi, v \rangle_{H' \times H} \qquad \forall v \in H .$$

Moreover, if a is symmetric, u is characterized by

$$\frac{1}{2} a(u,u) - \langle \varphi, u \rangle_{H' \times H} = \min_{v \in H} \left\{ \frac{1}{2} a(v,v) - \langle \varphi, v \rangle_{H' \times H} \right\} .$$

Proof: from the Riesz-Frechét theorem we have the existence of two elements, $f \in H$ such that

$$\langle \varphi, v \rangle = (f, v) \quad \forall v \in H \qquad \begin{pmatrix} \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{H' \times H} \\ (\cdot, \cdot) = (\cdot, \cdot)_H \end{pmatrix}$$

and, since for every u $v \longmapsto a(u,v)$ is a linear form on H and then is an element in H', $g_u \in H$ such that

$$a(u,v) = (g_u, v) \qquad \forall v \in H$$

To each element $u \in H$ corrisponds an element g_u that we can call Au, so as to define an operator $A : H \longrightarrow H$.

Now the question is: given $f \in H$ is it possible to find $u \in H$ such that

$$(Au, v) = (f, v) \quad \forall\, v \in H \quad ? \quad \left(\begin{array}{c} (\cdot,\cdot) \text{ scalar} \\ \text{product of } H \end{array} \right)$$

This in some sense means:

is it possible to invert A? If yes we are done and we have found u:

$$u = A^{-1} f$$

First observe that A is a linear and bounded operator.

1. A is linear $\quad (A(\lambda u_1 + \mu u_2), v) = a(\lambda u_1 + \mu u_2, v) =$

$$= \lambda\, a(u_1, v) + \mu\, a(u_2, v) =$$

$$= \lambda\, (Au_1, v) + \mu\, (Au_2, v)$$

for $\lambda, \mu \in \mathbb{R}$, $u_1, u_2, v \in H$

2. A is bounded (and then continuous) $\quad \|Au\|^2 = (Au, Au) = a(u, Au)$

$$\leq \beta \|u\| \|Au\|$$

$$\Rightarrow \text{ if } Au \neq 0 \quad \|Au\| \leq \beta \|u\|$$

Now we show that is invertible.

3. A injective

$$\alpha \|u\|^2 \leq a(u,u) = (Au, u) \leq \|Au\| \, \|u\|$$

$$\Rightarrow \quad \alpha \|u\| \leq \|Au\| \quad \text{if } u \neq 0$$

This gives injectivity because of the linearity of A:

if $Au = 0 \Rightarrow u = 0$ and this means A injective.

4. The range of A $R(A)$ is closed in H.

$R(A)$ is closed means that if $\{y_n\}_n \subseteq R(A)$ and $y_n \to \bar{y} \in H \Rightarrow \bar{y} \in R(A)$.

$y_n = Au_n$ for some $u_n \in H$ and $Au_n \to \bar{y}$

Since

$$\|u_n - u_m\| \leq \frac{1}{\alpha} \|Au_n - Au_m\| = \frac{1}{\alpha} \|y_n - y_m\|$$

we derive that $\{u_n\}$ is a Cauchy sequence, and then converging to some $\bar{u} \in H$ (because H is a Banach space)

By the continuity of A

$$Au_n \to A\bar{u} = \bar{y} \quad \text{and then } \bar{y} \in R(A).$$

5. A is surjective, i.e. $R(A) = H$.

If this were not true there would exist $w \neq 0$, $w \in (R(A))^{\perp} \subseteq H$. Then

$$\alpha \|w\|^2 \leq a(w,w) = (Aw, w) = 0$$

and this is impossible.

Now we see the second part. If a is symmetric $a(\cdot,\cdot)$ defines a new scalar product in H.

Indeed:

i) $a(u,u) \geq 0$ and $a(u,u) = 0$ iff $u = 0$

$$(\alpha \|u\|^2 \leq a(u,u) \leq \beta \|u\|^2)$$

ii) $a(u,v) = a(v,u) \quad \forall\, u,v \in H$

(a symmetric)

iii) $a(u+v, w) = a(u,w) + a(v,w)$

$a(\lambda u, v) = \lambda\, a(u,v)$ (a bilinear)

$\forall\, u,v,w \in H$, $\lambda \in \mathbb{R}$.

Then H is a Hilbert space also with the scalar product $a(\cdot,\cdot)$ and the norm $\|u\|_a := \sqrt{a(u,u)}$

Then there exists $g \in H$ such that (Riesz-Frechet)

$$\langle \varphi, v \rangle = a(g,v) \quad \forall\, v \in H$$

and since we have proved there is u (unique) such that

$$\langle \varphi, v \rangle = a(u,v) \quad \forall\, v \in H$$

We derive

$$a(g-u, v) = 0 \quad \forall\, v \in H.$$

that is u is orthogonal to g w.r.t. $a(\cdot,\cdot)$.

From (2) and (3) this means that u solves the minimum problem ($K = H$ in (2) and (3))

$$\min_{v \in H} \sqrt{a(g-v, g-v)} \qquad \text{i.e.}$$

$$a(g-u, g-u) = \min_{v \in H} a(g-v, g-v)$$

This amount to minimising

$$H \ni v \mapsto a(v,v) - 2\, a(g,v) + a(g,g) =$$
$$= a(v,v) - 2 \langle \varphi, v \rangle + a(g,g)$$

or equivalently

$$H \ni v \mapsto \frac{1}{2} a(v,v) - \langle \varphi, v \rangle .$$

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Observe that, clearly, the minimum of

$$q(v) = a(g-v, g-v) \quad \text{in } H \quad \text{is } \underline{\text{zero}}!$$

But if we modify $q(v)$ (removing the constant $a(g,g)$) this is not true any more. Anyway being q a quadratic form the function

$$v \mapsto \frac{1}{2} a(v,v) - \langle \varphi, v \rangle \quad \text{is convex.}$$

Let's see now a concrete example: we want to solve an elliptic partial differential equation,

$$Lu = f \quad \text{in } \Omega \subseteq \mathbb{R}^n$$

where

$$\boxed{Lu = -\operatorname{div}(a \cdot Du) + b \cdot Du + cu}$$

Here and in the following a dot $\cdot$ will denote both the product between a matrix and a vector and the scalar product between two vectors. The scalar product in $\mathbb{R}^n$ will also denotes, as usual, by $(,)$.

The coefficients a, b, c are as follows:

$\underline{a}$ is a $n \times n$ $\underline{\text{matrix}}$

$\underline{b}$ is a $\underline{\text{vector}}$ in $\mathbb{R}^n$

$\underline{c}$ is a $\underline{\text{scalar}}$

$$-\operatorname{div}(a\, Du) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}(x)\right)$$

$$b\, Du = \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x)$$

$$cu = c(x)\, u(x)$$

The coefficients $a = (a_{ij})_{i,j=1}^{n}$, $b = (b_1, .. b_n)$, c satisfy

$$a_{ij},\ b_i,\ c \in L^\infty(\Omega). \qquad (4)$$

For the sake of simplicity, but this is not necessary, we will suppose

$$a \text{ symmetric, i.e } a_{ij} = a_{ji} \qquad (5)$$

In this way all eigenvalues are real and we will suppose that all the eigenvalues are positive and we suppose there are two positive constants α, β such that

$$\alpha |\xi|^2 \leq (a(x)\cdot\xi, \xi) \leq \beta |\xi|^2 \qquad \forall \xi \in \mathbb{R}^n \text{ for a.e. } x \in \Omega \qquad (6)$$

REMARK Being a symmetric and satisfying (6) a induced a scalar product and then satisfies

$$(a\cdot\xi, \eta) \leq (a\cdot\xi, \xi)^{1/2} (a\cdot\eta, \eta)^{1/2} \qquad \forall \xi, \eta \in \mathbb{R}^n, \text{ a.e. in } \Omega \qquad (7)$$

About c we will only suppose that

$$c \in L^\infty(\Omega), \quad c \geq 0$$

The assumptions about b need some preliminary computations. The reason is that we will consider a bilinear form, that we denote by $\mathcal{a}$ just not to confuse it with the matrix a, and $\mathcal{a}$ will play the role of the bilinear form a of the Lax-Milgram theorem. Consider

$$\mathcal{a} : H^1_0(\Omega) \times H^1_0(\Omega) \longrightarrow \mathbb{R} \qquad \text{defined as}$$

$$\mathcal{a}(u,v) := \int_\Omega (a(x)\cdot Du, Dv)\,dx + \int_\Omega b\cdot Du\, v\,dx + \int_\Omega c\,u\,v\,dx$$

REMARK We repeat that with a little additional assumption one can consider also non-symmetric matrices a and $\mathcal{a}$ remains a bilinear form. It is sufficient to replace (7) by

$$(a\cdot\xi,\eta) \leq M\,(a\cdot\xi,\xi)^{1/2}(a\cdot\eta,\eta)^{1/2} \qquad (7')$$
$$\forall\, \xi,\eta\in\mathbb{R}^n, \text{ a.e. in } \Omega$$

Let's start taking a s.t.

- $a_{ij} = a_{ji}$ (a symmetric)
- $\alpha |\xi|^2 \leq (a \cdot \xi, \xi) \leq \beta |\xi|^2$ $\alpha, \beta > 0$
- $c \geq \lambda > 0$
- $b \equiv (0, \dots 0)$

a bounded

$$|a(u,v)| \leq \beta \int |Du||Dv| dx + \|c\|_\infty \int uv \, dx \leq$$

$$\leq \beta \left(\int |Du|^2 dx \right)^{1/2} \left(\int |Dv|^2 dx \right)^{1/2} + \|c\|_\infty \left(\int u^2 dx \right)^{1/2} \left(\int v^2 dx \right)^{1/2}$$

$$\leq \max \{ \beta, \|c\|_\infty \} \|u\|_{H^1} \|v\|_{H^1}$$

a coercive

$$a(u,u) \geq \alpha \int_\Omega |Du|^2 dx + \lambda \int_\Omega u^2 dx \geq$$

$$\geq \min \{ \alpha, \lambda \} \|u\|^2_{H^1(\Omega)}$$

Then for every $f \in H^{-1}(\Omega)$ there exists a unique $u \in H^1_0(\Omega)$ such that

$$\mathcal{a}(u,v) = \langle f, v \rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} \qquad \forall\, v \in H^1_0(\Omega)$$

i.e.

$$\int_\Omega (a\cdot Du, Dv)\,dx + \int_\Omega c\,u\,v\,dx = \langle f, v\rangle \qquad \forall\, v \in H^1_0(\Omega)$$

If $f \in L^2(\Omega)$

$$\int_\Omega (a\cdot Du, Dv)\,dx + \int_\Omega c\,u\,v\,dx = \int_\Omega f\,v\,dx \qquad \forall\, v \in H^1_0(\Omega)$$

This means : $\exists!\; u \in H^1_0(\Omega)$ s.t.

$$\begin{cases} -\operatorname{div}(a\cdot Du) + c\,u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

If Ω is bounded, by Poincaré inequality,

$\left(\int_\Omega |Du|^2 dx\right)^{\frac{1}{2}}$ is a norm and then

c can be chosen to satisfy $c \geq 0$, possibly $c \equiv 0$

Indeed $\quad a(u,u) \geq \alpha \int_\Omega |Du|^2 dx + \int_\Omega c u^2 dx \geq$

$$\geq \alpha \int_\Omega |Du|^2 dx = \alpha \|u\|^2_{H^1_0(\Omega)}$$

Then also the problem

$$\begin{cases} -\operatorname{div}(a \cdot Du) = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

admits a unique solution $u \in H^1_0(\Omega)$ $\left(\forall\, f \in H^{-1}\,!\right)$

and (a symmetric) u satisfies

$$\frac{1}{2} \int_\Omega (a \cdot Du, Du)\, dx - \langle f, u \rangle = \min_{v \in H^1_0} \left[\frac{1}{2} \int (a Dv, Dv) - \langle f, v \rangle \right]$$

Case $\quad b \neq (0, \dots 0)$

It is clear that $\mathcal{a}$ is bilinear.

Let's see that $\mathcal{a}$ is **bounded**:

$$|\mathcal{a}(u,v)| = \left| \int (a\cdot Du, Dv) + \int b\, Du\, v + \int c\, u\, v \right| \leq$$

$$((7)) \quad \leq \int (a\cdot Du, Du)^{1/2} (a Dv, Dv)^{1/2} + \int |b|\,|Du|\,|v| +$$

$$+ \int c\,|u|\,|v| \leq$$

$$(\text{Hölder}) \quad \leq \left(\int (a\cdot Du, Du)\,dx \int (a Dv, Dv) \right)^{1/2} +$$

$$+ \|b\|_\infty \left(\int |Du|^2 \int v^2 \right)^{1/2} + \|c\|_\infty \left(\int u^2 \int v^2 \right)^{1/2}$$

$$((6)) \quad \leq \beta\, \|Du\|_{L^2} \|Dv\|_{L^2} + \|b\|_\infty \|Du\|_{L^2} \|v\|_{L^2} + \|c\|_\infty \|u\|_{L^2} \|v\|_{L^2}$$

$$\leq \max\{\beta, \|b\|_\infty, \|c\|_\infty\}\, \|u\|_{H^1} \|v\|_{H^1}$$

Let's see if $\mathcal{a}$ is *coercive*:

$$\mathcal{a}(u,u) = \int (a\cdot Du, Du) + \int b\, Du\, u + \int c\, u^2$$

We use the following estimate: given $x, y \in \mathbb{R}$

one has $2xy \le x^2 + y^2$. This can be generalised taking $\varepsilon > 0$ as follows:

$$2xy = 2\sqrt{2\varepsilon}\, x \, \frac{1}{\sqrt{2\varepsilon}}\, y \le 2\varepsilon x^2 + \frac{1}{2\varepsilon} y^2$$

by which

$$xy \le \varepsilon x^2 + \frac{1}{4\varepsilon} y^2 \qquad \forall x, y \in \mathbb{R} \quad \forall \varepsilon > 0.$$

Then

$$\left| \int_\Omega b \cdot Du\, u \right| \le \|b\|_\infty \left(\varepsilon \int |Du|^2 + \frac{1}{4\varepsilon} \int u^2 \right)$$

which implies

$$\int b \cdot Du\, u\, dx \ge -\varepsilon \|b\|_\infty \int |Du|^2 - \frac{\|b\|_\infty}{4\varepsilon} \int u^2$$

Then we have

$$a(u,u) \overset{(6)}{\ge} \alpha \int |Du|^2 - \varepsilon \|b\|_\infty \int |Du|^2 + $$

$$- \frac{\|b\|_\infty}{4\varepsilon} \int u^2 dx + \int c u^2$$

For $\alpha - \varepsilon \|b\|_\infty > 0$, i.e. for $b \ne 0$ $\boxed{\varepsilon < \frac{\alpha}{\|b\|_\infty}}$ $\circledast$

we have

$$\alpha \int |Du|^2 - \varepsilon \|b\|_\infty \int |Du|^2 = \left(\alpha - \varepsilon \|b\|_\infty\right) \|Du\|_{L^2}$$

while

$$\int c u^2 - \int \frac{\|b\|_\infty}{4\varepsilon} u^2 = \int \left(c(x) - \frac{\|b\|_\infty}{4\varepsilon}\right) u^2 \geq 0$$

provided that $c(x) - \dfrac{\|b\|_\infty}{4\varepsilon} \geq 0$ a.e. in Ω

$$\Rightarrow \boxed{\varepsilon c > \frac{\|b\|_\infty}{4}} \overset{\otimes}{\Rightarrow} \boxed{c(x) > \frac{\|b\|_\infty^2}{4\alpha}} \qquad (8)$$

Then if $\boxed{\operatorname*{ess\,inf}_{x\in\Omega} c(x) > \dfrac{\|b\|_\infty^2}{4\alpha}, \quad \alpha > 0}$

the quadratic form $u \mapsto \mathcal{Q}(u,u)$ is coercive.

Notice that if $b \equiv 0$ everything is simpler and $\mathcal{Q}$ is coercive if

$$\alpha > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{\Omega} c > 0$$

If Ω is bounded then (by the Poincaré inequality)

$\left(\int_\Omega |Du|^2 dx \right)^{1/2}$ is a norm in $H^1_0(\Omega)$.

Then $c \geq \dfrac{\|b\|_\infty^2}{4\alpha}$ is sufficient to have a coercive

and, for $b \equiv 0$, $c \geq 0$ is sufficient.

Then (here we chose ε s.t. $\alpha - \varepsilon \|b\|_\infty = \frac{\alpha}{2}$, assuming $c - \frac{\|b\|_\infty^2}{2\alpha} \geq 0$, just as example)

$$a(u,u) \geq \frac{\alpha}{2} \int |Du|^2 + \int \left(c - \frac{\|b\|_\infty^2}{2\alpha}\right) u^2 \geq$$

$$\geq \frac{\alpha}{2} \int |Du|^2 = \alpha/2 \, \|u\|^2_{H^1_0(\Omega)}$$

If $b \equiv 0$ and $c \geq 0$ we have

$$a(u,u) \geq \alpha \int_\Omega |Du|^2 dx + \int_\Omega c(x) (u(x))^2 dx \geq \alpha \|u\|^2_{H^1_0(\Omega)}$$

Notice that a is symmetric if $b = (0, \dots 0)$ (and a symmetric).

Consider now the problem

$$(P) \quad \begin{cases} -\operatorname{div}(a \cdot Du) + b \cdot Du + cu = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

Theorem Given Ω bounded with Lipschitz boundary, a, b, c satisfying (4), (5), (6), (8). Then for every $f \in H^{-1}(\Omega)$ problem (P) has a unique solution $u \in H^1_0(\Omega)$. Moreover if $b \equiv 0$ (a is symmetric) the solution is the minimum of the functional

$$F v := \frac{1}{2} \int_\Omega (a \cdot Dv, Dv)\, dx + \int_\Omega c v^2 dx - \langle f, v \rangle$$

where $F : H^1_0(\Omega) \longrightarrow \mathbb{R}$. If $f \in L^2(\Omega)$

$$F v := \frac{1}{2} \int_\Omega (a \cdot Dv, Dv)\, dx + \int_\Omega c v^2 dx - \int_\Omega f v \, dx .$$

REMARK Consider $c(x) \equiv \lambda \in \mathbb{R}$ (a positive constant) and the elliptic equation

$$Lu = -\operatorname{div}(a \cdot Du) + \lambda u = f \quad \text{in } \Omega$$

If Ω is bounded we can consider $\lambda = 0$ since

$$\langle Lu, u \rangle = \int (a \cdot Du, Du)\,dx + \lambda \int u^2$$

and, by Poincaré inequality (which holds only if Ω has finite measure), the quantity

$$\left(\int |Du|^2 dx \right)^{1/2} \text{ is a norm in } H_0^1(\Omega)$$

and then if

$$Lu = -\operatorname{div}(a \cdot Du)$$

We have

$$\langle Lu, u \rangle_{H^{-1} \times H_0^1} = \int (a \cdot Du, Du) \geq \alpha \int |Du|^2 dx$$

and the bilinear form associated to L is coercive also if $\lambda = 0$.

If Ω is unbounded the quantity $\left(\int_\Omega |Du|^2 dx \right)^{1/2}$ is not a norm anymore.

REMARK Suppose you want to solve

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases}$$

and there is $\phi \in W^{1,2}(\Omega)$ such that $T\phi = \varphi$ in $\partial\Omega$

Then taking $v := u - \phi$ we have

$$-\operatorname{div}(a \cdot Dv) + b \cdot Dv + cv =$$

$$= -\operatorname{div}(a \cdot Du) + b \cdot Du + cu + \underbrace{+\operatorname{div}(a \cdot D\phi) - b \cdot D\phi - c\phi}_{= g \in H^{-1}(\Omega)} =$$

Then, if v is the solution of

$$\begin{cases} Lv = f + g & \Omega \\ v = 0 & \partial\Omega \end{cases}$$

$\Rightarrow$ $u = v + \phi$ is the solution of the original problem.

In particular if $f = 0$ one can pass

from $\begin{cases} Lu = 0 \\ u = \phi \end{cases}$ to $\begin{cases} Lu = g \\ u = 0 \end{cases}$ provided $\exists \phi \in W^{1,2}$ whose trace in $\partial\Omega$ is φ

ENERGY ESTIMATE

Given the problem
$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases}$$

With L as above, $f \in H^{-1}(\Omega)$, $\varphi \in L^2(\partial\Omega)$.

Consider u the solution. Then there is a constant $c > 0$ such that

$$\|u\|_{H^1(\Omega)} \leq c \left(\|f\|_{H^{-1}(\Omega)} + \|\varphi\|_{L^2(\partial\Omega)} \right)$$

proof: only for $\varphi \equiv 0$.

By coerciveness of the quadratic form Q (defined by $Q(u,v) = \langle Lu, v \rangle$) we get

$$\alpha \|u\|^2_{H^1} \leq Q(u,u) = \langle f, u \rangle \leq \|f\|_{H^{-1}} \|u\|_{H^1}$$

by which the thesis.

$\square$

Now let's come back to the example in dimension 1

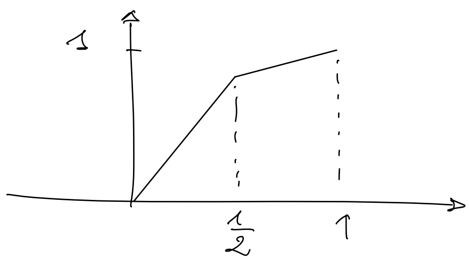
$$\begin{cases} -\dfrac{d}{dx}\left(a\dfrac{du}{dx}\right) = 0 \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

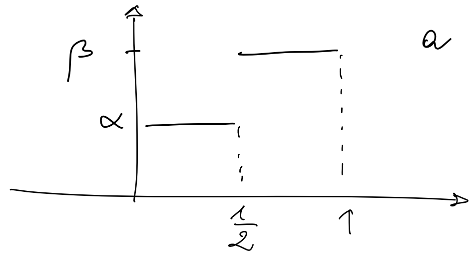
where $a = \begin{cases} \alpha & (0,1/2) \\ \beta & (1/2,1) \end{cases}$

and suppose

$$0 < \alpha < \beta$$

The solution $\bar{u}$ is





By the Lax-Milgram the solution has to minimise

$$Fu = \int_0^1 a(x)\,(u'(x))^2\,dx \qquad \text{with } u \in H^1(0,1)$$

$$u(0) = 0, \quad u(1) = 1$$

Roughly speaking, since minimising F means minimising the quantity $a(x)\,(u'(x))^2$, for the minimum $\bar{u}$ it is "convenient" to grow less (u' small) where a is higher and more (u' big) where a is lower

EULER - LAGRANGE EQUATIONS

Suppose you want to minimise a functional

$\mathcal{F} : X \to \mathbb{R}$, bounded by below, X vectorial space, and suppose to know that $\bar{u} \in X$ is the minimum point. Then

$$\mathcal{F}(\bar{u}) \leq \mathcal{F}(u) \qquad \forall u \in X$$

and in particular

$$\mathcal{F}(\bar{u}) \leq \mathcal{F}(\bar{u} + \varepsilon v)$$

for every $v \in X$, $\varepsilon \in \mathbb{R}$.

If the function

$$\mathbb{R} \ni \varepsilon \mapsto \mathcal{F}(\bar{u} + \varepsilon v) = F(\varepsilon) \quad \text{is differentiable}$$

one has that

$$F'(0) = 0.$$

Consider (a symmetric, coercive, bounded, $c \geq 0$, bounded)

$$\mathcal{F}u = \frac{1}{2}\int_\Omega (a \cdot Du, Du)\,dx + \frac{1}{2}\int_\Omega c u^2 - \int_\Omega f u \,dx \quad (f \in L^2(\Omega))$$

and suppose $\bar{u} \in H^1_0(\Omega)$ is a minimum point for $\mathcal{F}$.

Consider $\varphi \in C^1_c(\Omega)$ and $\varepsilon \in \mathbb{R}$. Then

$$\frac{d}{d\varepsilon} \mathcal{F}(\bar{u} + \varepsilon \varphi) = \frac{d}{d\varepsilon}\left[\frac{1}{2} \int_\Omega \left(Q\cdot(D\bar{u} + \varepsilon D\varphi), D\bar{u} + \varepsilon D\varphi \right) dx \right.$$

$$\left. + \int_\Omega c\,(\bar{u} + \varepsilon\varphi)^2 - \int_\Omega f\,(\bar{u} + \varepsilon \varphi)\, dx \right]$$

$$\left(Q\cdot(D\bar{u} + \varepsilon D\varphi), D\bar{u} + \varepsilon D\varphi \right) = \sum_{i,j=1}^{n} a_{ij} (D_j \bar{u} + \varepsilon D_j \varphi)(D_i \bar{u} + \varepsilon D_i \varphi)$$

$$\frac{d}{d\varepsilon} \left(Q\cdot(D\bar{u} + \varepsilon D\varphi), D\bar{u} + \varepsilon D\varphi \right) =$$

$$= \sum_{i,j=1}^{n} \left[a_{ij}\, D_j \varphi\, (D_i \bar{u} + \varepsilon D_i \varphi) + a_{ij} (D_j \bar{u} + \varepsilon D_j \varphi)\, D_i \varphi \right]$$

Then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{F}(\bar{u} + \varepsilon\varphi) = \frac{1}{2} \int_\Omega \left[(Q\cdot D\varphi, D\bar{u}) + (Q\cdot D\bar{u}, D\varphi) \right] dx$$

$$+ \int_\Omega c\, \bar{u}\, \varphi\, dx - \int_\Omega f \varphi\, dx$$

and by the symmetry of a

see below

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}(\bar{u}+\varepsilon\varphi) \overset{\circledast}{=} \int_\Omega \left[(a D\bar{u}, D\varphi) + c\,\bar{u}\,\varphi - f\varphi \right] dx = 0$$

$$\forall \varphi \in C^1_c(\Omega)$$

that is

$$-\operatorname{div}(a D\bar{u}) + c\,\bar{u} = f \qquad \text{in } \Omega$$

$$\bar{u} \in H^1_0(\Omega)$$

RECALL $\circledast$

Lemma Consider $f : \Omega \times I \to \mathbb{R}$, Ω open of $\mathbb{R}^n$, I interval ($x \in \Omega$, $t \in I$),
$f(\cdot, t) \in L^1(\Omega)$ for every $t \in I$
$f(x, \cdot)$ differentiable for almost every $x \in \Omega$.
Then

$$\frac{d}{dt} \int_\Omega f(x,t)\, dx = \int_\Omega \frac{\partial f}{\partial t}(x,t)\, dx$$

ON THE BOUNDARY CONDITION FOR THE LAPLACIAN

Consider the simplest equation (Ω bdd, $\partial\Omega$ lip)

$$-\Delta u = f \qquad \text{in } \Omega$$

Suppose to have u regular enough. Multiplying by v and integrating we have

$$-\Delta u = f \Rightarrow \int_\Omega (Du, Dv)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\mathcal{H}^{n-1} + \langle f, v\rangle \qquad (*)$$

Dirichlet problem Suppose you want to find a solution of

$$(D) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

To give a meaning to $(*)$ one would need to trace $\frac{\partial u}{\partial \nu}$ and v in $\partial\Omega$, therefore $u \in H^2(\Omega)$ (why?) and $v \in H^1(\Omega)$.

If we start from $u \in H^2(\Omega)$ and want u to satisfy (D) and multiply the equation by $v \in H^1_0(\Omega)$ we in fact get

$$\int_\Omega (Du, Dv)\,dx = \langle f, v\rangle \quad \forall\, v \in H^1_0(\Omega) \qquad (*D)$$

$$\text{and } Tu = 0 \text{ in } \partial\Omega \Rightarrow u \in H^1_0(\Omega)$$

Then we would have $(*D)$ for every $v \in H^1_0(\Omega)$ and $u \in H^2(\Omega) \cap H^1_0(\Omega)$. But in fact there is no need in $(*D)$ to require $u \in H^2(\Omega)$.

Note that $(*D)$ would not hold for $v \in H^1(\Omega)$ because a contribution at the boundary would appear.

Then $(*D)$ is the weak formulation for (D) and the boundary conditions are expressed in $(*D)$.

Neumann problem Suppose now you want to solve

$$(N) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega \end{cases}$$

If u is a solution in $H^2(\Omega)$ of (N), multiplying by $v \in H^1(\Omega)$ the equation we obtain $\circledast$ and since $T \dfrac{\partial u}{\partial \nu} = 0$ we get

$$\int_\Omega (Du, Dv)\, dx = \langle f, v \rangle \qquad \forall\, v \in H^1(\Omega) \qquad (*N)$$

But if $u \in H^1(\Omega)$ is a solution of $(*N)$ (i.e. $\forall v \in H^1$) confining to consider $v \in H^1_0(\Omega) \subset H^1(\Omega)$ we retrieve that $-\Delta u = f$ in Ω in a weak sense and starting from $-\Delta u = f$ in Ω and multiplying by $v \in H^1(\Omega)$ we derive that necessarily $\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\mathcal{H}^{n-1}$ is to be zero!

and the information $u \in H^2(\Omega)$ is not necessary to formulate the weak problem.

REMARK It can be given a meaning to the trace of $\frac{\partial u}{\partial \nu}$ for those functions $u \in H^1(\Omega)$ for which $\Delta u \in L^2(\Omega)$ (but this is to be proven!). Another way (we will briefly see this fact) to give a meaning to $T \frac{\partial u}{\partial \nu}$ is to prove a regularity result for the solutions: if

$f \in L^2(\Omega) \Rightarrow$ the solution u of $-\Delta u = f$ in Ω belongs to $H^2(\Omega)$ and then $Du \in H^1(\Omega)$ has a trace in $\partial\Omega$.

Conclusion: given a Hilbert space $V \subset L^2(\Omega) \subset V'$

we write

$$a(u,v) = \langle f, v\rangle_{V' \times V} \qquad \forall\, v \in V$$

as a weak formulation for both the problems

$$(D) \quad \begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \qquad \lambda \geq 0 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$(N) \quad \begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \qquad \lambda \geq 0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

The only difference is that in the first case $V = H^1_0(\Omega)$, in the second $V = H^1(\Omega)$.

Another difference is that, to have uniqueness, in the first case λ may be equal to 0 if Ω is bounded, in the second λ has to be positive. If $\lambda = 0$, even if Ω is bounded, and u is a solution then

$u + c$ is a solution $(c \in \mathbb{R})$.

To get uniquess for the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega \end{cases}$$

one have to look for a solution in , for instance,

$$\left\{ u \in H^1(\Omega) \;\middle|\; \int_\Omega u \, dx = 0 \right\}$$