FORMAL DERIVATION OF AN ESTIMATE

We have seen that for each $f \in H^{-1}(\Omega)$ there is a unique solution $u \in H^1_0$ (under suitable assumptions) of the problem

$$\begin{cases} -\operatorname{div}(a \cdot Du) + b \cdot Du + cu = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

where the equality is to be intended in $H^{-1}(\Omega)$.

Now suppose $f \in L^2(\Omega)$ and consider

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

and suppose $u \in C^\infty_c(\Omega)$ ($u \in H^3(\Omega) \cap H^2_0(\Omega)$ is sufficient)

Then

$$\int_\Omega f^2 \, dx = \int_\Omega (\Delta u)^2 \, dx = \int_\Omega \sum_{i=1}^n D_{ii} u \sum_{j=1}^n D_{jj} u \, dx$$

$$= -\int_\Omega \sum_{i,j=1}^n D_i u \, D_{ijj} u \, dx = \int_\Omega \sum_{i,j} (D_i D_j u)^2 \, dx$$

$$= \int_\Omega |D^2 u|^2 \, dx .$$

This simple computation shows that the H^2 norm of the solution of $-\Delta u = f$ can be estimated by the L^2 norm of f (in this case is equal), even if we have assumed more regularity on u –

Suppose $f \in H^1$: differentiating formally $-\Delta u = f$ one gets

$$-\Delta D_k u = D_k f$$

and by the same argument one gets

$$\| D^2 D_k u \|_{L^2} = \| D_k f \|$$

and then

$$\| D^3 u \|_{L^2} = \| Df \|_{L^2} .$$

In fact one can prove the following result.
Consider the problem

$$(P) \quad \begin{cases} -\operatorname{div}(a Du) + b Du + cu = f & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases}$$

Theorem Consider $\varphi \in H^1(\Omega)$, a, b, c as usual and moreover $a_{ij} \in W^{1,\infty}(\Omega)$, $f \in L^2(\Omega)$ and u a solution of (P). Then for every $\omega \subset\subset \Omega$ $u \in H^2(\Omega)$ and

[see G-T section 8.3]

there is $c > 0$ (c depending also on $\mathrm{dist}(\omega, \partial\Omega)$) such that

(*) $$\|u\|_{H^2(\omega)} \leq c \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

If $a_{ij} \in C^{k,1}(\overline{\Omega})$, $b_i, c \in C^{k-1,1}(\overline{\Omega})$ and $f \in H^k$, $k \in \mathbb{N}^*$, then

$$\|u\|_{H^{k+2}(\omega)} \leq c \left(\|u\|_{H^1(\Omega)} + \|f\|_{H^k(\Omega)} \right).$$

If $a_{ij}, b_i, c, f \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega)$.

REMARK Observe that in (*) $\|u\|_{H^1(\Omega)}$ in turn can be estimated by $\|\varphi\|_{H^1}$ and $\|f\|_{L^2}$.

Theorem Same assumptions as above and moreover $\partial\Omega$ is of class C^2 and $\varphi \in H^2(\Omega)$. Then

$u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq c \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\varphi\|_{H^2(\Omega)} \right).$$

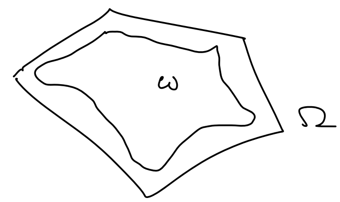
If $\partial\Omega$ is of class C^{k+2} and $\varphi \in H^k(\Omega)$ then

$u \in H^{k+2}(\Omega)$ and

$$\|u\|_{H^{k+2}(\Omega)} \leq c\left(\|u\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)} + \|\varphi\|_{H^{k+2}(\Omega)}\right).$$

REMARK Suppose $\partial\Omega$ Lipschitz.

With the first result we have at least inner regularity.



Idea of the proofs: use the **difference quotients**

$$D_k^h u(x) := \frac{u(x+he_k) - u(x)}{h}, \qquad h \in \mathbb{R},\ h \neq 0.$$

Observe that if $u \in L^2$ then $D_k^h u \in L^2$, but one can prove that

- if $u \in W^{1,p}(\Omega)$, $p \in [1, +\infty)$ then

$$\|D^h u\|_{L^p(\omega)} \leq c\,\|Du\|_{L^p(\Omega)}$$

for $\omega \subset\subset \Omega$ with $|h| < \frac{1}{2}\,\mathrm{dist}(\omega, \partial\Omega)$.

$$\left(D^h u = (D_1^h u, \ldots D_n^h u) \right)$$

D_k^h "behaves" like a derivative: for instance, if $\phi \in C_c^\infty(\omega)$

$$\int_\omega u\, D_k^h \phi\, dx = -\int_\omega D_k^{-h} u\, \phi\, dx$$