MAXIMUM PRINCIPLES

Before stating the results we need a notion of inequality at the boundary for functions in $H^1(\Omega)$.

We will say that $u \in H^1(\Omega)$ satisfies

$$u \leq 0 \quad \text{in } \partial\Omega$$

if $\quad u_+ \in H^1_0(\Omega) \qquad (u_+ := \max\{u, 0\})$.

Clearly if $u \in C^0(\overline{\Omega})$ $\quad u \leq 0$ in $\partial\Omega$ holds in the classical pointwise sense.

Then $\quad u \leq v$ in $\partial\Omega$ means $(u-v)_+ \in H^1_0(\Omega)$

and $\quad u \geq 0$ in $\partial\Omega$ holds if $-u \leq 0$ in $\partial\Omega$.

REMARK Since $u = u_+ - u_-$ we have

$$u - u_+ = -u_- \quad \Rightarrow \quad (u-u_+)_+ \equiv 0$$

(and in particular if $u \in H^1(\Omega) \Rightarrow (u-u_+)_+ \in H^1_0(\Omega)$).

Consider $\quad Lu := -\operatorname{div}(a \cdot Du) + b \cdot Du + cu$

Theorem (Weak maximum principle)

Suppose $c \equiv 0$. Let $u \in H^1(\Omega)$ satisfy

i) $Lu \leq 0$, then $\sup\limits_{\Omega} u = \sup\limits_{\partial\Omega} u$;

ii) $Lu \geq 0$, then $\inf_{\Omega} u = \inf_{\partial\Omega} u$.

Suppose now $c \geq 0$. Then if u satisfies

iii) $Lu \leq 0$, then $\sup_{\Omega} u \leq \sup_{\partial\Omega} u_+$;

iv) $Lu \geq 0$, then $\inf_{\Omega} u \geq \inf_{\partial\Omega} u_-$.

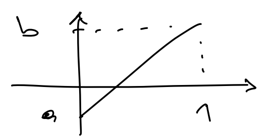
REMARK In particular if $Lu = 0$ we have that

if $c \equiv 0$ $\quad \sup_{\Omega} u = \sup_{\partial\Omega} u$ and

$$\inf_{\Omega} u = \inf_{\partial\Omega} u$$

if $c \geq 0$ $\quad \sup_{\Omega} |u| = \sup_{\partial\Omega} |u|$

Example If we solve $\begin{cases} -u'' = 0 \\ u(0) = a \\ u(1) = b \end{cases}$



We find $u(x) = (b-a)x + a$. If we solve

$$-u'' + cu = 0 \qquad (c \in \mathbb{R},\ c > 0)$$

We find: consider the polynomial

$$\lambda^2 - c = 0 \quad \Rightarrow \quad \lambda_1 = \sqrt{c}\,, \quad \lambda_2 = -\sqrt{c}$$

and a generic solution is

$$u(x) = k_1 e^{\sqrt{c}\,x} + k_2 e^{-\sqrt{c}\,x}\,, \qquad k_1, k_2 \in \mathbb{R}\,.$$

Imposing

$$u(-1) = k_1 e^{-\sqrt{c}} + k_2 e^{\sqrt{c}} = a \qquad (a = b)$$
$$u(1) = k_1 e^{\sqrt{c}} + k_2 e^{-\sqrt{c}} = a$$

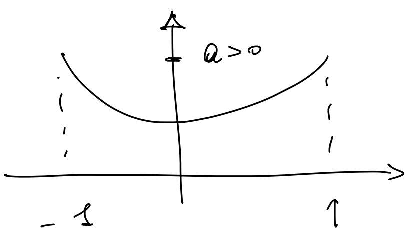
$$k_1 e^{-\sqrt{c}} + k_2 e^{\sqrt{c}} = k_1 e^{\sqrt{c}} + k_2 e^{-\sqrt{c}}$$

$$k_1 \left(e^{-\sqrt{c}} - e^{\sqrt{c}}\right) = k_2 \left(e^{-\sqrt{c}} - e^{\sqrt{c}}\right)$$

$$\Leftrightarrow \quad k_1 = k_2 = \frac{a}{e^{\sqrt{c}} + e^{-\sqrt{c}}}$$

Then the solution is even.

If $a > 0$ the solution has a minimum in 0,
if $a < 0$ " " " " maximum in 0.



while (if $a = b$) the solution of $-u'' = 0$ is constant

Then a solution of an equation like $(b \in \mathbb{R})$

$$-u'' + bu' = 0 \quad \text{in } (\alpha, \beta)$$

has its minimum and its maximum in α e in β,

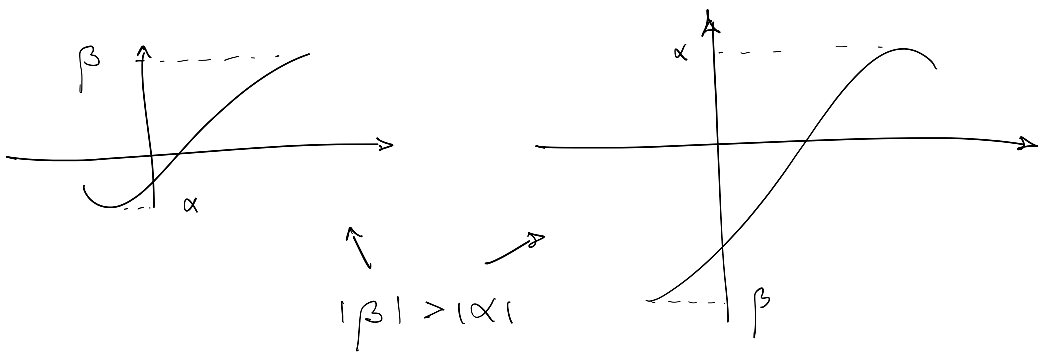
while for a solution of

$$(\bullet) \qquad -u'' + bu' + cu = 0 \qquad (c > 0)$$

not necessarily holds. Since one has

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|$$

the solution of $(\bullet)$ may have a graph like the following:



Theorem (Harnack) Consider $u \in H^1_{loc}(\Omega)$ a non-negative solution of $(*)$ and $\omega \subset\subset \Omega$ a connected open set. Then there is a constant C, depending only on ω and the coefficients a, b, c, such that

$$\sup_{\omega} u \leq C \inf_{\omega} u .$$

$(*)$ (a, b, c as usual, L^∞ with a "coercive")

$$-\operatorname{div}(a \cdot Du) + b \cdot Du + cu = 0 \quad \text{in } \Omega$$