VECTOR VALUED FUNCTIONS

Consider B a Banach space and I an interval.

We want to consider measurable functions $f : I \longrightarrow B$.

A step function $f : I \to B$ is a function

$$f(t) = \sum_{i=1}^{N} b_i \chi_i(t)$$

i.e. a finite sum of characteristic functions

where

$$I = A_1 \cup \dots \cup A_N \quad \text{is a partition of } I$$

and $b_i \in B$, $\chi_i = \chi_{A_i}$.

A step function f is measurable if A_j are measurable.

A function $f : I \to B$ is measurable if there is a sequence $\{f_n\}_{n \in \mathbb{N}}$ of measurable step functions such that

$$f_n(t) \longrightarrow f(t) \quad \text{for a.e. } t \in I.$$

Theorem $f : I \longrightarrow B$ is integrable if and only if f is measurable and $I \ni t \longmapsto \|f(t)\|_B$ is integrable.

Moreover if $\{f_n\}_n$ is a sequence of integrable functions such that

- $\|f_n(t)\|_B \leq g(t)$ $\forall n \in \mathbb{N}$, for a.e. $t \in I$ and for some $g : I \to \mathbb{R}$, $g \in L^1(\mathbb{R})$

- and $f_n(t) \to f(t)$ strongly in B for a.e. $t \in I$

then f is integrable and

$$\lim_{n\to\infty} \int_I \|f_n - f\|_B \, dt = 0$$

$$\lim_{n\to+\infty} \int_I f_n(t)\, dt = \int_I f(t)\, dt. \quad \text{(Bochner integral)}$$

At this point, since B is a Banach space, we can consider a sequence of step functions $\{f_n\}$ converging to f ($\{f_n\}_n$ is a Cauchy sequence) and consider the space of integrable functions.

Given an interval (a,b) or $[a,b]$ we define

$L^p(a,b;B)$, $p \in [1,+\infty]$, the space

of (equivalence classes of) measurable functions

$f : (a,b) \longrightarrow B$ such that

$(a,b) \ni t \longmapsto \| f(t) \|_B$ belongs to $L^p(a,b) = L^p(a,b;\mathbb{R})$

endowed with the norms

$$\| f \|_{L^p(a,b;B)} := \left(\int_a^b \| f(t) \|_B^p \, dt \right)^{1/p} \qquad p \in [1,+\infty)$$

$$\| f \|_{L^\infty(a,b;B)} := \operatorname{ess\,sup} \{ \| f(t) \|_B \mid t \in (a,b) \}$$

Theorem Given $p \in (1,+\infty)$, B reflexive, the space $\left(L^p(a,b;B) \right)'$ can be identified with $L^{p'}(a,b;B')$.

EXAMPLE $L^p(\Omega)$ is reflexive iff $p \in (1,+\infty)$.
$W^{1,p}(\Omega)$ " " " $p \in (1,+\infty)$.

$$\left(L^p(a,b; L^p(\Omega)) \right)' = L^{p'}(a,b; L^{p'}(\Omega))$$

$$\left(L^p(a,b; W_0^{1,p}(\Omega)) \right)' = L^{p'}(a,b; W^{-1,p}(\Omega))$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, $\quad p \in (1,+\infty)$.

Def For $u \in L^1(a,b;B)$ we can define a distributional derivative in the following sense: a function $v \in L^1(a,b;B)$ is called generalised derivative or distributional derivative of u if

$$\int_a^b u(t)\,\varphi'(t)\,dt = -\int_a^b v(t)\,\varphi(t)\,dt$$

$$\forall \varphi \in C_c^1((a,b);\mathbb{R})$$

These are Bochner integrals ← attention

Now we want to solve this problem: given Ω open set of $\mathbb{R}^n$, $T>0$, f and φ to be specified later. Consider

$$(P)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a \cdot Du) + b \cdot Du + cu = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega \times \{0\} \end{cases}$$

($u = \varphi$ ← initial condition)

$a = a(x,t)$, $b = b(x,t)$, $c = c(x,t)$

Setting of the problem. To simplify and short the notation we denote by A the operator

$$A(t) := -\operatorname{div}(a(\cdot,t)\cdot Du) + b(\cdot,t)\cdot Du + c(\cdot,t)\, u$$

As in the elliptic case we want

$$A(t) : H^1_0(\Omega) \longrightarrow H^{-1}(\Omega) \qquad t \in [0,T]$$

Now, if the function u (the solution of (P)) will depend also on time, i.e. beyond $x \in \Omega$ also on the variable $t \in [0,T]$, we can think of the function $u = u(x,t)$ as a function of only one variable ($t \in [0,T]$) and valued in $H^1_0(\Omega)$, i.e.

$$u : [0,T] \longrightarrow H^1_0(\Omega)$$

and we will write $u(t)$ to denote an element in $H^1_0(\Omega)$

In the same way we will write $f(t)$ for the datum f.

Also the derivative $\frac{\partial u}{\partial t}$ will be seen as the derivative of $u : [0,T] \to H^1_0(\Omega)$ and then denoted by

$$u'$$

Since in the "elliptic" case we have

$$\begin{array}{ccc} A\,u & = & f \\ \cap & & \cap \\ H^1_0(\Omega) & & H^{-1}(\Omega) \end{array}$$

in the same way, even if now u and f depend on time, it is natural to require

$$\begin{aligned} A(t)\, u(t) &\in H^{-1}(\Omega) \\ f(t) &\in H^{-1}(\Omega) \end{aligned} \qquad \text{for a.e. } t$$

and consequently

$$u' : [0,T] \longrightarrow \underline{H^{-1}(\Omega)}$$

Then, consider an evolution triple

$$V \subset H \subset V'$$

(think of H as $L^2(\Omega)$, V as $H^1_0(\Omega)$, but also $H^1(\Omega)$ or $W^{1,p}_0(\Omega)$ with $p > 2$ in the non-linear case and Ω bounded)

we consider the spaces

$$\mathcal{V} := L^2(0,T;V)$$

$$\mathcal{H} := L^2(0,T;H)$$

$$\mathcal{V}' := \left(L^2(0,T;V)\right)' = L^2(0,T;V')$$

(if V is reflexive, and $W^{1,p}(\Omega)$ is reflexive for every $p \in (1,+\infty)$)

Observe that as

$V \subset H \subset V'$ we also have $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$.

Finally we introduce the space where to look for the solution:

generalised derivative

$$\mathcal{W} := \{ u \in \mathcal{V} \mid u' \in \mathcal{V}' \}$$

(in our case if $V = H^1_0(\Omega)$

$$\mathcal{W} = \{ u \in L^2(0,T;H^1_0(\Omega)) \mid u' \in L^2(0,T;H^{-1}(\Omega)) \} \Big)$$

Theorem $\mathcal{W} \subseteq C^0([0,T];H)$

with continuous embedding, i.e. there is $c > 0$ such that

$$\| u(t) \|_H \leq c \, \| u \|_W \quad \forall \; t \in [0,T] .$$

$C^0([0,T]; B)$ with B Banach space is the space of functions u for which it makes sense to evaluate $u(t)$ for every $t \in [0,T]$ and

$$\lim_{t \to s} u(t) = u(s) \text{ in } B, \text{ i.e. } \lim_{t \to s} \| u(t) - u(s) \|_B = 0,$$

endowed with the norm

$$\| u \|_{C^0([0,T]; B)} := \max_{t \in [0,T]} \| u(t) \|_B .$$

Then we can give a meaning to the "initial" condition

$$u(x,0) = \varphi(x) \quad \text{in } \Omega$$

or

$$u = \varphi \quad \text{in } \Omega \times \{0\}$$

as

$$u(0) = \varphi \quad \text{in } H \quad \left(\text{i.e. in } L^2(\Omega) \text{ in our case} \right)$$

We are ready to write formally the problem (P) in an abstract way:

given $f \in L^2(0,T; H^{-1}(\Omega))$ (or V')

given $\varphi \in L^2(\Omega)$ (or H)

and a, b, c such that $(\alpha, \beta > 0)$

(H)
$$\begin{cases} \alpha |\xi|^2 \leq (a(x,t)\cdot\xi, \xi) \leq \beta |\xi|^2 & \text{for a.e. } (x,t) \text{ in } \Omega\times(0,T) \text{ and every } \xi \in \mathbb{R}^n \\ c = c(x,t) \geq 0 \\ a_{ij}, b_i, c \in L^\infty(\Omega\times(0,T)) \\ a_{ij} = a_{ji} \\ \text{and } b \text{ and satisfying } c > \dfrac{\|b\|_\infty^2}{4\alpha} \end{cases}$$

Then we have that, for almost every $t \in (0,T)$,

$$A(t) := -\operatorname{div}(a(\cdot,t)\cdot D) + b(\cdot,t)\cdot D + c(\cdot,t)$$

is a bounded, linear and coercive operator

$$A(t) : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

and the problem (P) may be written

(P_A)
$$\begin{cases} u'(t) + A(t)\, u(t) = f(t) & \text{for a.e. } t \in (0,T) \\ u(0) = \varphi \end{cases}$$

where the equality is to be intended in $H^{-1}(\Omega)$, (while the initial datum in $L^2(\Omega)$)

i.e. we require that

$$\langle u'(t) + A(t)\, u(t) \,,\, v \rangle_{H^{-1}\times H^1_0} = \langle f(t), v \rangle_{H^{-1}\times H^1_0}$$

for a. e. $t \in (0,T)$ and $\forall\, v \in H^1_0(\Omega)$

(and more generally $\langle\, ,\, \rangle_{V' \times V}$ and for every $v \in V$)

and u is to be sought in W.

ENERGY ESTIMATES Suppose u is a solution of (P_A). Then, multiplying the equation by u, we get

$$\langle u', u \rangle_{\mathcal{V}'\times \mathcal{V}} + \langle \mathcal{A} u, u \rangle_{\mathcal{V}'\times \mathcal{V}} = \langle f, u \rangle_{\mathcal{V}'\times \mathcal{V}}$$

where

$\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}'$ is defined as

$$\mathcal{A} v\, (t) := A(t)\, v(t) \quad \text{for } v \in \mathcal{V}.$$

Under assumptions (H) we have that $\mathcal{A}$ is coercive (as already seen in the elliptic case)

$$\langle \mathcal{A} u, u \rangle_{V' \times V} \geq c(\alpha) \|u\|_V^2$$

$c(\alpha)$ depends on α, could be $\frac{\alpha}{2}$ as done in the elliptic case, but also less than $\alpha/2$. Anyway $c(\alpha) > 0$

and

$$\langle \mathcal{A} u, v \rangle_{V' \times V} \leq$$

$$\leq \max\{\beta, \|b\|_\infty, \|c\|_\infty\} \|u\|_V \|v\|_V \qquad (1)$$

for every $u, v \in V$

Then, call c_1 and c_2 the two constants $c(\alpha)$ and $\max\{\beta, \|b\|_\infty, \|c\|_\infty\}$ and we have

$$\begin{cases} \langle \mathcal{A} u, u \rangle \geq c_1 \|u\|_V^2 \\ \|\mathcal{A} u\|_{V'} \leq c_2 \|u\|_V \end{cases} \qquad (2)$$

where the second one is obtained taking the supremum in (1) with respect to all $v \in V$ with $\|v\|_V \leq 1$.

REMARK If $f \in H$, $v \in V$ we have that

$$\langle f, v \rangle_{V' \times V} = (f, v)_H$$

and if $f \in \mathcal{H}$ and $v \in \mathcal{V}$ we have that

$$\langle f, v \rangle_{\mathcal{V}' \times \mathcal{V}} = (f, v)_{\mathcal{H}}.$$

Now suppose $w \in C^1([0,T]; H_0^1(\Omega))$, that is dense in $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{W}$. We have that

$$\frac{d}{ds} \| w(s) \|_H^2 = \frac{d}{ds} (w(s), w(s))_H =$$

$$= 2 (w'(s), w(s))_H = 2 \langle w'(s), w(s) \rangle_{V' \times V}$$

Integrating in $(0, t)$ we obtain

$$\| w(t) \|_H^2 - \| w(0) \|_H^2 = 2 \int_0^t \langle w'(s), w(s) \rangle_{V' \times V} \, ds \qquad (3)$$

that is (for $t = T$)

$$\langle w', w \rangle_{\mathcal{V}' \times \mathcal{V}} = \frac{1}{2} \| w(T) \|_H^2 - \frac{1}{2} \| w(0) \|_H^2 .$$

By density of $C^1([0,T]; H^1_0(\Omega))$ in W this holds also for functions in W. In particular, if $u \in W$ is the solution of (P_λ) we have

$$\langle u', u \rangle = \frac{1}{2} \| u(T) \|_H^2 - \frac{1}{2} \| u(0) \|_H^2 =$$

$$= \frac{1}{2} \int_\Omega (u(x,T))^2 dx - \frac{1}{2} \int_\Omega (\varphi(x))^2 dx$$

Then, if u is a solution of (P), we have

$$Au = f - u'$$

by which $\left(ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad \varepsilon > 0 \right)$

$$c_1 \| u \|_V^2 \le \langle Au, u \rangle = \langle f, u \rangle - \langle u', u \rangle \le$$

$$\le \| f \|_{V'} \| u \|_V + \frac{1}{2} \| \varphi \|_H^2 - \frac{1}{2} \| u(T) \|_H^2 \le$$

$$\le \frac{1}{4\varepsilon} \| f \|_{V'}^2 + \varepsilon \| u \|_V^2 + \frac{1}{2} \| \varphi \|_H^2 .$$

Choosing $\varepsilon = \frac{1}{2} c_1$ we get

$$\| u \|_V^2 \le \frac{1}{2c_1} \| f \|_{V'}^2 + \frac{1}{2c_1} \| \varphi \|_H^2 \qquad (4)$$

Finally, by $u' = f - Au$, we get

$$\|u'\|_{V'} \leq \|f\|_{V'} + \|Au\|_{V'} \leq \qquad (5)$$
$$\leq \|f\|_{V'} + c_2 \|u\|_V .$$

Putting together (4) and (5) we finally obtain that there is a constant $\tilde{c} > 0$, depending only on $\alpha, \beta, \|b\|_\infty, \|c\|_\infty$, such that

$$\|u\|_W \leq \tilde{c} \left[\|f\|_{V'} + \|\varphi\|_H \right] \qquad (6)$$

EXERCISE Consider $a, b > 0$.
Prove that, for each $p \in (1, +\infty)$, there is a constant $c = c(p) > 0$ such that

$$a^p + b^p \leq (a+b)^p \leq c(p) \left(a^p + b^p \right)$$

Moreover by (3) we have

$$\|u(t)\|_H^2 \leq \|\varphi\|_H^2 + 2 \|u'\|_{V'} \|u\|_V \qquad \forall t \in [0,T]$$

$$(\text{by } (5)) \quad \leq \|\varphi\|_H^2 + 2\|f\|_{V'}\|u\|_V + 2c_2\|u\|_V^2$$

$$\leq \|\varphi\|_H^2 + \|f\|_{V'}^2 + (1+2c_2)\|u\|_V^2$$

and by this and (6) we finally get that there is c depending on $\tilde{c}$ and c_2 and then only on $\alpha, \beta, \|b\|_\infty, \|c\|_\infty$, such that

$$\max_{t\in[0,T]} \|u(t)\|_H + \|u\|_V + \|u'\|_{V'} \leq c\left[\|f\|_{V'} + \|\varphi\|_H\right] \qquad (7)$$

Before showing existence and uniqueness of the solution of (P) we show that we can relax assumptions (H). In particular suppose that $c > \frac{\|b\|_\infty^2}{4\alpha}$ does not hold and suppose that

$$c \leq \frac{\|b\|_\infty^2}{4\alpha} \quad \text{in } \Omega \times (0,T) \qquad (8)$$

Assume to know that problem (P) admits a solution under assumptions (H) and suppose you want to solve (P) with, in the place of last request in (H) we have condition (δ).
Consider the functions u and v linked by

$$u(t) = e^{\lambda t} v(t), \qquad \lambda \in \mathbb{R}.$$

Notice that

$$u' = \lambda e^{\lambda t} v + e^{\lambda t} v'$$

and, by the linearity of A, one has that

$$Au = e^{\lambda t} A v.$$

Then
$$u' + Au = e^{\lambda t}\left[v' + Av + \lambda v\right] =$$

$$= e^{\lambda t}\left[v_t - \mathrm{div}(a\cdot Dv) + b\cdot Dv + (c+\lambda)v\right]$$

If we have c satisfying (δ) it is possible to find $\lambda > 0$ such that

$$c(x,t) + \lambda > \frac{\|b\|_\infty^2}{4\alpha}$$

Then, if v is the solution of

$$\begin{cases} v' + A v + \lambda v = e^{-\lambda t} f \\ v(0) = \varphi \end{cases},$$

the function $u = e^{\lambda t} v$ is the solution of

$$(P) \qquad \begin{cases} u' + A u = f \\ u(0) = \varphi \end{cases}.$$

UNIQUENESS OF THE SOLUTION

By the linearity of the problem if u_1 and u_2 are two solutions of (P), then $u_1 - u_2$ solves (P) with $f \equiv 0$ and $\varphi \equiv 0$. Then, by $(\mathcal{F})$, we get that $u_1 = u_2$ in W.

EXISTENCE OF THE SOLUTION

There are many ways to prove the existence of the solution of problem (P). Here we recall some of them:

- in the linear case a technique due to Lions (see chapter 3 in Showalter - Monotone operators in Banach spaces and ....)
- applying the theorem of Hille - Yosida (see, e.g. BREZIS)
- theory of semigroups
- use of maximal monotone operators
- Galerkin approximation

We see the last way (more constructive)

1° step) consider a sequence $\{w_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$

an orthogonal basis of $H_0^1(\Omega)$

and an orthonormal basis of $L^2(\Omega)$

(it is possible to find such a sequence because $L^2(\Omega)$ and $H_0^1(\Omega)$ are separable Hilbert spaces)

We want to prove the existence of a solution for

$$\begin{cases} u' + \mathcal{A}u = f & f \in L^2(0,T; H^{-1}(\Omega)) \\ u(0) = \varphi & \varphi \in L^2(\Omega) \end{cases}$$

For every $m \in \mathbb{N}^*$ we look for a function

$$u_m(t) := \sum_{k=1}^{m} d_m^k(t)\, w_k \qquad (9)$$

for which

•) $$d_m^k(0) = (\varphi, w_k)_{L^2} \qquad k = 1, \dots m$$

and

••) $$(u_m'(t), w_k)_{L^2} + \mathcal{Q}(t; u_m, w_k) = \langle f(t), w_k \rangle_{H^{-1} \times H_0^1} \qquad k = 1, \dots m, \quad t \in [0,T]$$

where

$$\mathcal{Q}(t; u, v) = \int_\Omega \Big[(a(x,t) \cdot Du(x,t), Dv(x,t)) + b(x,t) Du(x,t) v(x,t) + c(x,t) u(x,t) v(x,t) \Big] dx$$

$$\mathcal{Q}(u(t), v(t))$$

Then $\mathcal{Q}(t; u_m, w_k) = \mathcal{Q}(u_m(t), w_k)$.

Observe that, differentiating (9) w.r.t. t, we get

$$\langle u_m'(t), w_k \rangle_{H^{-1} \times H_0^1} = (u_m'(t), w_k)_{L^2} = (d_m^k)'(t)$$

Furthermore (by bilinearity of a)

$$a(u_m(t), w_k) = a\left(\sum_{h=1}^{m} d_m^h(t)\, w_k \,, w_k\right) =$$

$$= \sum_{h=1}^{m} d_m^h(t)\; a(t; w_h, w_k)$$

$$= \sum_{h=1}^{m} d_m^h(t)\; e_{hk}(t)$$

Finally, writing $f_k(t) := \langle f(t), w_k \rangle_{H^{-1} \times H^1_0}$ and recalling •), we get

$$(P_m) \begin{cases} (d_m^k)'(t) + \sum_{h=1}^{m} e_{hk}(t)\, d_m^h(t) = f_k(t) & t \in [0,T] \\ d_m^k(0) = (\varphi, w_k)_{L^2} & \end{cases} \qquad \forall\, k \in \{1, \dots m\}$$

Suppose, for the moment, $f \in C^0([0,T]; H^{-1}(\Omega))$

From the classical theory of ODEs we derive the existence of a unique solution

$$d_m(t) = (d_m^1(t), \dots d_m^m(t))$$

$d_m \in AC([0,T])$ and satisfying (P_m) for a.e. t.

Recall: find a solution $y \in C^1(I)$

$$\begin{cases} y' + a y = f \\ y(0) = y_0 \end{cases} \qquad \left(\begin{array}{l} y : I \to \mathbb{R} \\ \text{or } y : I \to \mathbb{R}^n \\ I \ni 0 \end{array} \right)$$

is equivalent to find $y \in C^0(I)$ s.t.

$$y(t) = y_0 + \int_0^t \left(f(s) - a(s) y(s) \right) ds$$

One can weaken the requirement about the solution and look only for $y \in AC(I)$.

$u \in AC([a,b])$ means that:

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $\forall N$, $x_1 < y_1 < x_2 < y_2 < \dots < x_n < y_n$, $x_i, y_i \in [a,b]$

and $\sum_{i=1}^N (y_i - x_i) < \delta \Rightarrow \sum_{i=1}^N |f(x_i) - f(y_i)| < \varepsilon$

IMPORTANT: $u \in AC(I) \Rightarrow u$ differentiable a.e.

and $u(x) - u(y) = \int_y^x u'(s)\, ds$

Moreover given $U(x) = \int_{x_0}^x u(s)\, ds$ for $u \in L^1(a,b) \Rightarrow U \in AC$

2° step) By (7) we get

$$\max_{t\in[0,T]} \|u_m(t)\|_H + \|u_m\|_V + \|u_m'\|_{V'} \leq$$

$$\leq c\left[\|f\|_{V'} + \|\varphi\|_H\right]$$

with c independent of m. Then we have the existence of a subsequence

$$\{u_{m_j}\}_{j\in\mathbb{N}} \subseteq \{u_m\}_{m\in\mathbb{N}} \quad \text{and} \quad u \in W \quad \text{s.t.}$$

(C)
$$\left|\begin{array}{llll} u_{m_j} \longrightarrow u & \text{in} & L^2(0,T; H_0^1(\Omega)) & \text{- weakly} \\ u_{m_j}' \longrightarrow u' & \text{in} & L^2(0,T; H^{-1}(\Omega)) & \text{- weakly} \end{array}\right.$$

(i.e. $u_{m_j} \to u$ weakly in W).

Then consider a function v of the form

$$v(t) = \sum_{k=1}^{N} d^k(t)\, w_k \in C^1([0,T]; H_0^1(\Omega))$$

(choose $d^k \in C^1([0,T])$)

and multiply the equation by v:

$$(*) \quad \int_0^T \left[(u_m'(t), v(t))_{L^2(\Omega)} + a(u_m(t), v(t)) \right] dt = \int_0^T \langle f(t), v(t) \rangle_{H^{-1}\times H_0^1} dt$$

Consider $m = m_j$ and send $j \to +\infty$. You get

$$\int_0^T \mathcal{a}(u_m(t), v(t)) = \int_0^T \int_\Omega (a(x,t) \cdot Du_m(x,t), Dv(x,t))\, dx\, dt +$$

$$+ \int_0^T \int_\Omega (b(x,t)\, Du_m(x,t) + c(x,t)\, u_m(x,t))\, dx\, dt$$

by weak convergence of $\{u_{m_j}\}_{j \in \mathbb{N}}$ in $\mathcal{V}$

we get that

$$\lim_{j \to +\infty} \mathcal{a}(u_{m_j}, v) = \lim_{j \to +\infty} \int_0^T a(t; u_{m_j}, v)\, dt =$$

$$= \mathcal{a}(u, v)$$

$$\int_0^T [\langle u'(t), v(t)\rangle + a(u(t), v(t))]\, dt = \int_0^T \langle f(t), v(t)\rangle_{H^{-1}\times H^1_0} dt$$

By density we get that this holds $\forall\, v \in L^2(0,T; H^1_0(\Omega))$

In the last equality consider now

$$v \in C^1([0,T]; H^1_0(\Omega)), \qquad v(T) = 0 .$$

You get

$$\text{i)} \int_0^T [-(v'(t), u(t))_{L^2} + A(u,v)(t)]\, dt = \int_0^T \langle f(t), v(t)\rangle_{H^{-1}\times H^1_0} dt + (u(0), v(0))_{L^2}$$

We have (from ⊛)

$$-\int_0^T (v'(t), u_m(t))_{L^2} dt + \int_0^T A(u_m, v)(t)\, dt =$$

$$= \int_0^T \langle f(t), u_m(t)\rangle_{H^{-1}\times H^1_0} dt + (u_m(0), v(0))_{L^2}$$

and taking the limit

$$\text{ii)} \int_0^T [-(v'(t), u(t))_{L^2} + A(u,v)(t)]\, dt = \int_0^T \langle f(t), u(t)\rangle_{H^{-1}\times H^1_0} dt + (\varphi, v(0))_{L^2}$$

By i) and ii) we finally derive $u(0) = \varphi$ in $L^2(\Omega)$.

Finally observe that each subsequence of $\{u_m\}_{m\in\mathbb{N}}$ is bounded and then it admits a further converging subsequence satisfying (C) for some $\tilde{u}$. But being $\tilde{u}$ a solution of

$$\begin{cases} u' + \mathcal{A}u = f \\ u(0) = \varphi \end{cases}$$

and by uniqueness of the solution of ( ) $\tilde{u} = u$.

Then we conclude that all the sequence $\{u_m\}$ satisfies

$$u_m \to u \quad \text{in } L^2(0,T; H_0^1(\Omega)) \text{ - weakly}$$

$$u_m' \to u' \quad \text{in } L^2(0,T; H^{-1}(\Omega)) \text{ - weakly}$$

and $$u_m \to u \quad \text{in } C^0([0,T]; L^2(\Omega)) \text{ - weakly}$$

Now we consider $f \in L^2(0,T; H^{-1}(\Omega)) = \mathcal{V}'$ and a sequence $\{f_j\}_{j\in\mathbb{N}} \in C^0([0,T]; H^{-1}(\Omega))$

$$f_j \to f \text{ in } \mathcal{V}' = L^2(0,T; H^{-1}(\Omega))$$

(Fact: B Banach, $C^0([0,T]; B)$ is dense in $L^2(0,T; B)$)

Let u_j be the solution of

$$\begin{cases} u_j' + A u_j = f_j \\ u_j(0) = \varphi \end{cases}$$

Then $u_j' - u_k' + A(u_j - u_k) = f_j - f_k$ by which

$$\| u_j - u_k \|_W \leq c \, \| f_j - f_k \|_{V'}$$

and then $\{u_j\}_W$ is a Cauchy sequence in W

(and then in $C^0([0,T]; L^2(\Omega))$.

Then we get that $\forall \ \varphi \in L^2(\Omega)$ and $f \in L^2(0,T; H^{-1}(\Omega))$ there is a unique solution $u \in W$ of (P).

$\square$

One can prove more :

$u_m \to u$ in $V = L^2(0,T; H^1_0(\Omega))$ and
in $C^0([0,T]; L^2(\Omega))$ (! strongly)

[See ZEIDLER, Nonlinear Functional Analysis and its Applications Volume II/A, chapter 23, section 23.9]

REMARK Everything can be done for

$$V \subset H \subset V'$$

with $H = L^2(\Omega)$ and $H^1_0(\Omega) \subseteq V \subseteq H^1(\Omega)$

- $V = H^1_0(\Omega)$ Dirichlet null boundary condition
- $V = H^1(\Omega)$ Neumann " " "
- $V = \{ u \in H^1(\Omega) \mid Tu = 0 \text{ in } \Gamma \subsetneq \partial\Omega, \ \Gamma \neq \emptyset \}$

REGULARITY RESULTS

$$\mathcal{A} u = -\operatorname{div}(a \cdot Du) + b\, Du + c\, u \qquad \text{satisfies } (H)$$

Theorem (space) Suppose a, b, c independent of t,

a symmetric, $\quad a_{ij} \in W^{1,\infty}(\Omega)$

$b_j, c \in L^\infty(\Omega)$

Assume $f \in L^2(0,T; L^2(\Omega))$

$\varphi \in H^1_0(\Omega)$

Then the solution of

$$\begin{cases} u' + \mathcal{A} u = f \\ u(0) = \varphi \end{cases} \qquad (P)$$

satisfies:

$$u \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1_0(\Omega))$$

$$u' \in L^2(0,T; L^2(\Omega))$$

[See Evans, PDEs, chapter 7]

Given $A(t) := -\operatorname{div}(a(\cdot,t)\cdot D) + b(\cdot,t)\cdot D + c(\cdot,t)\,u$

We define

$$A'(t) := -\operatorname{div}(a'(\cdot,t)\cdot D + b'(\cdot,t)\cdot D + c'(\cdot,t)\,u$$

where $(a')_{ij}(x,t) = \frac{\partial}{\partial t} a_{ij}(x,t)$

$$b'_j(x,t) = \frac{\partial}{\partial t} b_j(x,t), \qquad c'(x,t) = \frac{\partial c}{\partial t}(x,t).$$

If a_{ij}, b_j, c admit a weak derivative w.r.t. t

and $\frac{\partial a_{ij}}{\partial t}, \frac{\partial b_j}{\partial t}, c \in L^\infty(\Omega \times (0,T))$ then

$$\langle \mathcal{A}' u, v \rangle_{\mathcal{V}' \times \mathcal{V}} \leq \text{const} \, \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}'} \qquad (\bullet)$$

where

$$\mathcal{A}' u(t) := A'(t)\, u(t) \qquad \text{for } u \in \mathcal{V}.$$

Theorem (time) a, b, c satisfy (H). Moreover $\mathcal{A}'$ bounded from $\mathcal{V}$ to $\mathcal{V}'$ (as in $(\bullet)$)

$f \in H^1(0,T; V')$ $\qquad (H^1_0(\Omega) \subseteq V \subseteq H^1(\Omega))$

$\varphi \in V$

and $\quad f(0) - A(0)\varphi \in V$

Then the solution of (P) satisfies

$$u \in H^1(0,T; V) \,, \quad u'' \in L^2(0,T; V')$$

[see Wloka, PDEs theorem 27.2]

Theorem If A is symmetric ($b \equiv 0$)
$f \in L^2(0,T; L^2(\Omega))$, $\varphi \in L^2(\Omega)$.
Then u satisfies

$$t^{1/2} u' \in L^2(0,T; L^2(\Omega))$$

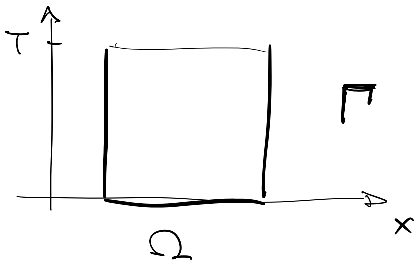
and for each $\delta > 0$

$$u \in H^1(\delta,T; L^2(\Omega)) \cap C^0([\delta,T]; V)$$

[see Showalter, Monotone Operators in Banach Spaces and Nonlinear PDEs, chapter VI, section 2]

WEAK MAXIMUM PRINCIPLE

Define $\Gamma := \left(\partial\Omega \times (0,T)\right) \cup \left(\Omega \times \{0\}\right)$



Notation $u \in C^2_1(\Omega \times (0,T))$ if

$u, u_t, u_{x_i}, u_{x_i x_j} \in C^0(\Omega \times (0,T))$

Theorem Assume $u \in C^2_1(\Omega \times (0,T)) \cap C^0\left(\overline{\Omega \times (0,T)}\right)$

Ω bounded and $c \equiv 0$.

i) if $u_t + Au \leq 0$ in $\Omega \times (0,T)$

then $\max\limits_{\overline{\Omega \times (0,T)}} u = \max\limits_{\Gamma} u$

ii) if $u_t + Au \geq 0$ in $\Omega \times (0,T)$

then $\min\limits_{\overline{\Omega \times (0,T)}} u = \min\limits_{\Gamma} u$

If $c \geq 0$:

i)' $u_t + \Delta u \leq 0 \Rightarrow \max\limits_{\overline{\Omega \times (0,T)}} u \leq \max\limits_{\Gamma} u_+$

ii)' $u_t + \Delta u \geq 0 \Rightarrow \min\limits_{\overline{\Omega \times (0,T)}} u \geq - \max\limits_{\Gamma} u_-$

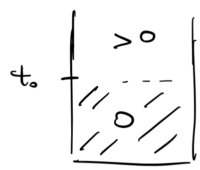
In particular for a solution $(c \geq 0)$

$$\max_{\overline{\Omega \times (0,T)}} u = \max_{\Gamma} |u|$$

Example : i) $\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u(x,t) = \psi(x,t) & \partial\Omega \times (0,T) \\ u(x,0) = 0 & \text{in } \Omega \end{cases}$

With $\psi = \begin{cases} 0 & \text{if } t \leq t_0 \\ > 0 & \text{for } t > 0 \end{cases}$

Then the solution is



HARNACK'S INEQUALITY

Assume $u \in C^2_1(\Omega \times (0,T))$ solves $u_t + \Delta u = 0$ and $u \geq 0$ in $\Omega \times (0,T)$. Suppose $\omega \subset\subset \Omega$. Then for every $t_1, t_2 \in (0,T)$, $t_1 < t_2$

$\exists\ c > 0$ s.t.

$$\sup_{\omega} u(\cdot, t_1) \leq c \inf_{\omega} u(\cdot, t_2)$$

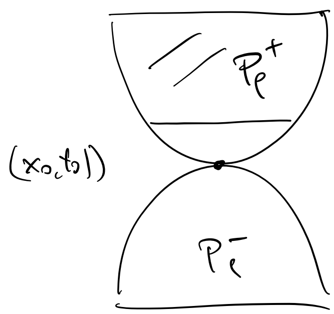
$c = c(\omega, t_2 - t_1)$.

Equivalent formulations: given $(x_0, t_0) \in \Omega \times (0,T)$ and $\rho > 0$ such that $B_\rho(x_0) \times (t_0 - \rho^2, t_0 + \rho^2) \subset\subset \Omega \times (0,T)$

$\exists\ c > 0$ such that

- $$\sup_{x \in B_\rho(x_0)} u(x, t_0 - \rho^2) \leq c\, u(x_0, t_0)$$
- $$u(x_0, t_0) \leq c \inf_{x \in B_\rho(x_0)} u(x, t_0 + \rho^2)$$
- $$\sup_{P_\rho^-(x_0, t_0)} u \leq c\, u(x_0, t_0)$$

$$= \quad u(x_0,t_0) \leq c \inf_{P_\rho^+(x_0,t_0)} u$$



$$P_\rho^+(x_0,t_0) = \bigcup_{0<t<\rho} \left(B_t(x_0) \times \{t_0 + t^2\} \right)$$

Why different times?

Take $G(t,x) = \dfrac{1}{\sqrt{4\pi t}} \exp\left(-\dfrac{x^2}{4t}\right)$ $\quad x \in \mathbb{R}, \; t \in (0,+\infty)$

and define $G_\xi(x,t) := G(x+\xi, t)$

Since $$\frac{G_\xi(0,1)}{G_\xi(x,1)} \xrightarrow[\xi \to +\infty]{} \begin{cases} 0 & \text{if } x<0 \\ +\infty & \text{if } x>0 \end{cases}$$

then the ratio cannot be bounded and $\sup G(\cdot,1)$ cannot be controlled by $\inf G(\cdot,1)$.

Also if you consider the solution of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(x,0) = \varphi(x) & \text{in } \mathbb{R}^n \end{cases}$$

$\varphi = \chi_K$, K compact set

the solution is $u(x,t) = \begin{cases} G_t * \varphi \; (x) & t > 0 \\ \varphi(x) & t = 0 \end{cases}$

$$G_t(x) = G(x,t)$$

Notice that $u(x,t) > 0 \quad \forall x \in \mathbb{R}^n$ if $t > 0$

while $u(x,0)$ is null outside of K.

Moreover $u(\cdot,0)$ is discontinuous while

$u(\cdot,t) \in C^\infty(\mathbb{R}^n)$ for $t > 0$.

$$\inf_A u(x,0) = 0 \quad \text{if } A \cap K \neq A \quad (A \not\subseteq K)$$

then we need to consider $\omega \subset\subset \Omega$

$(t_1,t_2) \subset\subset (0,T)$

Another example: (non-linear case)

Barenblatt function

$$T_p(x,t) = \frac{1}{t^{n/\lambda}} \left\{ \left[1 - \gamma_p \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right]_+ \right\}^{\frac{p-1}{p-2}}, \quad t > 0$$

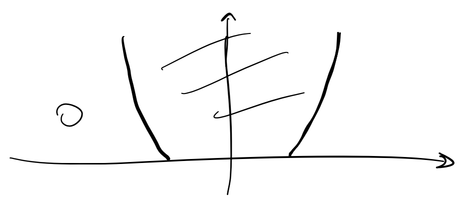
With $\lambda = n(p-2)$, γ_p a constant $\quad$ $p > 2$

$$\lim_{p \to 2^+} T_p(x,t) = G(x,t)$$

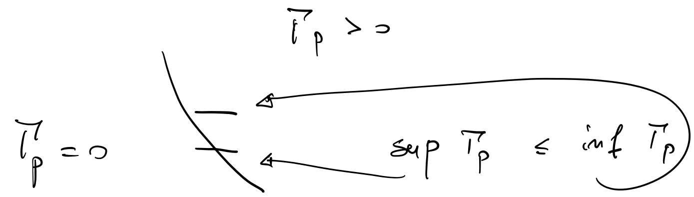
T_p solves $u_t - \operatorname{div}\left(|Du|^{p-2} Du\right) = 0$

If $\gamma_p |x| < t^{\frac{1}{\lambda}} \Rightarrow T_p(x,t) = 0$

T_p compactly supported (for fixed t)



Then it is not possible to control the supremum of T_p by the infimum at same time!



Meaning of parabolic Harnack: to control the supremum of u by its infimum you have to wait a little bit