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STABILITY OF $\mathscr{L}_4$ AND $\mathscr{L}_5$ IN THE RESTRICTED PLANAR THREE–BODY PROBLEM

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Contents

Introduction

The aim of this thesis is to examine the stability of the equilibria $\mathscr{L}_4$ and $\mathscr{L}_5$ of the Restricted Planar Three–Body Problem (RPTBP) by means of the KAM theory. The RPTBP is a particular case of the Three–Body Problem (TBP), which regards the study of the dynamics of a system of three heavy bodies subject only to mutual gravitational forces.

Attempts of solution of the TBP date back to Newton, and the triangular equilibria were found by Lagrange ([Card]). Whereas the conditions on the system's parameters for the stability of these equilibria in the linearized problem can be computed without difficulty (see [M&H, Szeb]), the problem of the Lyapunov stability, instead, was solved only in the second half of the last century using perturbation theory, and specifically the KAM approach ([Ar2, M&H, Pösch]), which makes use of Birkhoff Normal Forms ([Ben1, Ben2, FasLew]).

Physical interest of Restricted Three–Body Problem is attested in the study of Trojan asteroids, which hold the surroundings of $\mathscr{L}_4$ and $\mathscr{L}_5$ of the Sun–Jupiter system (see [Marz, Harv, Marchis, Morbi, Szeb]).

As of our work, first we obtain the Hamiltonian function and the equilibria of the system, along with the condition on the masses for the stability in the linearized problem. Classical Lyapunov methods, though, are found to be not sufficient to reach conclusions about the Lyapunov stability, and therefore modern techniques need to be used.

The second chapter outlines the properties of KAM theory and its behaviour near the equilibria of a system, after the Hamiltonian has been put in normal form by means of Birkhoff series, which are described in the third chapter.

The fourth chapter faces the specific problem of applying KAM theory to the triangular equilibria, gives a list of results in this field, and finally repeats in numerical form the construction of normal forms which allows to state the Lyapunov stability of $\mathscr{L}_4$ and $\mathscr{L}_5$.

Finally, a brief description of an astronomical case of Restricted Three–Body Problem is sketched in appendix, where some characteristics of the Trojan asteroids are reviewed.

Chapter 1

Equations of motion and equilibria

The Restricted Three–Body Problem concerns the study of the motion of one celestial body in the gravitational field of two other bodies (conventionally called the *primaries*) moving along circular Keplerian orbits around their center of mass. The third body has small mass with respect to the others', and is treated like a test particle whose motion results determined by the two bodies, yet without affecting their motion in turn. It is an approximation of the Three–Body Problem, which regards the study of the dynamics of three masses interacting by means of the gravitational force.

In this work we will consider the planar case, with two degrees of freedom; we remark that the problem is not integrable.

This simplification of the Three–Body Problem can be applied to several astronomical systems; the most studied cases are those in which one of the massive bodies is the Sun and the other Jupiter or the Earth, the third body being for example (in the case of Jupiter) an asteroid, but is also studied the problem with Earth and Moon as the two massive bodies, the third body being an artificial satellite.

1.1 Lagrangian and Hamiltonian functions

The Lagrangian of the Planar Restricted Three–Body Problem is:

$$L'(q',\dot{q}',t) = \frac{1}{2}m\,|\dot{q}'|^2 + \frac{Gmm_A}{\|q'-q_A(t)\|} + \frac{Gmm_B}{\|q'-q_B(t)\|}, \quad q' \in \mathbb{R}^2$$

where $q', \dot{q}'$ are coordinates and velocities of the point in study, $q_A(t), q_B(t), m_A, m_B$ are coordinates and masses of the primaries, and G is the universal gravitational constant.

We assume that the center of gravity of the system of the primaries is in the origin of the coordinates, and that they move on a circular Keplerian orbit, with angular velocity

$$\tilde{\omega} = \sqrt{\frac{G\mathcal{M}}{\mathcal{R}^3}}$$

where $\mathcal{R}$ is the distance between A and B, and $\mathcal{M} = m_A + m_B$.

Then, with a rotation:

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos\tilde{\omega}t & -\sin\tilde{\omega}t \\ \sin\tilde{\omega}t & \cos\tilde{\omega}t \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix}$$

we pass to the so–called *synodic* frame of reference in which both A and B are at rest on the q_1-axis. We introduce the parameter $\mu = \frac{m_A}{\mathcal{M}}$, $0 < \mu < 1$, so that the primaries have masses $\mathcal{M}\mu$, $\mathcal{M}(1-\mu)$, and their coordinates are (respectively) $(\mathcal{R}(1-\mu), 0)$ and $(-\mathcal{R}\mu, 0)$.

The resulting Lagrangian function is, neglecting the common factor m:

$$\begin{aligned} L(q_1, q_2, \dot{q}_1, \dot{q}_2) = {} & \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \tilde{\omega}(\dot{q}_1 q_2 - q_1 \dot{q}_2) + \frac{1}{2}\tilde{\omega}^2(q_1^2 + q_2^2) + \\ & + \frac{G\mathcal{M}\mu}{\sqrt{(q_1 - \mathcal{R} + \mathcal{R}\mu)^2 + q_2^2}} + \frac{G\mathcal{M}(1-\mu)}{\sqrt{(q_1 + \mathcal{R}\mu)^2 + q_2^2}} \end{aligned}$$

The passage to rotating coordinates makes the system autonomous at the cost of the appearance of new terms (Coriolis and centrifugal) in the Lagrangian.

Now we perform the Legendre transform $(q, \dot{q}) \xrightarrow{\Lambda} (q, p = \frac{\partial L}{\partial \dot{q}})$ (in particular, we have $p_1 = \dot{q}_1 + \tilde{\omega} q_2$, $p_2 = \dot{q}_2 - \tilde{\omega} q_1$) and obtain the Hamiltonian:

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2) - \tilde{\omega}(q_1 p_2 - q_2 p_1) - \frac{G\mathcal{M}\mu}{\sqrt{(q_1 - \mathcal{R} + \mathcal{R}\mu)^2 + q_2^2}} - \frac{G\mathcal{M}(1-\mu)}{\sqrt{(q_1 + \mathcal{R}\mu)^2 + q_2^2}} \tag{1.1}$$

1.2 Equilibria

Rescaling the coordinates and the momenta with the canonical transformation $q \mapsto \mathcal{R}q$, $p \mapsto \tilde{\omega}\mathcal{R}p$, we manage to remove the constants from (1.1), and rescaling the time with the transformation $t \mapsto \tilde{\omega}t$ we move $\tilde{\omega}$ out of H, passing to a dimensionless formulation of the problem in which the total mass of the system and the distance between the two primaries are both unity, and (1.1) becomes:

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2) - q \cdot \mathbb{J}_2 p - U(q) \tag{1.2}$$

where

$$U(q) = \frac{\mu}{d_1(q)} + \frac{1-\mu}{d_2(q)}$$

with

$$\begin{aligned} d_1(q) &= \sqrt{(q_1 - 1 + \mu)^2 + q_2^2} \\ d_2(q) &= \sqrt{(q_1 + \mu)^2 + q_2^2} \end{aligned}$$

and for any $n = 2, 4, \dots$

$$\mathbb{J}_n = \begin{pmatrix} 0_r & \mathbb{1}_r \\ -\mathbb{1}_r & 0_r \end{pmatrix}$$

where $r = n/2$ and $\mathbb{1}_r$ is the $r \times r$ identity matrix.

Proposition 1. *The system whose Hamiltonian is (1.2) has 5 equilibria: $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_4, \mathscr{L}_5$.*

- *$\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$, named the Euler collinear points, lay on the q_1–axis.*
- *$\mathscr{L}_4, \mathscr{L}_5$, named the Lagrange equilateral points, have coordinates:*

$$\begin{aligned} \mathscr{L}_4 &= (q_1^{\mathscr{L}_4}, q_2^{\mathscr{L}_4}, p_1^{\mathscr{L}_4}, p_2^{\mathscr{L}_4}) = (\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2} - \mu) \\ \mathscr{L}_5 &= (q_1^{\mathscr{L}_4}, -q_2^{\mathscr{L}_4}, p_1^{\mathscr{L}_4}, -p_2^{\mathscr{L}_4}) \end{aligned} \tag{1.3}$$

Proof. In order to find the equilibria, the partial derivatives of H as in (1.2) must be null, i.e.:

$$\begin{cases} \frac{\partial H}{\partial q} = -\mathbb{J}_2 p - \frac{\partial U}{\partial q} = 0 \\ \frac{\partial H}{\partial p} = p + \mathbb{J}_2 q = 0 \end{cases}$$

which is

$$q + \frac{\partial U}{\partial q} = 0 \tag{1.4}$$

If we set

$$V(q) = \frac{1}{2} \|q\|^2 + U(q) \tag{1.5}$$

we have $\frac{\partial V}{\partial q} = q + \frac{\partial U}{\partial q}$, so that equation (1.4) can be solved via the search of the critical points of V.

To do that, we observe that the map $q \mapsto d(q) = (d_1(q), d_2(q))$ is a local diffeomorphism at each point q with $q_2 \neq 0$. Therefore, in order to find the equilibria for $q_2 \neq 0$, instead of the critical points of $V(q)$ we consider those of:

$$\tilde{V}(d) = \frac{1}{2} \|d\|^2 + \tilde{U}(d)$$

with $\tilde{U}(d(q)) = U(q)$.

In particular, we note that

$$\|q\|^2 = q_1^2 + q_2^2 = \mu d_1^2 + (1-\mu) d_2^2 - \mu(1-\mu)$$

so that

$$\frac{\partial \tilde{V}}{\partial d}(d) = 0 \Leftrightarrow \begin{cases} \mu d_1 - \frac{\mu}{d_1^2} = 0 \\ (1-\mu) d_2 - \frac{(1-\mu)}{d_2^2} = 0 \end{cases} \tag{1.6}$$

system whose only solution is $d_1 = d_2 = 1$. Correspondently, in the q coordinates there are two solutions placed at the vertexes of two equilateral triangles whose common base is the segment joining the primaries. These are $\mathscr{L}_4$ and $\mathscr{L}_5$.

Besides, we consider the equilibria on the segment joining the primaries (i.e. with $q_2 = 0$). Because we have

$$\frac{\partial V}{\partial q_2}(q_1, 0) = 0 \quad \forall q_1 \in \mathbb{R}$$

the equilibria with $q_2 = 0$ are determined by the critical points of the function $q_1 \mapsto V(q_1, 0)$. Since

$$V(q_1, 0) = \frac{1}{2}q_1^2 + \frac{\mu}{|q_1 - 1 + \mu|} + \frac{1-\mu}{|q_1 + \mu|}$$

is smooth at all points $q_1 \neq -\mu, 1-\mu$, and

$$\lim_{q_1 \to \pm\infty} V(q_1, 0) = +\infty$$

$$\lim_{q_1 \to -\mu} V(q_1, 0) = +\infty$$

$$\lim_{q_1 \to 1-\mu} V(q_1, 0) = +\infty$$

it results that V has at least one critical point in each of the intervals $]-\infty, -\mu[$, $]-\mu, 1-\mu[$, $]1-\mu, +\infty[$.

Since in each of these intervals we have

$$\frac{d^2V}{dq_1^2}(q_1, 0) = 1 + \frac{\mu}{(|q_1 - 1 + \mu|)^3} + \frac{1-\mu}{(|q_1 + \mu|)^3} > 0$$

V is there convex, hence we conclude that V has exactly three critical points. These are $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$. $\square$

Figure (1.2) shows $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, $\mathscr{L}_4$, $\mathscr{L}_5$.

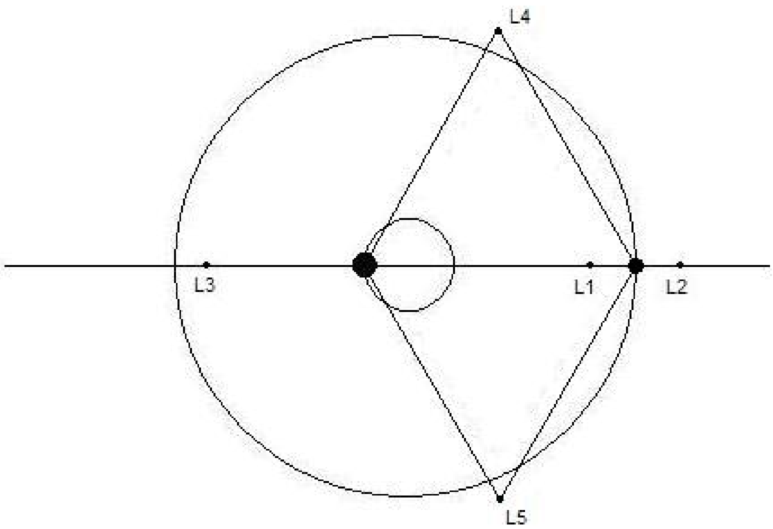

Figure 1.1: The five equilibria of the Restricted Three–Body Problem.

1.3 Linearization at $\mathscr{L}_4$

Now we turn to the study of the Lyapunov stability of the triangular points. Because of the symmetry of the problem with respect to the line joining the primaries, we can restrict our consideration to $\mathscr{L}_4$, defined by (1.3). We start with the first Lyapunov method (spectral method), which will give sufficient (though not necessary) conditions for the stability.

By definition, the Hamiltonian vector field X_H relative to a Hamiltonian H is:

$$X_H = \mathbb{J}_4 \nabla H$$

so that the linearization X'_H of X_H at an equilibrium $\mathcal{E} = (\mathcal{E}_q, \mathcal{E}_p)$ is:

$$X'_H(\mathcal{E}_q, \mathcal{E}_p) = \mathbb{J}_4 H''(\mathcal{E}_q, \mathcal{E}_p),$$

where H'' denotes the Hessian of H, namely

$$X'_H(\mathcal{E}_q, \mathcal{E}_p) = \begin{pmatrix} \mathbb{J}_2 & \mathbb{1}_2 \\ U''(\mathcal{E}_q) & \mathbb{J}_2 \end{pmatrix} \tag{1.7}$$

Now we study the eigenvalues of $X'_H(\mathcal{E}_q, \mathcal{E}_p)$ to apply the first Lyapunov method. From (1.5) we have $U_{ii} = V_{ii} - 1, U_{jk} = V_{jk}, j \neq k$ and the characteristic equation is:

$$\lambda^4 + [4 - V_{11}(\mathcal{E}_q) - V_{22}(\mathcal{E}_q)]\lambda^2 - [V_{12}(\mathcal{E}_q)]^2 + V_{11}(\mathcal{E}_q)V_{22}(\mathcal{E}_q) = 0$$

In the specific case of $\mathscr{L}_4$ we have $V_{11}((\mathscr{L}_4)_q) = \frac{3}{4}$, $V_{12}((\mathscr{L}_4)_q) = \frac{3\sqrt{3}}{4}(1-2\mu)$, $V_{22}((\mathscr{L}_4)_q) = \frac{9}{4}$, and the characteristic equation is:

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

which has four roots, given by:

$$\lambda^2 = \frac{1}{2}[-1 \pm \sqrt{1 - 27\mu(1-\mu)}].$$

Thus we conclude that:

- If $\mu > \mu_{Routh}$, where

  $$\mu_{Routh} := \frac{1}{2}(1 - \frac{\sqrt{69}}{9}) = 0.0385208965\ldots$$

  all the eigenvalues have nonzero real part so that, by Lyapunov first method, $\mathscr{L}_4$ is not stable.

- If $\mu \leq \mu_{Routh}$, we have purely imaginary eigenvalues $\pm i\hat{\omega}_1$, $\pm i\hat{\omega}_2$, where:

$$\hat{\omega}_1 = \sqrt{\frac{1}{2}[1 + \sqrt{1 - 27\mu(1-\mu)}]} \quad > \quad \hat{\omega}_2 = \sqrt{\frac{1}{2}[1 - \sqrt{1 - 27\mu(1-\mu)}]} \tag{1.8}$$

  and $\mathscr{L}_4$ is an elliptic equilibrium.

We will study the stability of $\mathscr{L}_4$ in the latter case. To this end, we must remark that it is not known whether the second Lyapunov method can be applied to study the stability of $\mathscr{L}_4$. In particular the Hamiltonian H cannot be used as a Lyapunov function, because H'' is not definite; from the point of view of the Lagrange–Dirichlet theorem, this corresponds to the fact that U does not have an isolated minimum at the equilibria.

However, since the planar problem has two degrees of freedom, we can approach the problem of the stability of $\mathscr{L}_4$ for $\mu \leq \mu_{Routh}$ by means of KAM theory.

Chapter 2

Elements of KAM theory

In this chapter we will review some properties of KAM theory we will later use to study the stability of the triangular equilibria.

2.1 KAM theory

First, let us recall an important definition:

Definition 1. *A Hamiltonian system with Hamiltonian $H(p,q)$ is called* completely integrable *in a domain $B \subset \mathbb{R}^{2n}$ if there exists a canonical diffeomorphism $w : B \to D \times \mathbb{T}^n$, where $D \subset \mathbb{R}^n$ and $\mathbb{T}^n$ is the n–dimensional torus, such that the transformed Hamiltonian $\tilde{H} = H \circ w^{-1}$ does not depend on the angles:*

$$\tilde{H}(I,\varphi) = K(I)$$

for some function $K : D \to \mathbb{R}$. I and φ are called action–angle *variables.*

At the International Mathematical Congress in Amsterdam, in 1954, A.N. Kolmogorov gave an address on "The general theory of dynamical systems and classical mechanics". This event has played an important role in the development of the so–called KAM (Kolmogorov–Arnold–Moser) theory, which has been subsequently studied and brought forth by Arnol'd, Moser and many others.

This theory refers to small perturbations of completely integrable Hamiltonian dynamical systems, with Hamiltonian (in action–angle variables) of the form:

$$H(I,\varphi) = K(I) + \varepsilon F(I,\varphi), \quad I \in D \subset \mathbb{R}^n, \varphi \in \mathbb{T}^n \tag{2.1}$$

The equations of motion relative to (2.1) are:

$$\begin{cases} \dot{I} = -\varepsilon \frac{\partial F}{\partial \varphi}(I,\varphi) \\ \dot{\varphi} = \omega(I) + \varepsilon \frac{\partial F}{\partial I}(I,\varphi) \end{cases} \tag{2.2}$$

where
$$\omega := \frac{\partial K}{\partial I}(I) : D \to \mathbb{R}^n \tag{2.3}$$
is the frequency map. If $\varepsilon = 0$, the actions I (slow variables) are constant, whereas the angles φ (fast variables) move linearly on the tori $I = \text{const}$ (the motion is *quasi–periodic*).

The basic result of the KAM theory is that, for analytic small perturbations of analytic integrable systems, if the frequencies are nondegenerate in a sense to specify, most of the invariant tori do not disappear, but are just a little bit deformed. The motions of the perturbed system are quasi-periodic on a deformed torus and fill it everywhere densely.

Let us set $\Omega := \omega(D)$, and $\forall\,\gamma > 0$ let us define the set of the γ*–Diophantine* frequencies as:
$$\Omega_\gamma = \{\bar{\omega} \in \Omega : |k \cdot \bar{\omega}| \geq \frac{\gamma}{\|k\|_\infty^n} \ \forall k \in \mathbb{Z}^n \setminus \{0\}\} \tag{2.4}$$
where:
$$\|k\|_\infty = |k_1| + |k_2| + \cdots + |k_n|$$
It is known that, if γ is sufficiently small (see [Ben1]), then:
$$\mu(\Omega \setminus \Omega_\gamma) = \mathcal{O}(\gamma)$$
where μ denotes the Lebesgue measure, so that Ω_γ is large for small γ.

A possible formulation of KAM theorem is (see [Pösch, Ben1]):

Theorem 1 (KAM)**.** *Consider a Hamiltonian H of the form (2.1). Suppose that:*

1. *H is analytic in the domain $D \times \mathbb{T}^n \ni (I, \varphi)$.*
2. *The frequency map (2.3) is a local diffeomorphism at each point, i.e. it satisfies (nondegeneracy condition):*
$$\det[\frac{\partial \omega}{\partial I}(I)] \neq 0 \quad \forall I \in D \tag{2.5}$$

Then there exist positive numbers δ and γ_0 independent of ε, such that for all $\gamma \leq \gamma_0$ and $|\varepsilon| \leq \gamma^2\delta$, there exists a diffeomorphism:
$$T : D \times \mathbb{T}^n \to D \times \mathbb{T}^n$$
which on $D_\gamma \times \mathbb{T}^n$, with $D_\gamma = \omega^{-1}(\Omega_\gamma) \setminus U_\gamma$ (where U_γ is a neighbourhood of ∂D with measure $\mathcal{O}(\gamma)$), conjugates the flow of the unperturbed system to that of the perturbed system expressed by equations (2.2).

Thus, the surviving tori are those of frequencies belonging to Ω_γ. Applying theorem 1 with $\gamma = \mathcal{O}(\sqrt{\varepsilon})$, and taking into account that the map ω is a local diffeomorphism, one concludes that the complement of the set of actions corresponding to preserved tori has measure $\mathcal{O}(\sqrt{\varepsilon})$ so that each point in D has distance at most $\mathcal{O}(\varepsilon^{\frac{1}{2n}})$ from a preserved torus.

2.2 KAM theory in two degrees of freedom

First, let us state an important definition:

Definition 2. *A system with Hamiltonian as (2.1) is said to be* isoenergetically non–degenerate *if:*

$$\det\begin{pmatrix} \frac{\partial^2 K}{\partial I^2}(I) & \frac{\partial K}{\partial I}(I) \\ \frac{\partial K}{\partial I}(I) & 0 \end{pmatrix} = \det\begin{pmatrix} \frac{\partial \omega}{\partial I}(I) & \omega(I) \\ \omega^T(I) & 0 \end{pmatrix} \neq 0 \quad \forall I \in D \tag{2.6}$$

Let us study the case of two degrees of freedom ($n = 2$): the phase space has dimension 4, so that the surfaces of constant energy of the unperturbed Hamiltonian are three–dimensional and have the form(see fig. 2.2):

$$K_e = \{(I, \varphi) : K(I) = e\}$$

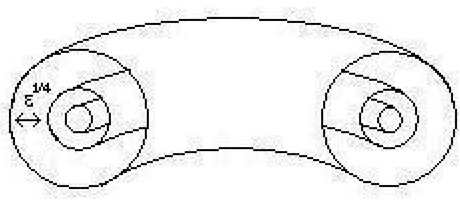

Figure 2.1: The tori relative to K

Now we remark an important distinction:

- If $n = 2$, since the phase space has dimension 4, the surfaces of constant energy $H = const$ are three–dimensional, so that the two–dimensional tori 'divide' these surfaces, and a motion starting in a gap between two deformed tori cannot 'escape' from this gap.
  Correspondently, the actions remain nearly constant and we can conclude that the motion is stable.

- If $n > 2$, instead, the surfaces of constant energy are $(2n - 1)$–dimensional, so that the n–dimensional tori do not "divide" them, and there may exist motions in which the actions evolve significantly. The unstable motion of the actions is called *Arnold diffusion.*

The first point develops into the following proposition:

Proposition 2. *Let the hypotheses of theorem 1 be verified, and let condition (2.6) hold. Then, along any motion* $t \mapsto (I_t, \varphi_t)$ *of the system with Hamiltonian (2.1) we have:*

$$|I_t - I_0| = \mathcal{O}(\sqrt[4]{\varepsilon}) \tag{2.7}$$

Proof (sketch). Let H be as in (2.1) and let us consider a motion on $H = \mathrm{const}$. First, we observe that condition (2.6) is equivalent to the fact that the images $K'_e = \omega(K_e)$ of the surfaces K_e of constant energy are transversal to the halflines out of the origin. Indeed, this transversality condition is equivalent to the relation

$$0 \neq u \in \mathbb{R}^2, \quad u \cdot \omega(I) = 0 \quad \Rightarrow \quad \frac{\partial \omega}{\partial I}(I) u \nparallel \omega$$

namely

$$\begin{cases} u \cdot \omega(I) = 0 \\ \frac{\partial \omega}{\partial I}(I) u \parallel \omega(I) \end{cases} \quad \Rightarrow \; u = 0$$

That is:

$$\begin{cases} u \cdot \omega(I) = 0 \\ \frac{\partial \omega}{\partial I}(I) u + \lambda \omega(I) = 0 \end{cases} \quad \Rightarrow \; u = 0, \, \lambda = 0 \tag{2.8}$$

The one above is a linear system in u and λ which reads:

$$\begin{pmatrix} \frac{\partial \omega}{\partial I} & \omega \\ \omega^T & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.9}$$

and (2.6) implies that (2.9) has only the trivial solution.

Proof of (2.7) develops now onto the following steps:

I. The frequencies of the image of the preserved tori in the frequencies' plane (ω_1, ω_2) have the structure of closed halflines out of the origin, each of which corresponds to a Diophantine frequency $\bar{\omega}$ and its multiples $\lambda\bar{\omega}$, $\lambda > 1$. In this framework, each level curve of K is mapped by the frequency map $\omega \circ \pi_I$ into the level curve $K' = K \circ \omega^{-1}$ ($\pi_I : (I, \phi) \mapsto I$ is the projection of the phase space onto the actions' space).

II. In the phase space (which is 4–dimensional), each closed halfline corresponds to a 3–dimensional 1–parametric family of tori $\mathbb{T}^2$. If in the frequencies' plane the rays are transversal to the level curves of K' (equivalently, if (2.6) holds), then in the phase space the 1-parametric families are transversal to the level surfaces of the unperturbed Hamiltonian K. For ε small enough, transversality holds with respect to the level surfaces of the perturbed Hamiltonian H, too.

    Since the gaps between the tori in the plane (ω_1, ω_2) (and, consequently, in the phase space) are $\mathcal{O}(\sqrt[4]{\varepsilon})$ wide, and the frequency map $\omega(I)$ is differentiable, it follows that on the level curves of H the gaps between two KAM tori are $\mathcal{O}(\sqrt[4]{\varepsilon})$-close to the unperturbed ones.

III. Finally, the tori projected back in the actions' plane (I_1, I_2) occupy a region $\mathcal{O}(\sqrt[4]{\varepsilon})$ wide, which proves (2.7).

A graphical illustration of steps I, II, III is given in figure 2.2.

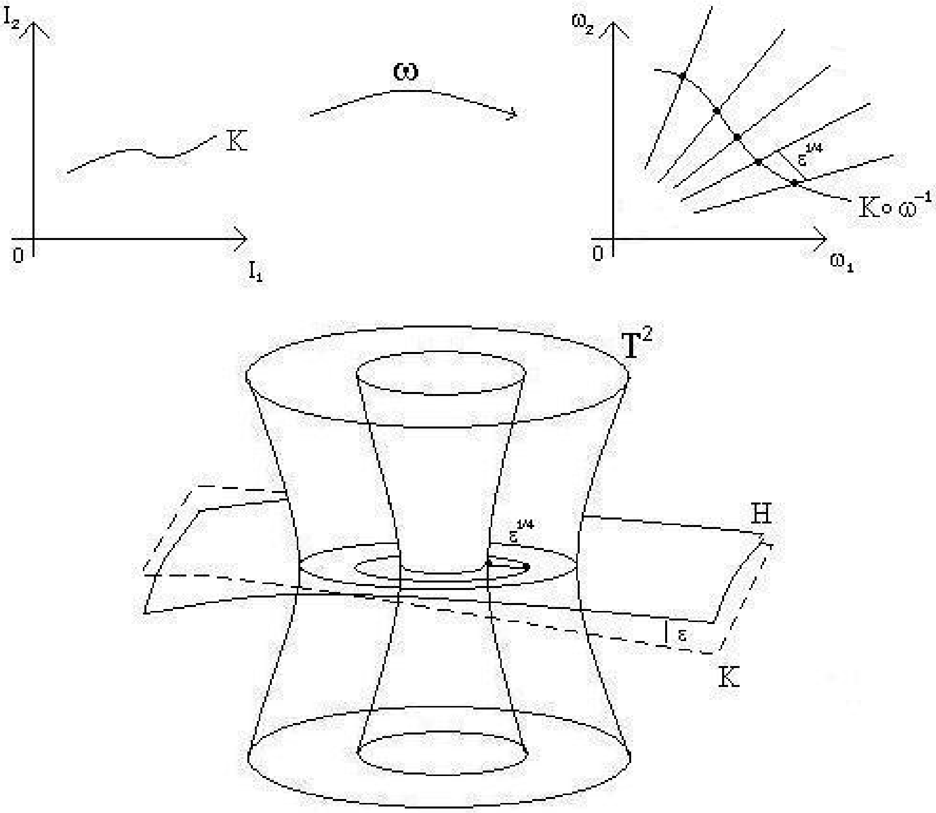

Figure 2.2: The steps I, II and III.

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Chapter 3

Symplectic diagonalization and Birkhoff normal forms

This chapter shows how to put the quadratic terms of the Hamiltonian function near an elliptic equilibrium in a diagonal form and then explains the elements of Birkhoff series theory necessary for our purpose.

3.1 Symplectic diagonalization

We show here how to diagonalize a real Hamiltonian matrix with a symplectic linear transformation, and then apply the result to our case.

Definition 3. – *A $2n \times 2n$ matrix S is called* ***symplectic*** *iff $S^T \mathbb{J}_{2n} S = \mathbb{J}_{2n}$.*

– *A $2n \times 2n$ matrix P is called* ***Hamiltonian*** *iff $P^T \mathbb{J}_{2n} = -\mathbb{J}_{2n} P$.*

– *A basis $\{x_1, x_2, \ldots, x_{2n}\}$ of $\mathbb{R}^{2n}$ is called* ***symplectic*** *iff $x_i \cdot \mathbb{J}_{2n} x_j = (\mathbb{J}_{2n})_{ij}$.*

Proposition 3. *Let P be a $2n \times 2n$ Hamiltonian matrix, with distinct, purely imaginary (hence nonzero) eigenvalues $\pm i\alpha_1, \ldots, \pm i\alpha_n$. Let $x_i^{\pm} = x_i' \pm i x_i''$ ($x_i', x_i'' \in \mathbb{R}^{2n}$) be the corresponding eigenvectors. Then, $x_i' \cdot \mathbb{J}_{2n} x_i'' \neq 0, i = 1, \ldots, n$.*
Moreover, denote $\beta_i = \sqrt{|x_i' \cdot \mathbb{J}_{2n} x_i''|}$ and

$$z_i' = \begin{cases} x_i'/\beta_i & \text{if } x_i' \cdot \mathbb{J}_{2n} x_i'' > 0 \\ x_i''/\beta_i & \text{if } x_i' \cdot \mathbb{J}_{2n} x_i'' < 0 \end{cases} \qquad z_i'' = \begin{cases} x_i''/\beta_i & \text{if } x_i' \cdot \mathbb{J}_{2n} x_i'' > 0 \\ x_i'/\beta_i & \text{if } x_i' \cdot \mathbb{J}_{2n} x_i'' < 0 \end{cases}$$

Then, the matrix

$$S = \mathrm{col}(z_1', \ldots, z_n', z_1'', \ldots, z_n'') \tag{3.1}$$

is symplectic and the canonical transformation $(q, p) \mapsto (\tilde{q}, \tilde{p}) = S^{-1}(q, p)$ conjugates the Hamiltonian of $P\binom{q}{p}$ to

$$\tilde{h}(\tilde{q}, \tilde{p}) = \sum_{i=1}^{n} \mathrm{sign}(x_i' \cdot \mathbb{J}_{2n} x_i'') \frac{\alpha_i}{2} (\tilde{q}_i^{\,2} + \tilde{p}_i^{\,2})$$

Proof. Since the $2n$ eigenvalues $\pm i\alpha_i$ are distinct, the eigenvectors $x'_i \pm ix''_i$ are linearly independent, and so are x'_i and x''_i, their real and imaginary part. If we denote by W_i the plane spanned by x'_i and x''_i, we have:

$$\mathbb{R}^{2n} = W_1 \oplus \cdots \oplus W_n$$

We define here the *symplectic skew product* of two vectors $u, v \in \mathbb{R}^{2n}$ as:

$$u \cdot \mathbb{J}_{2n} v$$

Each factor W_i of the above decomposition is symplectically orthogonal to the others (i.e. $W_i \cdot \mathbb{J}_{2n} W_j = 0 \quad \forall i \neq j$). In fact, recall that, if λ and $\mu \neq -\lambda$ are eigenvalues of a Hamiltonian matrix A, the corresponding eigenspaces x and y of A are symplectically orthogonal:

$$\lambda x \cdot \mathbb{J}_{2n} y = Ax \cdot \mathbb{J}_{2n} y = -x \cdot \mathbb{J}_{2n} Ay = -\mu x \cdot \mathbb{J}_{2n} y$$

Thus, the restriction of the symplectic skew product to every W_i is non–degenerate (i.e. W_i is symplectic): in fact, by contradiction, let us assume that $\mathbb{R}^{2n} = U \oplus V$ with $U \cdot \mathbb{J}_{2n} V = 0$ and the symplectic skew product degenerates on U. It follows that there exists a vector $0 \neq u \in U$ such that $u \cdot u' = 0 \ \ \forall u' \in U$; because each vector of $\mathbb{R}^{2n}$ is sum of an element of U and of one of V, it would follow that u is symplectically orthogonal to any vector in $\mathbb{R}^{2n}$, against the nondegeneracy of the symplectic skew product on $\mathbb{R}^{2n}$. The fact that the planes W_i are two–dimensional, along with the linear independence of x'_i and x''_i, allows us to conclude that $x'_i \cdot \mathbb{J}_{2n} x''_i \neq 0$, $\forall i = 1, \ldots, n$. Therefore we can define a new basis of $\mathbb{R}^{2n}$ made of the vectors $z'_1, \ldots, z''_n$, with z'_i, z''_i spanning the corresponding W_i. The fact that the matrix S as in (3.1) is symplectic follows from:

$$z'_i \cdot \mathbb{J}_{2n} z''_j = 0 \quad \text{if} \quad i \neq j$$

because the planes W_i are symplectically orthogonal, and from:

$$z'_i \cdot \mathbb{J}_{2n} z''_i = 1$$

which follows by definition of z'_i, z''_i. The Hamiltonian $h(q,p)$ of $P\binom{q}{p}$ is:

$$h(q,p) = \frac{1}{2}(q,p) \cdot M \begin{pmatrix} q \\ p \end{pmatrix}, \quad M = -\mathbb{J}_{2n} P;$$

which is conjugate to:

$$\tilde{h}(\tilde{q}, \tilde{p}) = \frac{1}{2}(\tilde{q}, \tilde{p}) \cdot S^T M S \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} = \sum_{i=1}^{n} \pm \frac{\alpha_i}{2}(\tilde{q}_i^2 + \tilde{p}_i^2)$$

since $S^T M S = -S^T \mathbb{J}_{2n} P S = -\mathbb{J}_{2n} S^{-1} P S = -\mathbb{J}_{2n}[\pm \mathrm{diag}(\alpha_1, \ldots, \alpha_n)\mathbb{J}_{2n}] = \mathrm{diag}(\pm\alpha_1, \ldots, \pm\alpha_n)$. In the previous lines, in each case, one takes the plus or the minus sign according to $x'_i \cdot \mathbb{J}_{2n} x''_i$ being positive or negative. $\square$

3.2 Birkhoff normal forms

If $n = 2$, it follows from proposition 3 that in the expansion of a Hamiltonian near an elliptic equilibrium (set at the origin)

$$H = \sum_{j=2}^{\infty} H_j \tag{3.2}$$

one has

$$H_2 = \sum_{i=1}^{n} \bar{\omega}_i \frac{p_i^2 + q_i^2}{2} \tag{3.3}$$

We would like to find a change of coordinates (Birkhoff series)

$$(q, p) \mapsto (\tilde{q}, \tilde{p})$$

which conjugates H to its *Birkhoff Normal Form* (BNF) of order r

$$H^{(r)} = H_2 + G + f \tag{3.4}$$

where $G = \sum_{j=2}^{r/2} G_{2j}$, and G_{2j} is a j–degree homogenous polynomial in the variables $I_k = \frac{\tilde{p}_k^2 + \tilde{q}_k^2}{2}$ $\quad k = 1, 2$ and f is a power series starting with terms of order $r + 1$.

3.2.1 Construction of BNF

Definition 4. *Let $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_n) \in \mathbb{R}^n$. A* resonance *of order r for $\bar{\omega}$ is a vector $k \in \mathbb{Z}^n$ such that*

$$\sum_{i=1}^{n} |k_i| = r, \quad k \cdot \bar{\omega} = 0$$

and we say $\bar{\omega}$ is nonresonant up to order r if it has no resonances of order $j = 1, \ldots, r$.

We now state the following result:

Theorem 2 (Birkhoff)**.** *Assume that the vector $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_n)$ is nonresonant up to order r. Then there exists a Birkhoff normal form of order r for any power series (3.2) defined in some neighbourhood of the origin with H_2 as in (3.3).*

Proof (sketch). To construct the symplectic transformation that puts (3.2) into BNF, we use the Lie method. This method uses the fact that Φ_χ^t, the map at time t of the flow of an analytic Hamiltonian vector field X with Hamiltonian χ, is analytic in t for small t. Thus, if $F : \mathbb{R}^{2n} \to \mathbb{R}$ is an analytic function, using the relation:

$$\frac{d}{dt}(F \circ \Phi_\chi^t) = \mathcal{L}_\chi F \circ \Phi_\chi^t$$

where

$$\mathcal{L}_\chi(\cdot) = \{\cdot, \chi\} = \sum_{i=1}^{n} \frac{\partial \chi}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial \chi}{\partial q_i}\frac{\partial}{\partial p_i}$$

is the "Poisson bracket" operator, we write:

$$F \circ \Phi_\chi^t = F + \sum_{j=1}^{\infty} \frac{t^j}{j!}\frac{d^j}{dt^j}(F \circ \Phi_\chi^t)_{|t=0} = F + \sum_{j=1}^{\infty} \frac{t^j}{j!}\mathcal{L}_\chi^j F \tag{3.5}$$

If χ is small then $F \circ \Phi_\chi^t$ is analytic up to $t = 1$. Under such hypotheses, we have:

$$\begin{aligned} F \circ \Phi_\chi^1 &= F + \mathcal{L}_\chi F + \frac{1}{2}\mathcal{L}_\chi^2 F + \dots \\ &=: F + \mathbf{R}_\chi^{\mathbf{1}}(F) \\ &=: F + \mathcal{L}_\chi F + \mathbf{R}_\chi^{\mathbf{2}}(F) \end{aligned}$$

where $\mathbf{R}_\chi^{\mathbf{1}}(F)$ and $\mathbf{R}_\chi^{\mathbf{2}}(F)$ are the remainders of first and second order of the series expansion (3.5), respectively. Now we turn to the construction of BNF. First of all, we pass to complex coordinates, with the immersion:

$$\begin{aligned} \mathbb{R}^{2n} &\hookrightarrow \mathbb{C}^{2n} \\ (q, p) &\mapsto (w, z) \end{aligned}$$

where

$$w_j = i\frac{q_j - ip_j}{\sqrt{2}} \quad j = 1, \dots, n$$

are the coordinates and

$$z_j = \frac{q_j + ip_j}{\sqrt{2}} \quad j = 1, \dots, n$$

are the momenta. In these coordinates (3.3) becomes:

$$H_2 = \sum_{j=1}^{n} i\bar{\omega}_j w_j z_j$$

and $\mathcal{L}_{H_2} = \{\cdot, H_2\}$ has the expression

$$\mathcal{L}_{H_2} = \sum_{j=1}^{n} i\bar{\omega}_j (w_j \frac{\partial}{\partial w_j} - z_j \frac{\partial}{\partial z_j}) \tag{3.6}$$

Now our aim is to build a sequence of Lie transforms ($s = r - 2$)

$$\Phi_{\chi_1}^1, \Phi_{\chi_2}^1, \dots, \Phi_{\chi_s}^1$$

such that $H \circ \Phi_{\chi_1}^1 \circ \Phi_{\chi_2}^1 \circ \dots \circ \Phi_{\chi_s}^1$ has the expression (3.4).

We start from $s=1$. Since we have:

$$H \circ \Phi^1_{\chi_1} = H_2 + H_3 + \{H_2, \chi_1\} + \dots \stackrel{!}{=} H_2 + G + f$$

the function χ_1 must satisfy the relation:

$$\{H_2, \chi_1\} = -(H_3 - G) \tag{3.7}$$

If we denote

$$f(w,z) = \sum_{\mu,\nu\in\mathbb{N}^n} f_{\mu\nu} w^\mu z^\nu$$

the Taylor series of a function f (with μ, ν multi–indices: $w^\mu z^\nu \equiv w_1^{\mu_1} w_2^{\mu_2} \cdot \dots \cdot z_1^{\nu_1} z_2^{\nu_2} \cdot \dots$), then (3.7) becomes (using the expression (3.6))

$$\sum_{\mu,\nu\in\mathbb{N}^n} i\bar{\omega}\cdot(\mu-\nu)(\chi_1)_{\mu\nu} w^\mu z^\nu = -(H_3 - G)$$

If $\bar{\omega}$ is nonresonant up to the third order, the equation above has the solution:

$$\chi_1 = \sum_{\substack{\mu\neq\nu\in\mathbb{N}^n \\ |\mu|+|\nu|=3}} \frac{(H_3)_{\mu\nu} w^\mu z^\nu}{i\bar{\omega}\cdot(\nu-\mu)} \tag{3.8}$$

$$G = \sum_{\mu\in\mathbb{N}^n} (H_3)_{\mu\mu} = 0$$

in which the last equality follows from the fact that H_3 is a homogeneous polynomial of degree 3, so that $(H_3)_{\mu\mu} = G = 0$, and χ_1 is analytic with the assumptions made. Moreover, f has the form:

$$f = \mathbf{R}^{\mathbf{1}}_{\chi_1}(H_3 + \dots) + \mathbf{R}^{\mathbf{2}}_{\chi_1}(H_2 + \dots)$$

We showed how to construct the BNF of order 3 for H. An analogous procedure iterated for s times leads to the proof of theorem 2. □

3.2.2 Fourth order normal form

Our aim is to find the expression of order 4 of fourth–order BNF of the Hamiltonian, which is

$$H^{(4)} = H_2 + G_4 + O(5) \tag{3.9}$$

With a procedure similar to the one previously described, if $\bar{\omega}$ is nonresonant up to the fourth order, one finds that:

$$G_4 = \overline{H_4 + \frac{1}{2}\{H_3, \chi_1\}} \tag{3.10}$$

where χ_1 is like in (3.8), that is:

$$\chi_1 = \sum_{\substack{\mu \neq \nu \in \mathbb{N}^n \\ |\mu|+|\nu|=3}} \frac{(H_3)_{\mu\nu} w^\mu z^\nu}{i\bar{\omega} \cdot (\nu - \mu)}$$

and $\overline{}$ in (3.10) means the average:

$$\overline{f} = \sum_{\substack{\nu \in \mathbb{N}^n \\ |\nu|=2}} f_{\nu\nu} w^\nu z^\nu .$$

Chapter 4

Application of KAM theory to the Restricted Three–Body Problem

This chapter applies what seen in chapter 2 to the problem of the stability of the triangular equilibrium points of the Restricted Three–Body Problem.

4.1 KAM at the equilibria

The application of KAM theorem to the stability of elliptic equilibria of Hamiltonian systems with two degrees of freedom develops into two steps: diagonalization of the quadratic part of the Hamiltonian and construction of normal forms.
If $(\tilde{p}, \tilde{q}) = (0, 0)$ is an equilibrium of the system with analytic Hamiltonian $H(\tilde{p}, \tilde{q})$, then the expansion of H in series near the origin is:

$$H = \sum_{s \geq 2} H_s \tag{4.1}$$

where H_s is a homogeneous polynomial of degree s in $(\tilde{p}, \tilde{q})$, and we took $H_0 = 0$.

If the equilibrium is elliptic and its frequencies $\bar{\omega} = (\bar{\omega}_1, \dots, \bar{\omega}_n)$ are nonresonant up to order 2, then by means of a linear symplectic transformation, we can pass to new coordinates (p, q) reducing H_2 to the diagonal form:

$$H_2 = \bar{\omega} \cdot I, \quad I = (I_1, \dots, I_n),\ I_i = \frac{p_i^2 + q_i^2}{2}$$

which is a completely integrable system.

In a small neighbourhood of the origin the terms $H_3, H_4, \dots$ are small compared with H_2, and can be regarded as a perturbation of the integrable system with Hamiltonian H_2.

As seen in chapter 3, given any $r > 1$, if the frequencies $\bar{\omega}$ are nonresonant up to order r, it is possible to construct a canonical transformation conjugating H with $H^{(r)}$, its *Birkhoff normal form* of order r:

$$H^{(r)} = H_2 + G + f \tag{4.2}$$

where $G = \sum_{j=3}^{r} G_j$ and G_j is a homogeneous polynomial of degree $j/2$ in the actions, so that in particular $G_j = 0$ for odd j. The Hamiltonian system with Hamiltonian $K = H_2 + G$ is completely integrable, and f is a series starting with terms of order $r+1$.

We would like to apply theorem 1 to $H^{(r)}$. As a matter of fact, there are technical differences with the theory explained in the previous sections, because here the action–angle variables are singular in the equilibrium, so that the construction of Normal Forms that leads to the KAM theorem can be done only in a neighbourhood deprived of the equilibrium point. The aim of the theorem in this framework is to find a neighbourhood in which the invariant tori survive on any surface of constant energy.

In our case the Hamiltonian H written in fourth–order BNF reads:

$$H = H_2 + G_4 + \mathcal{O}(5) \tag{4.3}$$

where:

$$H_2 = \bar{\omega} \cdot I, \quad G_4 = \frac{1}{2} I \cdot LI$$

for some matrix L.

Condition of isoenergetic non–degeneracy is sufficient for the stability, and actually, referring to Hamiltonian (4.3), we can even weaken this condition. To understand how, first we observe that, since:

$$\omega(I) = \frac{\partial(H_2 + G_4)}{\partial I}(I) = \bar{\omega} + LI \tag{4.4}$$

condition (2.6) for $I = 0$ reads:

$$\mathcal{D}(I) := \det \begin{pmatrix} L & \omega(I) \\ \omega(I)^T & 0 \end{pmatrix} = \det \begin{pmatrix} l_{11} & l_{12} & \omega_1(I) \\ l_{12} & l_{22} & \omega_2(I) \\ \omega_1(I) & \omega_2(I) & 0 \end{pmatrix}$$

In order to show that $\mathcal{D} \neq 0$ in a neighbourhood of the equilibrium, it is sufficient to show that $\mathcal{D} \neq 0$ at the equilibrium itself. The computation of the last determinant yields:

$$-\begin{pmatrix} -\bar{\omega}_2 \\ \bar{\omega}_1 \end{pmatrix} \cdot \begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix} \begin{pmatrix} -\bar{\omega}_2 \\ \bar{\omega}_1 \end{pmatrix} \overset{(4.4)}{=} -2G_4(-\bar{\omega}_2, \bar{\omega}_1)$$

so that:

$$\mathcal{D} = 0 \iff G_4(-\bar{\omega}_2, \bar{\omega}_1) = 0$$

which happens if and only if H_2 divides G_4, since H_2 vanishes in $(I_1, I_2) = (-\omega_2, \omega_1)$. Generalizing these considerations to BNF of order r, one can prove the following version of the KAM theorem for the equilibria (see [M&H]):

Theorem 3 (Arnold's Stability Theorem)**.** *Let $n = 2$. The origin is stable for the system with Hamiltonian written in BNF of order r:*

$$H^{(r)} = H_2 + G_4 + \cdots + G_r + f$$

(the hypotheses on $H^{(r)}$ being the same as in 4.2), provided for some $k \in \mathbb{N}$, $2 \leq k \leq [r/2]$, $D_{2k} := G_{2k}(-\omega_2, \omega_1) \neq 0$, or, equivalently, provided H_2 does not divide G_{2k}. Moreover, arbitrarily close to the origin, there are invariant tori and the motion on these invariant tori is linear.

4.2 Diagonalization at $\mathscr{L}_4$

We remark the fact that, in order to apply the constructions seen in the last chapter to the specific case of $\mathscr{L}_4$, the parameter μ has to be such that the frequencies are nonresonant up to the fourth order. To simplify the computations, first note that for $\mu \leq \mu_{Routh}$ we have (as seen at the end of chapter 1) $0 < \hat{\omega}_2 < \frac{\sqrt{2}}{2} < \hat{\omega}_1$. Moreover, $\hat{\omega}_1^2 + \hat{\omega}_2^2 = 1$ and $\hat{\omega}_1^2\hat{\omega}_2^2 = \frac{27\mu(1-\mu)}{4}$.

We set $\hat{\gamma} := 3\sqrt{3}(1-2\mu)$. As $\mu \mapsto \hat{\omega}_1(\mu)$ and $\mu \mapsto \hat{\omega}_2(\mu)$ are local diffeomorphisms, we can express $\hat{\gamma}$ in terms of $\hat{\omega}_1$ and $\hat{\omega}_2$, in fact:

$$\frac{16\hat{\omega}_1^2\hat{\omega}_2^2}{(\hat{\omega}_1^2+\hat{\omega}_2^2)^2} = 27 - \hat{\gamma}^2$$

so that

$$\hat{\gamma}^2 = \frac{27\hat{\omega}_1^4 + 38\hat{\omega}_1^2\hat{\omega}_2^4 + 27\hat{\omega}_2^4}{(\hat{\omega}_1^2+\hat{\omega}_2^2)^2} \tag{4.5}$$

If we compute the components of (1.7) in $\mathscr{L}_4$, the linearization at this point results:

$$X'((\mathscr{L}_4)_q, (\mathscr{L}_4)_p) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -\frac{1}{4} & -\frac{\hat{\gamma}}{4} & 0 & 1 \\ \frac{\hat{\gamma}}{4} & \frac{5}{4} & -1 & 0 \end{pmatrix} \tag{4.6}$$

Now we apply the techniques of diagonalization we will see in the last chapter to investigate the KAM stability of $\mathscr{L}_4$. The eigenvalues of the matrix (4.6) are found to be $\pm i\hat{\omega}_1, \pm i\hat{\omega}_2$. After some computations (see [M&H]), one verifies that in this case:

$$z_1' = \frac{\sqrt[4]{\hat{\omega}_1^2+\hat{\omega}_2^2}}{2\xi_1\sqrt{\hat{\omega}_1}} \begin{pmatrix} 13\hat{\omega}_1^2 + 9\hat{\omega}_2^2 \\ -\hat{\gamma}(\hat{\omega}_1^2+\hat{\omega}_2^2) \\ \hat{\gamma}(\hat{\omega}_1^2+\hat{\omega}_2^2) \\ \frac{\sqrt[4]{(\hat{\omega}_1^2+\hat{\omega}_2^2)^3(5\hat{\omega}_1^2+9\hat{\omega}_2^2)}}{\sqrt{\hat{\omega}_1^3}} \end{pmatrix} \qquad z_1'' = \frac{\sqrt{\hat{\omega}_1}\sqrt[4]{\hat{\omega}_1^2+\hat{\omega}_2^2}}{2\xi_1} \begin{pmatrix} 0 \\ 8\sqrt{\hat{\omega}_1^2+\hat{\omega}_2^2} \\ \frac{5\hat{\omega}_1^2+\hat{\omega}_2^2}{\sqrt{\hat{\omega}_1^2+\hat{\omega}_2^2}} \\ -\hat{\gamma}\sqrt{\hat{\omega}_1^2+\hat{\omega}_2^2} \end{pmatrix}$$

$$z_2' = \frac{\sqrt[4]{\hat{\omega}_1^2+\hat{\omega}_2^2}}{2\xi_2} \begin{pmatrix} 0 \\ 8\hat{\omega}_2\sqrt{\hat{\omega}_1^2+\hat{\omega}_2^2} \\ \frac{\hat{\omega}_2(\hat{\omega}_1^2+5\hat{\omega}_2^2)}{\sqrt{\hat{\omega}_1^2+\hat{\omega}_2^2}} \\ -\hat{\gamma}\hat{\omega}_2\sqrt{\hat{\omega}_1^2+\hat{\omega}_2^2} \end{pmatrix} \qquad z_2'' = \frac{\sqrt[4]{\hat{\omega}_1^2+\hat{\omega}_2^2}}{2\xi_2} \begin{pmatrix} 9\hat{\omega}_1^2 + 13\hat{\omega}_2^2 \\ -\hat{\gamma}(\hat{\omega}_1^2+\hat{\omega}_2^2) \\ \hat{\gamma}(\hat{\omega}_1^2+\hat{\omega}_2^2) \\ 9\hat{\omega}_1^2 + 5\hat{\omega}_2^2 \end{pmatrix}$$

where $\xi_1 = \sqrt{(\hat\omega_1^2 - \hat\omega_2^2)(13\hat\omega_1^2 + 9\hat\omega_2^2)}$, $\xi_2 = \sqrt{\hat\omega_2(\hat\omega_1^2 - \hat\omega_2^2)(9\hat\omega_1^2 + 13\hat\omega_2^2)}$.

Thus, using Proposition 3 of chapter 3 and relations (4.5), together with the fact that $x_1' \cdot \mathbb{J}_4 x_1'' > 0$, $x_2' \cdot \mathbb{J}_4 x_2'' < 0$, we know that the matrix (3.1) is symplectic and $(q,p) \mapsto (\tilde q, \tilde p) = S^{-1}(q,p)$ conjugates the Hamiltonian of (4.6) to the Hamiltonian:

$$K(\tilde p, \tilde q) = \frac{\hat\omega_1}{2}(\tilde q_1^2 + \tilde p_1^2) - \frac{\hat\omega_2}{2}(\tilde q_2^2 + \tilde p_2^2) \tag{4.7}$$

The minus sign reflects the fact that the Hessian of H is not positive definite.

4.3 Normal form and stability in $\mathscr{L}_4$

In section 1.3 we showed that the linearization in $\mathscr{L}_4$ has four purely imaginary eigenvalues $\pm i\hat\omega_1$, $\pm i\hat\omega_2$ given by (1.8) if $\mu \le \mu_{Routh} = 0.0385\ldots$, and in section 4.2 we found symplectic coordinates in which the quadratic part of the Hamiltonian has the form (in action–angle variables):

$$H_2 = \hat\omega_1 I_1 - \hat\omega_2 I_2$$

We define μ_p the value of μ for which $\hat\omega_1/\hat\omega_2 = p$. In particular we have (using the expressions obtained in chapter 1) $\mu_2 \approx 0.0243, \mu_3 \approx 0.0135$, so that:

$$\mu_3 < \mu_2 < \mu_1 = \mu_{Routh}$$

Thus when $0 < \mu < \mu_1$, provided $\mu \ne \mu_2, \mu_3$ (so that $\hat\omega$ is nonresonant up to the fourth order, see chapter 3) the Hamiltonian (1.2) can be written in BNF of fourth order, becoming:

$$H^{(4)} = \hat\omega_1 I_1 - \hat\omega_2 I_2 + G_4 + \ldots$$

Deprit and Deprit–Bartholome, in order to apply the Arnold's Stability Theorem as in section (4.1), computed (by hand) G_4 and derived that: (see [M&H])

$$D_4 = G_4(\hat\omega_2, \hat\omega_1) = -\frac{36 - 541\hat\omega_1^2 + \hat\omega_2^2 + 644\hat\omega_1^4\hat\omega_2^4}{8(1 - 4\hat\omega_1^2\hat\omega_2^2)(4 - 25\hat\omega_1^2\hat\omega_2^2)}$$

which vanishes at $\mu = \mu^0 \approx 0.0109$. Thus, the hypotheses of theorem 3 hold for $\mu \ne \mu_0, \mu_1, \mu_2, \mu_3$.

Meyer and Schmidt (see [M&S]), using an algebraic processor, brought the normalization to sixth order, obtaining:

$$D_6 = \frac{P}{Q}$$

where:

$$\begin{aligned}
P = &-\frac{3105}{4}+\frac{1338449}{48}\sigma-\frac{489918305}{1728}\sigma^2+\frac{7787081027}{6912}\sigma^3-\frac{2052731645}{1296}\sigma^4-\\
&-\frac{1629138643}{324}\sigma^5+\frac{1879982900}{81}\sigma^6+\frac{368284375}{81}\sigma^7\\
Q = &\,\hat{\omega}_1\hat{\omega}_2(\hat{\omega}_1^2-\hat{\omega}_2^2)^5(4-25\sigma)^3(9-100\sigma)\\
\sigma = &\,\hat{\omega}_1^2\hat{\omega}_2^2
\end{aligned}$$

and then

$$D_{6|\mu=\mu^0} \approx 66.64$$

so by Arnold's theorem it is possible to conclude that $\mathscr{L}_4$ is *stable* if $0 < \mu < \mu_1$, provided $\mu \neq \mu_2, \mu_3$. Using particular techniques, in [Alf1] and [Alf2] it is proved that $\mathscr{L}_4$ is unstable for $\mu = \mu_2, \mu_3$, and in [Sok] that it is unstable for $\mu = \mu_1$.

4.4 Numerical results

Using the software *Mathematica*®, we repeated the passages that led to the fourth–order normal form (3.9), evaluating it numerically for different values of μ. The commands shown below, in order:

- Implement the Hamiltonian function (1.1), find its linearization in $\mathscr{L}_4$ (4.6) and compute its eigenvalues and eigenvectors (see (1.8) in section 1.3).
- Diagonalize the Hamiltonian with the techniques described in Proposition 3 of chapter 3 and in section 4.2.
- Expand the Hamiltonian (1.1) translated in $\mathscr{L}_4$ in Taylor series up to fourth–order terms, apply the symplectic diagonalization, pass to complex coordinates and reach an explicit expression for terms of order 2, 3 and 4 of the transformed Hamiltonian (H_2, H_3 and H_4) as in the proof of theorem 2.
- Compute the Normal Form of order 4, $H^{(4)}$ (as in (3.9)).
- Plot $G_4(-\omega_2, \omega_1)$ as a function of μ.

Appendix

An astronomical example: the Trojan asteroids

The aim of this brief chapter is to study the triangular equilibrium points $\mathscr{L}_4$ and $\mathscr{L}_5$ of the Restricted Three–Body Problem within the framework of the system in which the two primaries are Sun ($\odot$) and Jupiter ($♃$) (we refer to [Marz, Marchis, Morbi, Szeb]).

Possible relevance of the triangular equilibrium points $\mathscr{L}_4$ and $\mathscr{L}_5$ in celestial mechanics became evident in 1906, when the German astronomer Max Wolf discovered the asteroid *588 Achilles* in the region near the equilateral point $\mathscr{L}_4$ of the system formed by Sun and Jupiter. With subsequent observations, it has been found that the neighbourhoods of $\mathscr{L}_4$ and $\mathscr{L}_5$ of the aforecited system are populated by two swarms of asteroids, altogether called *Trojans*. The Trojans are defined as celestial objects having orbital period between 0.97 and 1.03 relatively to Jupiter, absolute values of angular distance from Jupiter between 40° and 90° and eccentricities less than 0.15 (see [Morbi]). On behalf of distinction, the members of the two groups were later named after the Greek ($\mathscr{L}_4$) and Trojan ($\mathscr{L}_5$) heroes of the Homer's Iliad (the exceptions being 617 *Patroclus* and 624 *Hector*, which are in the "enemy's field").

Two basic remarks must be mentioned (see [Szeb]):

– Since the biggest Trojan asteroid, *624 Hector* ($\mathfrak{H}$), has mass $m_{\mathfrak{H}} \approx 1.4 \cdot 10^{19}\, kg$, we have: ($m_{♃} \approx 1898.8 \cdot 10^{24}\, kg$)

$$\frac{m_{\mathfrak{H}}}{m_{♃}} \approx 7.37 \cdot 10^{-9}$$

so that the error made neglecting the presence of $\mathfrak{H}$ in the equations of motion of $\odot$ and $♃$ is less than one hundred thousand millionth, estimate which enables us to considerate this as a reasonable example of Restricted Three–Body Problem. Nevertheless we notice that, because the eccentricities are nonzero (see table 4.4), the physical problem is three–dimensional, whereas in the previous discussion we studied the planar problem.

– We can apply in this framework the considerations on the stability of equilibria studied in the previous chapters because we have:

$$\mu_{\odot♃} = \frac{m_{♃}}{m_{\odot} + m_{♃}} \approx 0.000954$$

so that $\mu_{\odot♃} < \mu_{Routh}$, and the equilibria of the system are elliptic.

– Finally, we observe that the KAM theory reviewed in the second chapter of this work does not allow under no circumstances to reach a realistic estimate about a physically significant area of stability. Results in this field are obtained with other perturbative techniques, which, differently from what we did in chapter 2, aim at finding finite times of stability.

Figure 4.1 shows the position of the Greek and the Trojan camp of the Trojan asteroids. The Greeks rotate 60 degrees ahead of Jupiter and the Trojans trail 60 degrees back.

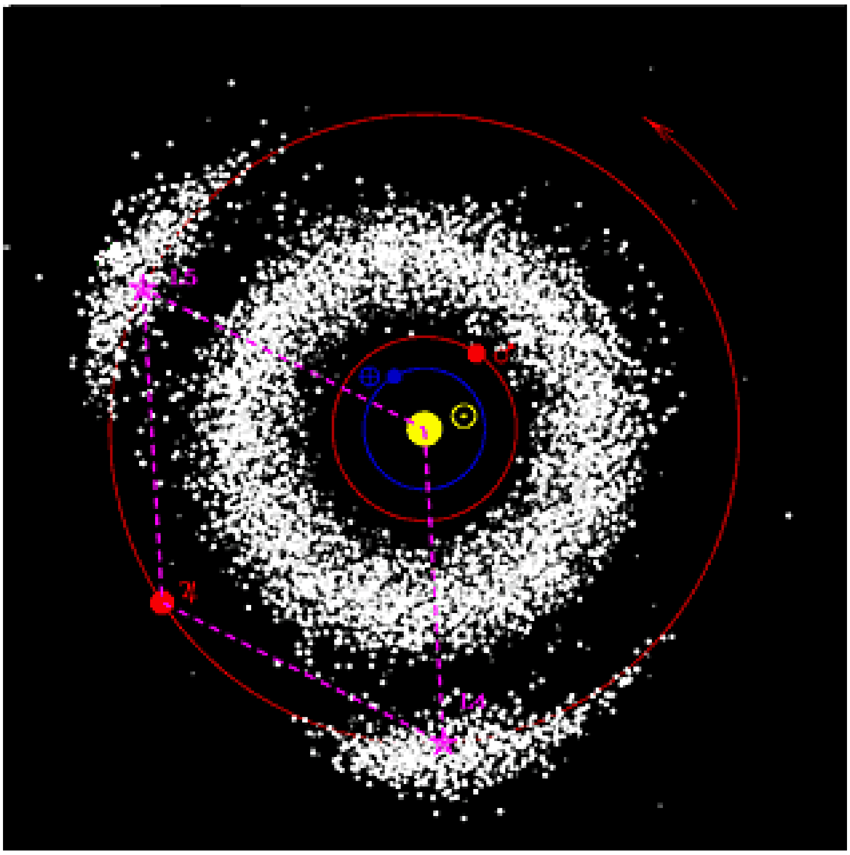

Figure 4.1: Inner Solar System, Jupiter and the Trojan asteroids (*Credit: UC Berkeley–IMCCE–Observatoire de Paris*)

Though from the figure the two swarms seem coorbital with Jupiter, most of the asteroids actually revolve around the Sun following orbits much more inclined and eccentric than Jupiter's.

Different hypotheses hold regarding the Trojans' origin: the most accepted one states that originally they orbited near Jupiter and then, after Jupiter's mass growth and subsequent collisions, they have been captured in the regions near $\mathscr{L}_4$ and $\mathscr{L}_5$ (see [Marz]).
However, recent observations (at the Keck Observatory in Mauna Kea, Hawai'i, see [Marchis]) have found that *617 Patroclus*, which in reality is a binary asteroid, formed by two bodies of similar sizes, Patroclus and Moenetius (see fig. 4.2), has a density lower than water ice, which suggests the idea that as a matter of fact some Trojans can be fragments of comets or other celestial bodies coming from the periphery of the Solar System and the Kuiper belt (see also [Morbi]). Therefore, the study of these asteroids can tell more also about the formation of the Solar System.

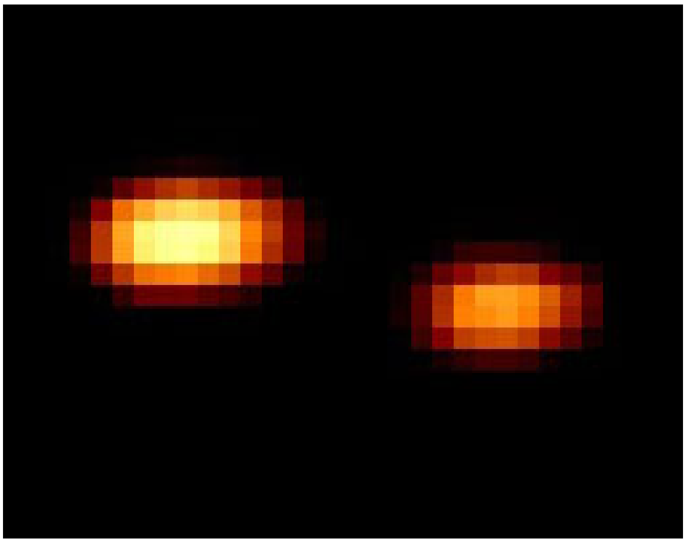

Figure 4.2: *617 Patroclus* observed with the Keck 10–m telescope on the Mauna Kea summit in Hawai'i and its adaptive optics system (*Credit: [Marchis]*)

As of 2007, Jan. 7, 1149 Jupiter Trojans are known to be near $\mathscr{L}_4$ and 929 near $\mathscr{L}_5$ (a daily updated list can be found in [Harv]).

Table 4.4 shows the data relative to the first discovered 16 Trojan asteroids ordered by date of first observation, where:

- **Designation(and name)**: contains object's number and name.
- **Ln**: indicates whether the asteroid is in L4 or in L5.
- **q and Q**: perihelion and aphelion distance (in AU).
- **H**: absolute visual magnitude.
- **Incl.**: Inclination of the orbit to the ecliptic, in degrees.

- **e**: orbital eccentricity.
- **a**: major semiaxis (in AU).

Designation and name		Ln	q	Q	H	Incl.	e	a
(588)	Achilles	L4	4.433	5.956	8.67	10.3	0.147	5.195
(617)	Patroclus	L5	4.504	5.949	8.19	22.0	0.138	5.227
(624)	Hektor	L4	5.104	5.350	7.49	18.2	0.024	5.227
(659)	Nestor	L4	4.587	5.798	8.99	4.5	0.117	5.192
(884)	Priamus	L5	4.534	5.788	8.81	8.9	0.121	5.161
(911)	Agamemnon	L4	4.908	5.600	7.89	21.8	0.066	5.254
(1143)	Odysseus	L4	4.784	5.733	7.93	3.1	0.090	5.258
(1172)	Aneas	L5	4.656	5.727	8.33	16.7	0.103	5.192
(1173)	Anchises	L5	4.593	6.053	8.89	6.9	0.137	5.323
(1208)	Troilus	L5	4.761	5.714	8.99	33.6	0.091	5.237
(1404)	Ajax	L4	4.701	5.905	9.0	18.0	0.114	5.303
(1437)	Diomedes	L4	4.936	5.384	8.30	20.5	0.043	5.160
(1583)	Antilochus	L4	4.840	5.374	8.60	28.6	0.052	5.107
(1647)	Menelaus	L4	5.115	5.349	10.3	5.6	0.022	5.232
(1749)	Telamon	L4	4.614	5.730	9.2	6.1	0.108	5.172
(1867)	Deiphobus	L5	4.909	5.357	8.61	26.9	0.044	5.133

Table 4.1: List of Trojan asteroids(*Credit: [Harv]*)

We recall that 1 AU$= 1.496 \cdot 10^8$ km (the mean distance between Sun and Earth), and that, for Jupiter, $\mathbf{q}_{♃} = 4.952$ AU, $\mathbf{Q}_{♃} = 5.455$ AU, $\mathbf{a}_{♃} = 5.20$ AU, $\mathbf{e}_{♃} = 0.048$. Moreover, we recall that **H**, the absolute visual magnitude, by definition is the apparent magnitude that a celestial object would have if it were at a distance of 10 parsec (1 parsec$= 2.06 \cdot 10^5$ AU).

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