Energy-preserving methods and B-series

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joint work with
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Abstract

Given the system of ordinary differential equations (ODEs)

$$\dot{y} = f(y), \quad y(0) = y_0 \in \mathbf{R}^n,$$

assume $H : \mathbf{R}^n \to \mathbf{R}$ is an invariant or the energy of the system, and that we have $f(y) = S\,\nabla H(y)$ with S a $n \times n$, skew-symmetric matrix[1].

To derive energy-preserving numerical integration methods for these problems one can use discrete gradient techniques, [3], [4]. Such methods rely on appropriate approximations of $\nabla H(y)$ and S (in the case S depends on y), and in general can not be expanded in a B-series. One surprising exception is the average vector field (AVF) method, namely

$$y_{n+1} - y_n = h \int_0^1 f((1-\xi)y_n + \xi y_{n+1})\, d\xi, \quad y_n \approx y(t_n),\, n = 0, 1, \ldots. \tag{1}$$

Under the assumption that S is a constant matrix, this is both a B-series method and a discrete gradient method, as recently observed by Quispel and McLaren [5].

Faou, Hairer and Pham, [2], and Chartier, Faou and Murua, [1], characterized precisely all energy preserving B-series for Hamiltonian problems ($\dot{y} = S\,\nabla H(y)$, with S the canonical symplectic matrix).

The recent work on (1) brings new possibilities for the construction of concrete energy-preserving, B-series methods.

After a brief introduction of skew-gradient systems and B-series, we will discuss possible extensions of (1) within the class of B-series methods. To illustrate the performance of the methods some numerical experiments on partial differential equations will be presented.

[1]This implies that the energy H is preserved along the solution of the ODE, that is

$$\frac{dH(y(t))}{dt} = \nabla H(y)^T \cdot f(y) = \nabla H(y)^T \cdot S\,\nabla H(y) = 0.$$

References

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