

Nearly-integrable almost-symplectic Hamiltonian systems*

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Abstract

Integrable Hamiltonian systems on almost-symplectic manifolds have recently drawn some attention. Under suitable properties, they have a structure analogous to those of standard symplectic-Hamiltonian completely integrable systems. Here we study small Hamiltonian perturbations of these systems. Preliminarily, we investigate some general properties of these systems. In particular, we show that if the perturbation is ‘strongly Hamiltonian’ (namely, its Hamiltonian vector field is also a symmetry of the almost-Hamiltonian structure) then the system reduces, under an almost-symplectic version of symplectic reduction, to a family of nearly integrable standard symplectic-Hamiltonian vector fields on a reduced phase space, of codimension not less than 3. Therefore, we restrict our study to non-strongly Hamiltonian perturbations. We will show that KAM theorem on the survival of strongly nonresonant quasi-periodic tori does not apply, but that a weak version of Nekhoroshev theorem on the stability of actions is instead valid, even though for a time scale which is polynomial (rather than exponential) in the inverse of the perturbation parameter.

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1 Introduction

An almost-symplectic manifold is a generalization of a symplectic manifold, in which the nondegenerate 2-form is not closed. Hamiltonian systems on almost-symplectic manifolds arise, for instance, in nonholonomic mechanics [2]; moreover, their study might be an intermediate step to the study of the more general case of (generalized) Hamiltonian

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systems on (twisted) Dirac manifolds considered in [5, 19, 21]. The main difference between Hamiltonian systems on symplectic and almost-symplectic manifolds is that in the almost-symplectic case, due to the non-closedness of the 2-form, Hamiltonian vector fields are not necessarily symmetries of the almost-symplectic structure. Those which are symmetries of the almost-symplectic structure have special properties, and resemble more closely the Hamiltonian vector fields of the standard symplectic case; they were called ‘strongly Hamiltonian’ in [9].

We are aware of only a few articles dedicated to the almost-symplectic case. Our previous work [9] focussed on the integrability of Hamiltonian systems on almost-symplectic manifolds. Reference [20], by I. Vaisman, studies general properties of strongly Hamiltonian systems on almost-symplectic manifolds and provides examples. Some geometric aspects are studied in [18, 10].

The question underlying the present work is how different is the dynamics of an almost-symplectic Hamiltonian system from the standard symplectic-Hamiltonian ones. This is a broad question, which probably does not have a single, definite answer. For instance, in the almost-symplectic case, non-strongly Hamiltonian systems need not conserve the volume in phase space, while strongly Hamiltonian systems do (see Section 2). In this paper, we begin this investigation by considering a special case, that of almost-symplectic Hamiltonian systems that are small perturbations of almost-symplectic integrable Hamiltonian systems. The question we ask is whether the great theorems of Hamiltonian perturbation theory—KAM [1] and Nekhoroshev [15, 4, 17] theorems—retain their validity in the almost-symplectic framework. Not surprisingly, the answer depends crucially on whether the perturbation is assumed to be Hamiltonian or strongly Hamiltonian.

Preliminarily to this study, we need to further investigate some properties of nearly integrable Hamiltonian systems on almost-symplectic manifolds. We will do this in Sections 2 and 3. The first question we investigate are the properties—and in a way the very existence—of strongly Hamiltonian perturbations. We will investigate this question under certain hypotheses of genericity on the almost-symplectic structure and on the perturbation and show that, at least under such hypotheses, strongly Hamiltonian nearly-integrable almost-symplectic Hamiltonian systems are in some way not deeply different from standard symplectic-Hamiltonian systems. Specifically, each of them reduce, under an analog of the standard Meyer-Marsden-Weinstein symplectic reduction that was studied in [20], to a family of nearly integrable standard symplectic-Hamiltonian systems. Versions of KAM and Nekhoroshev theorem for strongly Hamiltonian perturbations could be easily obtained, but have limited novelty.

More interestingly, we will see that if the perturbation is Hamiltonian, but not strongly Hamiltonian, then KAM theorem does not apply: quasi-periodic motions do not survive small perturbations. However, a weaker version of Nekhoroshev theorem that gives *stability of all motions for polynomial times* does hold. Specifically, we will prove that, if the unperturbed system has the standard (quasi) convexity property of Nekhoroshev theory, then the variations of the actions in all motions is bounded by quantities of order ε^{c_1} , with a positive constant c_1 , over times ε^{-c_2} , with a constant $c_2 > 1$, where ε is the size of the perturbation. This time scale is much shorter than Nekhoroshev’s stability time scale for symplectic-Hamiltonian systems, which is exponential in $1/\varepsilon^{\text{const}}$.

The basic ideas of perturbation theory in the almost-symplectic context will be ex-

plained in Section 4, in a form that should make plausible (and self-evident for the reader expert in Hamiltonian perturbation theory) our almost-symplectic version of Nekhoroshev theorem. Since the proof of this result is almost identical to that of the symplectic case (as it can be found, e.g., in [17]) we will not reproduce it here.

2 Hamiltonian systems on almost-symplectic manifolds

2.1 Hamiltonian and strongly Hamiltonian vector fields. We consider a connected manifold M of even dimension $2n$ equipped with a nondegenerate 2-form σ . (M, σ) is a symplectic manifold if σ is closed. We will say that (M, σ) is an *almost-symplectic manifold* if σ is not closed. The nondegeneracy requires $n \geq 2$.

Following [9], we say that a vector field X on an almost-symplectic manifold (M, σ) is

- i. *Hamiltonian* if $i_X\sigma$ is exact. Thus $i_X\sigma = -df$ for some function $f \in C^\infty(M)$ that we call a Hamiltonian of X , and we will write X_f for X .
- ii. *Strongly Hamiltonian* if it is Hamiltonian and, moreover, it is a symmetry of σ , that is,

$$L_X\sigma = 0,$$

where L denotes the Lie derivative. By Cartan's magic formula $L_X\sigma = i_X(d\sigma) + d(i_X\sigma)$, the strong Hamiltonianity of a Hamiltonian vector field is equivalent to

$$i_Xd\sigma = 0. \tag{1}$$

In both cases, we will call n the number of degrees of freedom of the system. Moreover, following partly [20], we say that

- iii. A *strongly Hamiltonian function* is any function on M whose Hamiltonian vector field is strongly Hamiltonian. We will denote by $S^\infty(M)$ the subset of $C^\infty(M)$ consisting of strongly Hamiltonian functions.

Remarks: (i) Reference [20] considered only the case of strongly Hamiltonian vector fields, and called them Hamiltonian. We adhere here to the terminology that we used in [9] because we will consider both classes of Hamiltonian and strongly Hamiltonian vector fields and need to distinguish among them.

(ii) Most of the following could be generalized, in analogy with the standard symplectic case, to 'local' Hamiltonian and strongly Hamiltonian vector fields, as done in [20]. We do not consider such a greater generality because we will not need it in the study of integrable systems and their perturbations.

At an algebraic level, the reason for considering strongly Hamiltonian vector fields is the following. The almost-symplectic form σ induces an almost-Poisson bracket on smooth functions of M , which is defined by

$$\{f, g\} := -\sigma(X_f, X_g) \quad \forall f, g \in C^\infty(M). \tag{2}$$

Because of the nonclosedness of σ , this bracket does not satisfy the Jacobi identity. Therefore, it does not make $C^\infty(M)$ a Lie algebra and does not induce a (anti-)homomorphism

between functions and Hamiltonian vector fields on M . However, all this holds true for the restriction of the bracket to $S^\infty(M)$. This is a consequence of the following Lemma (from [9], where its statement contains however an obvious flaw):

Lemma 1. *Let Y and Z be two vector fields on (M, σ) . If Y is Hamiltonian and Z is a symmetry of σ then $[Y, Z]$ is Hamiltonian with Hamiltonian $-\sigma(Y, Z)$:*

$$[Y, Z] = -X_{\sigma(Y, Z)}.$$

Proof. Since $d(i_Y\sigma) = 0$ and $L_Z\sigma = 0$, $d(\sigma(Y, Z)) = d(i_Z i_Y\sigma) = L_Z(i_Y\sigma) - i_Z d(i_Y\sigma) = i_Y(L_Z\sigma) + i_{[Z, Y]}\sigma = i_{[Z, Y]}\sigma = -i_{[Y, Z]}\sigma$. \square

Applied to strongly Hamiltonian vector fields, Lemma 1 gives

$$[X_f, X_g] = -X_{\{f, g\}} \quad \forall f, g \in S^\infty(M).$$

Since the Lie bracket of two symmetries of σ is still a symmetry of σ , this shows that the set of strongly Hamiltonian vector fields is a Lie subalgebra of the algebra of vector fields on M . Correspondingly, $S^\infty(M)$ is a Lie algebra when equipped with the bracket (2) and $f \mapsto X_f$ is an anti-homomorphism between these two Lie algebras.

As pointed out in [20], in view of (1) and of Lemma 1, strongly Hamiltonian vector fields form a distribution \tilde{S} on M . This distribution is a subdistribution of the distribution $\mathcal{K}_{d\sigma}$ whose fiber is, at each point, the kernel of $d\sigma$ at that point.¹ In the present case, given that $d\sigma$ is closed, $\mathcal{K}_{d\sigma}$ is integrable and coincides with the so called characteristic distribution of $d\sigma$ [11, 14]. Some properties of the distribution \tilde{S} have been studied in [20].

Reference [20] remarks that the class of strongly Hamiltonian vector fields (which may be identified, modulo constants, with the class $S^\infty(M)$ of strongly Hamiltonian functions) might be much smaller than that of Hamiltonian vector fields (which may be identified, modulo constants, with $C^\infty(M)$), and that it is not even a priori clear whether $S^\infty(M)$ contains any non-constant function. In order to show that this is not the case, reference [20] provided some examples. We add here a simple, quantitative remark on this question, that we will use in the sequel:

Lemma 2. *At any point at which $d\sigma$ is nonzero there exist at most $2n - 3$ germs of functionally independent strongly Hamiltonian functions.*

Equivalently: if $d\sigma(m) \neq 0$ at some $m \in M$, there exists a coordinate system in a neighbourhood V of m such that the restriction to V of any strongly Hamiltonian function does not depend on three coordinates.

Proof. We use the algebraic fact that at a point at which a 3-form is nonzero, the codimension of its kernel is ≥ 3 . This must of course be known, but since we could not find a reference we provide a proof. Let η be a 3-form on M and $\eta_m \neq 0$ at some $m \in M$. Then there exists a vector $v \in T_m M \setminus \{0\}$ such that the 2-form $i_v \eta_m \neq 0$. Since any nonzero 2-form, being antisymmetric, has positive even rank, this implies $\text{rank } i_v \eta_m \geq 2$.

¹We recall that the kernel of a 3-form η at a point $m \in M$ is the kernel of the linear map $T_m M \rightarrow \Lambda^2(T_m M)$ given by contraction with η_m , namely the map $v \mapsto i_v \eta_m$, $v \in T_m M$. Here, $\Lambda^2(T_m M)$ denotes the space of all covariant antisymmetric 2-tensors on $T_m M$.

The conclusion now follows observing that $\ker \eta_m \subset \ker i_v \eta_m$ given that the latter contains v while the former does not; hence $\dim(\ker i_v \eta_m \setminus \ker \eta_m) \geq 1$ and the conclusion follows.

Assume now $d\sigma \neq 0$ at a point m . Then, $d\sigma$ is everywhere nonzero in any sufficiently small neighbourhood of m . If V_0 is one such neighbourhood, then in V_0 the leaves of $\mathcal{K}_{d\sigma}$ have dimension $\leq 2n - 3$ and any set of sections of $\mathcal{K}_{d\sigma}$ that are linearly independent at each point of V_0 has cardinality $\leq 2n - 3$. In turn, by the nondegeneracy of σ , there are at most $2n - 3$ strongly Hamiltonian functions which are defined in a neighbourhood $V \subseteq V_0$ of m and are everywhere functionally independent. The statement in terms of coordinates follows by restricting V if necessary and completing a set of functionally independent strongly Hamiltonian functions to a coordinate system. \square

The upper bound of Lemma 2 is *de facto* met in all examples in [20].

Dynamically, strongly Hamiltonian vector fields have special properties among the class of Hamiltonian vector fields. For instance, Hamiltonian vector fields need not preserve the volume σ^n . An example is the vector field $X = -\frac{\partial}{\partial x_3}$ on $M = \mathbb{R}^4 \setminus \{0\}$ with almost-symplectic form $\sigma = x_3 dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Instead, since $L_X \sigma^n = \sigma^{n-1} \wedge L_X \sigma$, we have the following

Proposition 3. *Every strongly Hamiltonian vector field on an almost-symplectic manifold (M, σ) preserves the volume σ^n .*

2.2 Strongly Hamiltonian completely integrable systems. As shown in [9], the well known notion of complete integrability of the symplectic case and the resulting structure described by the Liouville-Arnold theorem, are a particular case of a more general situation, that holds in the almost-symplectic case. We begin recalling the following almost-symplectic version of the Liouville-Arnold theorem:

Proposition 4. [9] *Let (M, σ) be an almost-symplectic manifold of dimension $2n$ and $\pi = (\pi_1, \dots, \pi_n) : M \rightarrow \mathbb{R}^n$ a submersion with compact and connected fibers whose components π_1, \dots, π_n are*

- *strongly Hamiltonian functions*
- *pairwise in involution with respect to the almost-Poisson bracket (2), namely $\{\pi_i, \pi_j\} = 0$ for $i, j = 1, \dots, n$.*

Then the fibers of π are diffeomorphic to \mathbb{T}^n and each of them has a neighbourhood V equipped with coordinates $(a, \alpha) : V \rightarrow \mathcal{A} \times \mathbb{T}^n$, with $\mathcal{A} \subseteq \mathbb{R}^n$, such that $\pi = \pi(a)$ and the local representative σ_{aa} of σ in these coordinates has the form

$$\sigma_{aa} = \sum_{i=1}^n da_i \wedge d\alpha_i + \frac{1}{2} \sum_{i,j=1}^n A_{ij}(a) da_i \wedge da_j \quad (3)$$

where A is an $n \times n$ antisymmetric matrix that depends smoothly on a .

The hypothesis that π_1, \dots, π_n are strongly Hamiltonian functions is essential for this result to hold. It ensures that the Hamiltonian vector fields of these functions, besides being tangent to the fibers of π on account of the involutivity hypothesis ($L_{X_{\pi_i}} \pi_j = -\sigma(X_{\pi_i}, X_{\pi_j}) = -\{\pi_i, \pi_j\} = 0$), do pairwise commute ($[X_{\pi_i}, X_{\pi_j}] = -X_{\{\pi_i, \pi_j\}} = 0$) and

thus give the fibers of π the structure of the n -dimensional torus; for details and comments see [9].

The coordinates (a, α) will be called action-angle coordinates relative to π . From (3) it follows that, in these coordinates, the Hamiltonian vector field $X_f = \sum_{i=1}^n (X_f^{a_i} \partial_{a_i} + X_f^{\alpha_i} \partial_{\alpha_i})$ of a function $f(a, \alpha)$ has components

$$X_f^a = -\frac{\partial f}{\partial \alpha}, \quad X_f^\alpha = \frac{\partial f}{\partial a} + A \frac{\partial f}{\partial \alpha} \quad (4)$$

and that the almost-Poisson brackets (2) have the expression

$$\{f, g\}_{\text{aa}} = \sum_{i=1}^n \left(\frac{\partial f}{\partial a_i} \frac{\partial g}{\partial \alpha_i} - \frac{\partial f}{\partial \alpha_i} \frac{\partial g}{\partial a_i} \right) + \sum_{i,j=1}^n A_{ij} \frac{\partial f}{\partial \alpha_i} \frac{\partial g}{\partial \alpha_j}. \quad (5)$$

Moreover,

$$d\sigma_{\text{aa}} = \frac{1}{2} \sum_{i,j,k=1}^n \frac{\partial A_{ij}}{\partial a_k} da_k \wedge da_i \wedge da_j = \sum_{i,j,k=1}^n C_{ijk} da_k \otimes da_i \otimes da_j \quad (6)$$

where

$$C_{ijk}(a) = \frac{\partial A_{ij}}{\partial a_k}(a) + \frac{\partial A_{ki}}{\partial a_j}(a) + \frac{\partial A_{jk}}{\partial a_i}(a). \quad (7)$$

The condition that σ_{aa} is not symplectic is precisely that the skew-symmetric 3-tensor field C with components C_{ijk} does not vanish.

As expression (6) shows, $d\sigma_{\text{aa}}$ is a basic 3-form with respect to the bundle $\mathcal{A} \times \mathbb{T}^n \rightarrow \mathcal{A}$. Correspondingly, we will regard the tensor field C as defined on \mathcal{A} . Thus, the argument used in the proof of Lemma 2 implies that

Lemma 5. *At a point $a \in \mathcal{A}$ at which $C(a) \neq 0$,*

$$\ker C(a) := \{u \in \mathbb{R}^n : C_{ijk}(a)u_k = 0\}$$

is a subspace of \mathbb{R}^n of dimension $\leq n - 3$.

As in the standard symplectic-Hamiltonian case, the action-angle coordinates relative to a given fibration π need not be defined globally and are not unique. But exactly as in that case, any two different sets (a, α) and $(\tilde{a}, \tilde{\alpha})$ of action-angle coordinates with overlapping domains are related to each other by transformations of the form

$$\tilde{a} = Za + z, \quad \tilde{\alpha} = Z^{-T}\alpha + \mathcal{F}(a) \quad (8)$$

for some unimodular matrix Z with integer entries, some $z \in \mathbb{R}^n$ and some invertible map \mathcal{F} [9].

Definition 6. *Given a submersion π as in Proposition 4, a function h on M is called completely integrable with respect to π if it is in involution with all functions π_1, \dots, π_n :*

$$\{h, \pi_i\} = 0, \quad i = 1, \dots, n.$$

Since the π_i 's depend only on the actions, the involutivity conditions of Definition 6 and expression (5) imply that *a function h is completely integrable with respect to π if and only if it is a function of the actions alone*. Moreover, under such hypothesis, by (4),

$$X_h(a) = - \sum_{i=1}^n \frac{\partial h}{\partial a_i}(a) \partial_{\alpha_i} \quad (9)$$

so that the flow of h is linear on the tori $a = \text{const}$. Furthermore, *every completely integrable function is strongly Hamiltonian*: if $h = h(a)$, then $i_{X_h} d\sigma_{aa} = 0$ because $d\sigma_{aa}$ contains no differential of the angles.

From the point of view of complete integrability the case $n = 2$ is special, and has no interest:

Proposition 7. *If on a 4-dimensional almost-symplectic manifold (M, σ) there is a submersion $\pi = (\pi_1, \pi_2)$ as in Proposition 4—and hence a completely integrable Hamiltonian system—then σ is symplectic.*

Indeed, every totally antisymmetric 3-tensor on a 2-dimensional space is identically zero; hence $C = 0$ and $d\sigma_{aa} = 0$ in the domain of any system of action-angle coordinates.

Therefore, from now on we will assume $n \geq 3$.

Remarks. (i) Restricted to the domain of an action-angle chart, completely integrable almost-symplectic systems are dynamically indistinguishable from the completely integrable systems of the standard symplectic case. Even more so, the restriction (9) of a completely integrable almost-symplectic vector field to a domain of action-angle coordinates is Hamiltonian with respect to the symplectic structure $d\alpha_i \wedge d\alpha_i$ [9]. (It is not known if things might be different globally, that is, if there is an almost-symplectic manifold (M, σ) with a strongly Hamiltonian vector field that is not Hamiltonian with respect to *any* symplectic form on M).

(ii) The conclusions of Lemma 5 and Proposition 7 may be reached in more geometric terms. The map $\pi : M \rightarrow \pi(M)$ is a fibration with fiber \mathbb{T}^n . The transition functions (8) among the local systems of action-angle coordinates show that there is a symplectic form σ_s on M with local representatives $da_i \wedge d\alpha_i$. In these charts the bundle map $\pi : M \rightarrow \pi(M)$ is $(a, \alpha) \mapsto a$ and (3) shows that there is a 2-form μ on $\pi(M)$ such that $\sigma = \sigma_s + \pi^* \mu$. Hence $d\sigma = \pi^* d\mu$. If $n = 2$, $d\mu = 0$.

(iii) Reference [9] considers a more general situation of that described in Proposition 4, which extends from the symplectic to the almost-symplectic context not only the notion of complete integrability, but also that of ‘noncommutative integrability’ or ‘superintegrability’ (in which the invariant tori may be isotropic, not just Lagrangian).

3 Nearly-integrable almost-symplectic Hamiltonian systems

3.1 Hamiltonian perturbations. Our goal in this paper is to study small perturbations of an almost-symplectic completely integrable system (M, σ, h) with $n \geq 3$ degrees of freedom.

Specifically, we aim to investigate the persistence of the invariant tori of the unperturbed system and the existence of bounds on the variations of the actions on finite but long time scales. Therefore, our approach may be consistently done in a neighbourhood of an invariant torus of the unperturbed system, that is, as we will say ‘semi-globally’. In particular, we may consistently restrict the analysis to the domain $\mathcal{A} \times \mathbb{T}^n$ of a set of action-angle coordinates (a, α) , with \mathcal{A} connected.

Thus, from now on we will restrict our study to Hamiltonian systems of the type

$$h(a) + \epsilon f(a, \alpha), \quad (a, \alpha) \in \mathcal{A} \times \mathbb{T}^n \quad (10)$$

where ϵ is a small parameter, h and f are two functions, and the almost-symplectic 2-form σ_{aa} on $\mathcal{A} \times \mathbb{T}^n$ is as in (3). As is typical in perturbation theory, we will work in the real analytic category.

It is interesting to note that, independently of the smallness of the parameter ϵ , the dynamics of these systems is subject to constraints that appear to come from the almost-symplectic geometry of the manifold:

Proposition 8. *Any Hamiltonian vector field on the almost-symplectic manifold $(\mathcal{A} \times \mathbb{T}^n, \sigma_{aa})$, with $\mathcal{A} \subseteq \mathbb{R}^n$ and σ_{aa} as in (3), preserves the volume σ^n .*

Proof. Assume X_f is Hamiltonian. Then $L_{X_f}(\sigma_{aa})^n = (\sigma_{aa})^{n-1} \wedge L_{X_f} \sigma_{aa} = (\sigma_{aa})^{n-1} \wedge i_{X_f} d\sigma$ because $i_{X_f} \sigma_{aa}$ is closed. Since $\sigma_{aa} = \sum_{i=1}^n da_i \wedge d\alpha_i + \frac{1}{2} \sum_{i,j=1}^n A_{ij} da_i \wedge da_j$ and $i_{X_f} d\sigma_{aa} = \sum_{i,j,k=1}^n C_{ijk} \frac{\partial f}{\partial \alpha_k} da_i \wedge da_j$, $(\sigma_{aa})^{n-1} \wedge i_{X_f} d\sigma_{aa}$ is a sum of terms each of which contains the wedge product of at least $n+1$ differentials of the n actions, and therefore vanishes. \square

This seems to imply that Hamiltonian vector fields on an almost-symplectic manifold that hosts a completely integrable system are, under certain aspects, special, not generic, among all Hamiltonian vector fields on almost-symplectic manifolds.

3.2 Strongly Hamiltonian perturbations. Clearly, there are no obstructions whatsoever to the existence of nearly-integrable Hamiltonian systems (10), because any function $f(a, \alpha)$ gives one.

As we now discuss, there are instead much stronger conditions on the properties of strongly Hamiltonian nearly-integrable systems, to the point that it is not even clear if there exist any such system which either is not completely integrable or that does not reduce, in a sense that will be made precise below, to a standard nearly-integrable symplectic-Hamiltonian system.

We have not been able to investigate in full generality the structure of all strongly Hamiltonian functions on almost-symplectic manifold of the particular type $(\mathcal{A} \times \mathbb{T}^n, \sigma_{aa})$. We will do this only under certain hypotheses of genericity on the strongly Hamiltonian function f and on the almost-symplectic 2-form σ_{aa} .

Preliminarily to this analysis, we recall an almost-symplectic version of the standard symplectic reduction procedure [14] studied in [5], who consider the more general almost-Dirac case, and in [20]. Following [5] we will say that an action Φ of a Lie group G on an almost-symplectic manifold (M, σ) is *strongly Hamiltonian* if its infinitesimal generators

are strongly Hamiltonian vector fields.² Clearly, any strongly Hamiltonian action has a momentum map $J : M \rightarrow \mathfrak{g}^*$, with \mathfrak{g} the Lie algebra of G . Such a momentum map is constant along the flow of any strongly Hamiltonian system whose Hamiltonian is invariant under Φ [5]. Denote now by G_μ the isotropy group of $\mu \in \mathfrak{g}$ relative to the coadjoint action of G .

Proposition 9. [5, 20] *Consider a strongly Hamiltonian action Φ on an almost-symplectic manifold (M, σ) whose momentum map J is equivariant with respect to the action Φ on M and to the coadjoint action on \mathfrak{g}^* . Let $\mu \in \mathfrak{g}^*$ be a regular value of J and assume that the Φ -action on $J^{-1}(\mu)$ is free and proper. Let $\pi : J^{-1}(\mu) \rightarrow J^{-1}(\mu)/G_\mu$ be the canonical projection and $i : J^{-1}(\mu) \hookrightarrow M$ the immersion. Then*

- (i) *The smooth manifold $J^{-1}(\mu)/G_\mu$ has an almost-symplectic (or symplectic) structure $\bar{\sigma}_\mu$ such that $\pi^*\bar{\sigma}_\mu = i^*\sigma$.*
- (ii) *If f is a strongly Hamiltonian function on (M, σ) , then the function \bar{f} such that $\pi^*\bar{f} = f$ is a strongly Hamiltonian function on $(J^{-1}(\mu)/G_\mu, \bar{\sigma}_\mu)$.*

Proof. The proof of item (i) is given in [20]. As for item (ii), which is not noticed in [20], $0 = L_X\sigma = L_{\pi^*X_{\bar{f}}}\pi^*\bar{\sigma} = \pi^*L_{X_{\bar{f}}}\bar{\sigma}$. Thus $L_{X_{\bar{f}}}\bar{\sigma} = 0$. \square

3.3 The Fourier spectrum of a strongly Hamiltonian function. The origin of the obstruction to the existence of strongly Hamiltonian perturbations traces back to Lemma 2, according to which it is always possible to choose the coordinates, at least locally, in such a way that a strongly Hamiltonian function is independent of at least 3 coordinates. Lemma 2 does not guarantee that these coordinates may be chosen to be action-angle coordinates, and that the strongly Hamiltonian function is independent of (at least) three angles, but we will show that this happens under certain conditions, and that it has further consequences.

In order to investigate this question we will resort to Fourier series techniques. Any function $f : \mathcal{A} \times \mathbb{T}^n \rightarrow \mathbb{R}$ can be expanded in the Fourier series

$$f(a, \alpha) = \sum_{\nu \in \mathbb{Z}^n} \hat{f}_\nu(a) E_\nu(\alpha) \quad (11)$$

where $E_\nu(\alpha) = e^{\sqrt{-1}\nu \cdot \alpha}$. We call “spectrum” of f at a point $a \in \mathcal{A}$ the set

$$\text{Sp}(f, a) := \{\nu \in \mathbb{Z}^n : \hat{f}_\nu(a) \neq 0\}.$$

The following Lemma gives a link between the spectrum of a strongly Hamiltonian perturbation and the kernel of the 3-tensor C defined in (7).

Lemma 10. *Consider a strongly Hamiltonian function f on the almost-symplectic manifold $(\mathcal{A} \times \mathbb{T}^n, \sigma_{\text{aa}})$, with $\mathcal{A} \subseteq \mathbb{R}^n$ and σ_{aa} as in (3). Then*

$$\text{Sp}(f, a) \subseteq \mathbb{Z}^n \cap \ker C(a) \quad \forall a \in \mathcal{A}.$$

²In the standard symplectic case, this term is sometimes used with a different meaning (e.g. in [14]).

Proof. Since $i_{X_f}d\sigma_{aa} = \sum_{i,j,k=1}^n C_{ijk}X_f^{a_k} da_i \otimes da_j$, the condition for f to be strongly Hamiltonian is

$$\sum_{k=1}^n C_{ijk}(a) \frac{\partial f}{\partial \alpha_k}(a, \alpha) = 0 \quad \forall i, j = 1, \dots, n, \quad a \in \mathcal{A}_C, \quad \alpha \in \mathbb{T}^n \quad (12)$$

that is,

$$\frac{\partial f}{\partial \alpha}(a, \alpha) \in \ker C(a) \quad \forall a \in \mathcal{A}, \quad \alpha \in \mathbb{T}^n. \quad (13)$$

Expanding f in Fourier series, conditions (12) become

$$\sum_{\nu \in \mathbb{Z}^n} \sum_{k=1}^n C_{ijk}(a) \nu_k \hat{f}_\nu(a) E_\nu(\alpha) = 0 \quad \forall i, j = 1, \dots, n, \quad a \in \mathcal{A}, \quad \alpha \in \mathbb{T}^n$$

that is,

$$\sum_{k=1}^n C_{ijk}(a) \nu_k \hat{f}_\nu(a) = 0 \quad \forall i, j = 1, \dots, n, \quad a \in \mathcal{A}$$

Thus, for each $\nu \in \mathbb{Z}^n$ and at each point $a \in \mathcal{A}_C$, if $\hat{f}_\nu(a) \neq 0$ then $\sum_{k=1}^n C_{ijk}(a) \nu_k = 0$ for all i, j , namely $\nu \in \ker C(a)$. \square

3.4 Constraints on strongly Hamiltonian perturbations. For systems with 3 degrees of freedom, Lemma 10 has the following immediate consequence:

Proposition 11. *Let f be a strongly Hamiltonian function on $(\mathcal{A} \times \mathbb{T}^n, \sigma_{aa})$ with $n = 3$. Assume that the basic 3-form $d\sigma_{aa}$ is everywhere nonzero in an open and dense subset of \mathcal{A} . Then*

- i. f is independent of the angles α .*
- ii. X_f is Hamiltonian with respect to the symplectic structure $\sum_{i=1}^3 da_i \wedge d\alpha_i$ on $\mathcal{A} \times \mathbb{T}^3$ (and, moreover, completely integrable with respect to $\pi : (a, \alpha) \mapsto a$).*

Proof. i. Let \mathcal{A}_C be the subset of \mathcal{A} where $d\sigma_{aa}$, and hence C , are not zero. By Lemma 5, since $n = 3$, the kernel of C is zero-dimensional at all points of \mathcal{A}_C . Hence condition (13) implies that $\frac{\partial f}{\partial \alpha}(a, \alpha) = 0$ for all $a \in \mathcal{A}_C$ and $\alpha \in \mathbb{T}^n$. By continuity, $\frac{\partial f}{\partial \alpha} = 0$ in all of $\mathcal{A} \times \mathbb{T}^n$.

- ii. This has already been noticed in Remark (i) at the end of Section 2.2. \square

The case with $n \geq 4$ is less clear and we will study it by supplementing the hypothesis of the density of the non-zero set of $d\sigma_{aa}$ with conditions of genericity of the function f . We will consider two such conditions.

First, we make an assumption on f which is a well known condition introduced by Poincaré in his study of the non-existence of first integrals in nearly integrable Hamiltonian systems ([16], vol. 1, cap. 5; see also [3]). We say that a function $f : \mathcal{A} \times \mathbb{T}^n \rightarrow \mathbb{R}$ is *Fourier-generic* in \mathcal{A} if for any $\bar{\nu} \in \mathbb{Z}^n \setminus \{0\}$, either $\hat{f}_{\bar{\nu}} = 0$ or, for any $a \in \mathcal{A}$, there exists a $\nu \in \mathbb{Z}^n$ which is ‘parallel’ to $\bar{\nu}$ and is such that

$$f_\nu(a) \neq 0.$$

By saying that two vectors ν and $\bar{\nu}$ of \mathbb{Z}^n are ‘parallel’ we mean that $\nu = k\bar{\nu}$ for some $k \in \mathbb{Q} \setminus \{0\}$.

We note that the property of being Fourier-generic is independent of the choice of action-angle coordinates: that is, if it is satisfied by a function f , it is also satisfied by $\tilde{f} := f \circ \mathcal{C}^{-1}$ with $\mathcal{C} : \mathcal{A} \times \mathbb{T}^n \rightarrow \tilde{\mathcal{A}} \times \mathbb{T}^n$ any change of action-angle coordinates, which has the form (8). Indeed, the Fourier components of the two functions f and \tilde{f} are related by

$$\hat{f}_\nu(Za + z) = e^{-\sqrt{-1}\nu \cdot \mathcal{F}(a)} \hat{f}_{Z^{-1}\nu}(a)$$

(see [8]) and the linear map Z preserves the ‘parallelism’ of integer vectors.

Lemma 12. *Consider a strongly Hamiltonian function f on the almost-symplectic manifold $(\mathcal{A} \times \mathbb{T}^n, \sigma_{\text{aa}})$ with $n \geq 4$. Assume that $d\sigma_{\text{aa}}$ is everywhere nonzero in an open and dense subset \mathcal{A}_C of \mathcal{A} and that f is Fourier-generic in \mathcal{A} .*

Then there is a change of action-angle coordinates $\mathcal{C} : \mathcal{A} \times \mathbb{T}^n \rightarrow \tilde{\mathcal{A}} \times \mathbb{T}^n$, $(a, \alpha) \mapsto (\tilde{a}, \tilde{\alpha})$ of the type (8) such that $f \circ \mathcal{C}^{-1}$ depends on at most $n - 3$ angles $\tilde{\alpha}$.

Proof. By Lemma 5, at each point of \mathcal{A}_C the kernel of C has dimension $\leq n - 3$. Since f is strongly Hamiltonian, then by Lemma 10

$$\text{Sp}(f, a) \subseteq \mathcal{L}_a \quad \forall a \in \mathcal{A}_C,$$

with

$$\mathcal{L}_a =: \mathbb{Z}^n \cap \ker C(a).$$

Since at the points a of \mathcal{A}_C , $\ker C(a)$ is a subspace of \mathbb{R}^n of dimension $\leq n - 3$, at each of these points the set \mathcal{L}_a is a sublattice of \mathbb{Z}^n of rank $r \leq n - 3$.

We now observe that, under the hypotheses of item ii., if we define

$$\mathcal{L} := \bigcap_{a \in \mathcal{A}_C} \mathcal{L}_a = \left(\bigcap_{a \in \mathcal{A}_C} \ker C(a) \right) \cap \mathbb{Z}^n$$

then we have

$$\text{Sp}(f, a) \subseteq \mathcal{L} \quad \forall a \in \mathcal{A}_C.$$

Indeed, assume that $\bar{\nu} \in \text{Sp}(f, \bar{a})$ for some $\bar{a} \in \mathcal{A}_C$, so that $\bar{\nu} \in \mathcal{L}_{\bar{a}}$. Since f is Fourier-generic in \mathcal{A} , and hence in \mathcal{A}_C , for any $a \in \mathcal{A}_C$ there exists $\nu \in \mathbb{Z}^n$ parallel to $\bar{\nu}$ such that $\nu \in \text{Sp}(f, a)$ and hence $\nu \in \mathcal{L}_a$. But \mathcal{L}_a , being the intersection of \mathbb{Z}^n with a subspace of \mathbb{R}^n , contains all integer vectors parallel to ν . Thus $\bar{\nu} \in \mathcal{L}_a$. This proves that $\bar{\nu} \in \bigcap_{a \in \mathcal{A}_C} \mathcal{L}_a$.

Since each $\ker C(a)$ with $a \in \mathcal{A}_C$ is a subspace of \mathbb{R}^n of dimension $\leq n - 3$, the intersection $\bigcap_{a \in \mathcal{A}_C} \ker C(a)$ is also a subspace of \mathbb{R}^n of dimension $\leq n - 3$ and \mathcal{L} is a sublattice of \mathbb{Z}^n of rank $r \leq n - 3$.

Now, a sublattice of \mathbb{Z}^n of rank r is the set of all the linear combinations with integer coefficients of r vectors $u_1, \dots, u_r \in \mathbb{Z}^n$, called a basis. Consider a basis $\{u_1, \dots, u_r\}$ of \mathcal{L} and complete it to a basis $\{u_1, \dots, u_r, u_{r+1}, \dots, u_n\}$ of \mathbb{Z}^n . That this is possible is guaranteed by the Elementary Divisor Theorem (see e.g. [13], Theorem 7.8) thanks to the fact that \mathcal{L} is not just a generic sublattice of \mathbb{Z}^n , but it is the intersection of \mathbb{Z}^n with a subspace of \mathbb{R}^n .

Specifically, the Elementary Divisors Theorem states that for any finitely generated submodule $\neq \{0\}$ (e.g., a lattice) of a free abelian module over a principal ideal domain (e.g., \mathbb{Z}^n) there exists a basis $\{u_1, \dots, u_n\}$ of the latter, an integer $1 \leq r \leq n$ and integers d_1, \dots, d_r such that $\{d_1 u_1, \dots, d_r u_r\}$ is a basis of the former. In our case, all $d_i = 1$ because the lattice \mathcal{L} is the intersection of \mathbb{Z}^n with a subspace of \mathbb{R}^n .

Since $\{u_1, \dots, u_r\}$ is a basis of \mathbb{Z}^n there exists a unimodular integer matrix Z such that $Zu_i = e_i$, the i -th unit vector, for all $i = 1, \dots, n$. The change of coordinates

$$\mathcal{C}_C : \mathcal{A}_C \times \mathbb{T}^n \rightarrow \tilde{\mathcal{A}}_C \times \mathbb{T}^n, \quad (a, \alpha) \mapsto (\tilde{a}, \tilde{\alpha}) = (Z^T a, Z^{-1} \alpha)$$

produces a new set of action-angle coordinates with the property that the spectrum of the representative \tilde{f} of f is contained in the lattice generated by e_1, \dots, e_r . Hence, \tilde{f} depends only on the first $r \leq n - 3$ angles. The change of action-angle coordinates \mathcal{C}_C extend by linearity to a change of action-angle coordinates \mathcal{C} which is defined in all of $\mathcal{A} \times \mathbb{T}^n$ and, by continuity, conjugates f to a function that depends only on the first r angles. \square

Lemma 12 has the following consequence:

Proposition 13. *Let f be a strongly Hamiltonian function on $(\mathcal{A} \times \mathbb{T}^n, \sigma_{\text{aa}})$ with $n \geq 4$. Under the hypotheses of Lemma 12 the system reduces, under a torus action, to a family of (possibly nonintegrable) symplectic-Hamiltonian systems with at most $n - 3$ degrees of freedom.*

Proof. Let $f(a, \alpha)$ be a strongly Hamiltonian function. By Lemma 12, there is a choice of action-angle coordinates such that f is independent of the last $r \geq 3$ angles. We denote by $(a, \alpha) = (I, J, \varphi, \psi) \in \mathbb{R}^{n-r} \times \mathbb{R}^r \times \mathbb{T}^{n-r} \times \mathbb{T}^r$ these coordinates, with f independent of the r angles ψ . The system is thus invariant under the \mathbb{T}^r -action given by translations of the angles ψ . This is a strongly Hamiltonian action, with equivariant momentum map given by the actions $J : M \rightarrow \mathbb{R}^r$. Fix a value of $J \in \mathbb{R}^r$. Then the reduced phase space is $\bar{\mathcal{A}}_J \times \mathbb{T}^{n-r} \ni (I, \varphi)$ with $\bar{\mathcal{A}}_J \subseteq \mathbb{R}^{n-r}$ and, if we write $A = \begin{pmatrix} \bar{A} & B \\ -B^T & D \end{pmatrix}$ with the block \bar{A} of dimension $(n - r) \times (n - r)$ etc., the reduced almost-symplectic 2-form is

$$\bar{\sigma}_{\text{aa}, J} = \sum_{i=1}^{n-r} dI_i \wedge d\varphi_i + \frac{1}{2} \sum_{i,j=1}^{n-r} (\bar{A}_J)_{ij} dI_i \wedge dI_j$$

with $\bar{A}_J(I) := \bar{A}(I, J)$. The reduced Hamiltonian is $\bar{f}_J(I, \varphi) := f(I, J, \varphi)$.

There are now three possibilities. If $\bar{\sigma}_{\text{aa}, J}$ is symplectic (what happens, in particular, if $n - r = 0, 1, 2$), then the reduced system is symplectic-Hamiltonian. If $n - r = 3$ and $d\bar{\sigma}_{\text{aa}, J} \neq 0$, then in view of Proposition 11 the reduced system is symplectic-Hamiltonian with respect to a modified symplectic form.

The last possibility is that $n - r \geq 4$ and $d\bar{\sigma}_{\text{aa}, J} \neq 0$. Note that the function f_J inherits from f the property of Fourier genericity and $d\bar{\sigma}_{\text{aa}, J}$ inherits from $d\sigma_{\text{aa}}$ the property of vanishing in a subset of the reduced action space $\bar{\mathcal{A}}_J$ whose complement is open and dense. Therefore, the reduced system $(\bar{\mathcal{A}}_J \times \mathbb{T}^{n-r}, \bar{\sigma}_{\text{aa}, J}, \bar{f}_J)$ satisfies the hypotheses of the present Proposition and we may apply to it the reduction procedure just described. This leads

to a family of reduced systems with at most $n - 6$ degrees of freedom, each of which is either symplectic, or has 3 degrees of freedom, or has more than 3 degrees of freedom and is almost-symplectic. The reduction procedure can be applied to the latter, etc. The iteration stops when all reduced systems obtained either are symplectic-Hamiltonian (in particular, if they have 0,1 or 2 degrees of freedom) or have 3 degrees of freedom. \square

The same conclusions about the structure of strongly Hamiltonian functions can be obtained under different hypotheses on these functions. Let $\text{Sp}(f) = \{\nu \in \mathbb{Z}^n : f_\nu \neq 0\} = \cup_{a \in \mathcal{A}} \text{Sp}(f, a)$. For each $\nu \in \text{Sp}(f)$, the set

$$\mathcal{F}_\nu := \{a \in \mathcal{A}_C : f_\nu(a) \neq 0\}$$

is open and nonempty, and $\nu \in \ker C(a)$ for all $a \in \mathcal{F}_\nu$. The set

$$\mathcal{F} := \bigcap_{\nu \in \text{Sp}(f)} \mathcal{F}_\nu$$

need not be open and nonempty. However, if it is nonempty, then any $\nu \in \text{Sp}(f)$ satisfies $\nu \in \ker C(a)$ for all $a \in \mathcal{F}$ and hence $\nu \in \left(\bigcap_{a \in \mathcal{F}} \ker C(a)\right) \cap \mathbb{Z}^n$. Thus

$$\text{Sp}(f) \subseteq \left(\bigcap_{a \in \mathcal{F}} \ker C(a)\right) \cap \mathbb{Z}^n.$$

Proposition 14. *The conclusion of Proposition 13 remains true if it is assumed that \mathcal{A}_C is dense in \mathcal{A} and, instead of the Fourier genericity of f , that for each $\nu \in \text{Sp}(f)$ the set \mathcal{F}_ν is dense in \mathcal{A} .*

Proof. \mathcal{F} is a countable intersection of open dense subsets of \mathcal{A} . Since \mathcal{A} is an open subset of \mathbb{R}^n , a straightforward application of the Baire category theorem guarantees that \mathcal{F} is an open dense subset of \mathcal{A} . The set $\left(\bigcap_{a \in \mathcal{F}} \ker C(a)\right) \cap \mathbb{Z}^n$ is a sublattice of \mathbb{Z}^n of rank $r \leq n - 3$ and is the intersection of \mathbb{Z}^n with a subspace of \mathbb{R}^n . We may thus proceed as in the proof of Lemma 12 and conclude that there is a system of action-angle coordinates (I, J, φ, ψ) in which f depends only on the $r \leq n - 3$ angles ψ . The statements as in Proposition 13 follow from here. \square

Combining the arguments used in the proofs of Lemma 12 and Proposition 14, one easily sees that the conclusions of Proposition 13 remain valid if the Fourier-genericity of the function f is weakened, by assuming that, for each $\bar{\nu} \in \text{Sp}(f)$, for each a in a dense subset of \mathcal{A} there is a ν parallel to $\bar{\nu}$ such that $f_\nu(a) \neq 0$.

In view of Propositions 11 and 13, and at least under the stated hypotheses, strongly Hamiltonian perturbations $h + \varepsilon f$ of completely integrable almost-symplectic systems reduce to families of symplectic-Hamiltonian nearly integrable systems and can therefore be studied via the standard results and techniques of Hamiltonian perturbation theory, applied to each reduced system. We will therefore restrict our study of perturbation theory for nearly-integrable almost-symplectic systems to the case of perturbations that are not strongly Hamiltonian.

We remark that, even though it reduces to a family of symplectic-Hamiltonian systems, the non-symplectic-Hamiltonian character of the system is encoded in the evolution of the angles that have been quotiented out in the reduction process. The corresponding ‘reconstruction’ equation is given by the equations of motion of the angles ψ , which using the notation of the proof of Proposition 13 is

$$\dot{\psi} = \frac{\partial f}{\partial J}(I, J, \varphi) - B(I, J) \frac{\partial f}{\partial \varphi}(I, J, \varphi), \tag{14}$$

and by the analogous equations at the other stages of the reduction procedure.

Examples. Examples of the situation described in this Section are easily constructed.

For instance, the matrix $A = \begin{pmatrix} 0 & a_4 & 0 & 0 \\ -a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ leads to an almost-symplectic structure

on $\mathbb{R}^4 \times \mathbb{T}^4$ of the form (3) that is not symplectic. It is immediate to check that the quantities $C_{ijk} \frac{\partial f}{\alpha_k}$ either are 0 or, up to the sign, equal one of the three derivatives $\frac{\partial f}{\alpha_1}, \frac{\partial f}{\alpha_2}, \frac{\partial f}{\alpha_3}$. Thus a function is strongly Hamiltonian function if and only if it is independent of

the three angles $\alpha_1, \alpha_2, \alpha_3$. An example with $n = 5$ has $A = \begin{pmatrix} 0 & a_1 a_3 & 0 & 0 & 0 \\ -a_1 a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$;

here too, the strongly Hamiltonian functions are the functions independent of $\alpha_1, \alpha_2, \alpha_3$; among them, $\frac{a_4^2}{2} + a_5 - (1 + \cos \alpha_5) \cos \alpha_4$ describes a periodically perturbed pendulum, which is nonintegrable.

4 Perturbation theory

4.1 A first look. We begin by investigating the possibility of a perturbation theory for nearly integrable almost-symplectic Hamiltonian systems, so as to determine the analogies and the differences from the standard symplectic-Hamiltonian case. Our treatment, at this initial stage, will be rather formal. More precise considerations will be made in subsection 4.2.

We start from the system

$$X_k + \varepsilon X_f, \tag{15}$$

with $k = k(a)$ and $f = f(a, \alpha)$, on the phase space $\mathcal{A} \times \mathbb{T}^n \ni (a, \alpha)$ with $\mathcal{A} \subseteq \mathbb{R}^n$ and almost-symplectic form σ_{aa} as in (3). We assume $n \geq 3$. Moreover, we assume k and f to be real analytic, and ε (suitably) small.

The equations of motion of system (15) are

$$\dot{a} = -\varepsilon \frac{\partial f}{\partial \alpha}(a, \alpha), \quad \dot{\alpha} = \frac{\partial k}{\partial a}(a) + \varepsilon \frac{\partial f}{\partial a}(a, \alpha) + \varepsilon A \frac{\partial f}{\partial \alpha}(a, \alpha).$$

The first of these equations gives the apriori estimate $|a_t - a_0| = \mathcal{O}(\varepsilon t)$ on the variation of the actions over a time t , and hence

$$|a_t - a_0| \leq \text{const } \varepsilon^{c_1} \quad \text{for } |t| \leq \text{const } \varepsilon^{-c_2}$$

with any pair of positive constants c_1 and c_2 such that $c_1 + c_2 = 1$. The goal of perturbation theory is to go beyond this apriori estimate.

Since the a -equation for the vector field (15) is the same as that of the symplectic-Hamiltonian case, it can be expected that it might be possible to build a perturbation theory which is to some extent similar to the symplectic-Hamiltonian one. The basic step is to look for the existence of a family of diffeomorphisms Φ_ε which depends smoothly on ε in an interval which contains zero, equals the identity for $\varepsilon \rightarrow 0$ and is such that

$$\Phi_\varepsilon^*(X_k + \varepsilon X_f) = X_k + \varepsilon G + \mathcal{O}(\varepsilon^2)$$

with a vector field G which is “as integrable as possible” or that, at least, moves the new actions $a \circ \Phi_\varepsilon^{-1}$ as little as possible. If this is the case, then the vector field $X_k + \varepsilon G + \mathcal{O}(\varepsilon^2)$ will be generically called a normal form.³ This procedure should then be iterated as many times as possible. We begin by looking at the first step.

Preliminarily, we recall that in the symplectic case the diffeomorphisms Φ_ε are constructed so as to be symplectic, and the normal form is accordingly built for the Hamilton function, rather than for the Hamiltonian vector field. However, in the almost-symplectic context there is no analog of a symplectic transformation, which conjugates Hamiltonian vector fields to Hamiltonian vector fields while conjugating as well the respective Hamiltonian functions. Hence, we are forced to work with the Hamiltonian vector fields.

A standard way of constructing the family of diffeomorphisms Φ_ε is through the maps Φ_ε^Z at time ε of the flow Φ^Z of a vector field Z . This is the so called Lie method, that we will apply to vector fields, see [7] for details. Recalling the basic identity $\frac{d}{dt}(\Phi_t^Z)^*Y = (\Phi_t^Z)^*(L_Z Y)$ between the pull back of a vector field Y under a flow and the Lie derivative (here we write $L_Z Y$ for $[Z, Y]$), one immediately sees that

$$\Phi_\varepsilon^* Y = Y + \varepsilon R_\varepsilon^1(Y) = Y + \varepsilon L_Z Y + \varepsilon^2 R_\varepsilon^2(Y)$$

where, if both Z and Y are real analytic,

$$R_\varepsilon^1(Y) = \sum_{s=1}^{\infty} \frac{\varepsilon^{s-1}}{s!} L_Z^s Y, \quad R_\varepsilon^2(Y) = \sum_{s=2}^{\infty} \frac{\varepsilon^{s-2}}{s!} L_Z^s Y$$

with $L_Z^1 Y = L_Z Y$ and $L_Z^{s+1} Y = L_Z^s(L_Z Y)$ for $s \geq 1$.

Applying the Lie method to (15), and observing that $R_\varepsilon^1(X_{\varepsilon f}) = \varepsilon R_\varepsilon^1(X_f)$, gives

$$\Phi_\varepsilon^*(X_k + \varepsilon X_f) = X_k + \varepsilon[Z, X_k] + \varepsilon X_f + \varepsilon^2 R_\varepsilon^2(X_k) + \varepsilon^2 R_\varepsilon^1(X_f)$$

³The fact that the remainder is order ε^2 is formal: due to the presence of resonances, the remainder might in fact be $\mathcal{O}(\varepsilon^p)$ with some $1 < p < 2$, see below. To simplify the exposition, however, in this Section we adopt this formal point of view.

and therefore, given that the last two terms are $\mathcal{O}(\varepsilon^2)$, the vector field Z should be selected so that

$$[Z, X_k] + X_f = G \quad (16)$$

with some G with the desired properties. Equation (16) is the so-called (*'vector'*) *homological equation* of perturbation theory. There are very well known obstructions to the existence of solutions to this equation, due to the presence of resonances, and it is well known that, in order to obtain a solution, the equation has to be modified.

Before seeing this, we point out that if we look for solutions Z, G of equation (16) which are Hamiltonian, then we are essentially in the standard symplectic-Hamiltonian case. To see this, we first note that, for Hamiltonian vector fields, equation (16) reduces to the standard homological equation for the Hamiltonian functions of the symplectic case:

Lemma 15. *If there exist functions z and g which satisfy the ('scalar') homological equation*

$$\{k, z\}_{\text{aa}} + f = g \quad (17)$$

then $Z = X_z$ satisfies the ('vector homological') equation (16) with $G = X_g$.

Proof. Since X_k is strongly Hamiltonian and $Z = X_z$ is Hamiltonian, by Lemma 1 the vector field $[Z, X_k]$ is Hamiltonian and equals $X_{\{k, z\}_{\text{aa}}}$. Therefore $[Z, X_k] - X_f - G = X_{\{k, z\}_{\text{aa}} - f - g} = 0$. \square

Let now $\omega = \frac{\partial k}{\partial \alpha}$ be the frequency map of the unperturbed system, so that $X_k = \sum_{j=1}^n \omega_j \partial_{\alpha_j}$. Since the function k depends only on the actions, the A -dependent terms in the almost Poisson brackets $\{k, z\}_{\text{aa}}$ are absent, see (5). Therefore, the scalar homological equation (17) reduces exactly to the standard homological equation of the symplectic case, namely

$$\omega \cdot \frac{\partial z}{\partial \alpha} = g - f.$$

Furthermore, given that the relation between the action-components of a Hamiltonian vector field and its Hamiltonian function is the same as in the symplectic case, one realizes that in the almost-symplectic case the normal form term g can be chosen exactly as in the symplectic case: namely, as (partial) average of the perturbation f . We will be more precise on this in the next Subsection.

In other words, in the almost-symplectic case that we consider, at the level of Hamiltonian functions things go exactly as in the symplectic case. Nevertheless, even in the first normalization step that we are considering here, there are differences at the level of the normal form vector fields. A (minor) difference from the standard symplectic case is that the α -components of Z and G contain extra A -dependent terms. More important, the vector field Z is Hamiltonian, but need not be strongly Hamiltonian. This has the consequence that the remainder $\varepsilon^2 R_\varepsilon^1(X_f) + \varepsilon^2 R_\varepsilon^2(X_k)$ need not be Hamiltonian, and the procedure just outlined cannot be iterated.

The conclusion of this elementary analysis is that it can be expected that all results on the variations of the actions that, in the symplectic case, follow from a single normalization step will retain their validity in the almost-symplectic case. As we will see, this includes a 'first-order' formulation of Nekhoroshev theorem. However, all results obtained through

iteration of the normal form procedure, in particular the KAM theorem, will not extend to the almost-symplectic case, unless the perturbation has special properties (e.g., it is strongly Hamiltonian, see Section 4.3).

4.2 An almost-symplectic Nekhoroshev-like theorem. In order to make more definite statements, we need to take into considerations the role of resonances. This requires the consideration of Fourier series of Hamiltonian vector fields and of some properties of their (partial) averages.

From now on, we will write the Fourier series (11) of a function on $\mathcal{A} \times \mathbb{R}^n$ as

$$y = \sum_{\nu \in \mathbb{Z}^n} y_\nu,$$

where the functions $y_\nu : \mathcal{A} \rightarrow \mathbb{R}$, that we call the *harmonics* of y , are given by $y_\nu(a, \alpha) = \hat{y}_\nu(a)E_\nu(\alpha)$. Similarly, if Y is a vector field, we will write

$$Y = \sum_{\nu \in \mathbb{Z}^n} Y_\nu$$

where, for each ν , the harmonic Y_ν is defined as the vector field whose components are the ν -th harmonics of the components of Y . Note that if Y is Hamiltonian, with Hamiltonian function y , then, for each $\nu \in \mathbb{Z}^n$, Y_ν is a Hamiltonian vector field, with Hamiltonian function y_ν . Furthermore, for any subset Λ of \mathbb{Z}^n we define projectors Π_Λ on the spaces of functions and vector fields as

$$\Pi_\Lambda y := \sum_{\nu \in \Lambda} y_\nu, \quad \Pi_\Lambda Y := \sum_{\nu \in \Lambda} Y_\nu.$$

Clearly, if Y is Hamiltonian with Hamiltonian y , then $\Pi_\Lambda Y$ is Hamiltonian with Hamiltonian $\Pi_\Lambda y$.

A point $a \in \mathcal{A}$ is said to be resonant with a vector $\nu \in \mathbb{Z}^n$ if $\omega(a) \cdot \nu = 0$. In that case, ν is called a resonance of a and $|\nu| := \sum_{i=1}^n |\nu_i|$ its order. The resonances of a point a form a sublattice Λ_a of \mathbb{Z}^n . Conversely, given a subset (not necessarily a sublattice) $\Lambda \subseteq \mathbb{Z}^n$, the Λ -resonant set is

$$\mathcal{A}_\Lambda := \{a \in \mathcal{A} : \omega(a) \cdot \nu = 0 \text{ for all } \nu \in \Lambda\}.$$

By expanding all functions in Fourier series, the scalar homological equation (17) becomes $\sqrt{-1}\omega \cdot \nu = g_\nu - f_\nu$ for all $\nu \in \mathbb{Z}^n$. Hence, at each point a , if Λ_a denotes as above the set of resonances of a , equation (17) has the solution

$$z(a, \alpha) = \sum_{\nu \notin \Lambda_a} \frac{f_\nu(a, \alpha)}{\sqrt{-1}\omega(a) \cdot \nu}, \quad g(a, \alpha) = \Pi_{\Lambda_a} f(a, \alpha).$$

(If $\Lambda \neq \{0\}$ then this solution is not unique, because there is arbitrariness in the choice of z_ν for $\nu \in \Lambda_a$ and of g_ν for $\nu \notin \Lambda_a$; however, this solution is the one which is usually considered in the symplectic case and there is no reason here to change it). Due to the resonances, the solution above has obvious and well known smoothness problems.

Specifically, if, as we will assume, the Hamiltonian k is such that the frequency map $\omega : \mathcal{A} \rightarrow \mathbb{R}^n$ is a local diffeomorphism, which happens if k satisfies Kolmogorov's nondegeneracy condition $\det \frac{\partial^2 k}{\partial a \partial a}(a) \neq 0$ for all $a \in \mathcal{A}$, then the set of resonant points is dense in \mathcal{A} . The way out depends to a certain extent on the result one is looking for, but for both KAM and Nekhoroshev theorems it is based on the approximation of the perturbation by a finite order Fourier truncation

$$f^{\leq N}(a, \alpha) := \sum_{\nu \in \mathbb{Z}^n, |\nu| \leq N} f_\nu(a), \quad (18)$$

so as to have to deal with only a finite number of resonances and avoid the density of resonances, and on the construction of (resonant) normal forms in neighbourhoods of the corresponding resonant sets. The parameter N is called a cutoff and $X_f^{>N} := X_f - X_f^{\leq N}$ the ultraviolet part of X_f . For real analytic vector fields, $X_f^{>N}$ decays with N as $\exp(-\text{const} \times N)$. Thus, suitably choosing N as a function of ε makes $X_f^{>N}$ of order ε^2 (or smaller, if needed).

We thus fix a cutoff N , a set $\Lambda \subseteq \mathbb{Z}_N^n := \{\nu \in \mathbb{Z}^n : |\nu| \leq N\}$ and a subset \mathcal{B}_Λ of \mathcal{A} whose points possibly resonate with the vectors of Λ but do not resonate with any other vector $\nu \in \mathbb{Z}_N^n \setminus \{\Lambda\}$. Since

$$\Phi_\varepsilon^*(X_k + \varepsilon X_f) = X_k + \varepsilon[Z, X_k] + \varepsilon X_f^{\leq N} + X_f^{>N} + \varepsilon^2 R_\varepsilon^2(X_k) + \varepsilon^2 R_\varepsilon^1(X_f) \quad (19)$$

if we take $Z = X_z$ in $\mathcal{B}_\Lambda \times \mathbb{T}^n$ with

$$z = \sum_{\nu \notin \Lambda} \frac{f_\nu}{\sqrt{-1} \omega \cdot \nu}, \quad g = \Pi_\Lambda f \quad (20)$$

we obtain the Λ -resonant normal form

$$\Phi_\varepsilon^*(X_k + \varepsilon X_f) = X_k + \varepsilon \Pi_\Lambda X_f^{\leq N} + \mathcal{O}(\varepsilon^2) \quad (21)$$

which is defined in $\mathcal{B}_\Lambda \times \mathbb{T}^n$. Note that the function z is now a sum of finitely many terms, and all denominators are nonzero, so there are no smoothness issues. The usefulness of these approximate normal forms in the standard symplectic case is due to the properties of the a -components of the averages $\Pi_\Lambda X_f$. These properties are valid in the almost-symplectic case as well:

Lemma 16. *Let Λ be a subset of \mathbb{Z}^n . Then, if Y is Hamiltonian, the a -component of $\Pi_\Lambda Y$ is parallel to Λ .*

Proof. If y is a Hamiltonian of Y then $\Pi_\Lambda y$ is a Hamiltonian of $\Pi_\Lambda Y$ and $Y_a = -\frac{\partial}{\partial \alpha} \sum_{\nu \in \Lambda} y_\nu = \sum_{\nu \in \Lambda} \sqrt{-1} \nu y_\nu$. \square

Remark: The statement in Lemma 16 that a vector field on \mathcal{A} is parallel to a set $\Lambda \subseteq \mathbb{R}^n$ is meaningful because, just in the standard symplectic case, the action space \mathcal{A} has an affine structure [9].

Lemma 16 has the consequence that, near a resonance set \mathcal{A}_Λ where the dynamics is described by the normal form (21), for times short with respect to ε^{-2} the motion of the (transformed) actions takes place approximately in an affine subspace parallel to Λ of the action space. This is the ‘fast drift’ subspace of Nekhoroshev theory. In the symplectic Hamiltonian case, Nekhoroshev theory provides a mechanisms of confinement of such fast drift, which requires certain properties of the unperturbed Hamiltonian. The most general among these properties are the so called steepness properties (see [12] for a recent, refined proof of Nekhoroshev theorem under such general conditions), but a simple case is provided by convexity, namely

$$\left| u \cdot \frac{\partial^2 k}{\partial a \partial a}(a) u \right| \geq \text{const} \|u\|^2 \quad \forall u \in \mathbb{R}^n, \quad a \in \mathcal{A}.$$

Under this hypothesis (or more generally, under the so-called hypothesis of quasi-convexity, see [17]), in the symplectic case the confinement of the actions’ movement along the fast drift hyperplane is provided by the conservation of the Hamiltonian.

In our case, the normal form vector field $(\Phi_\varepsilon^Z)^*(X_k + \varepsilon X_f) = X_k + \varepsilon \Pi_\Lambda X_f + \mathcal{O}(\varepsilon^2)$ is not Hamiltonian. However, the original system $X_k + \varepsilon X_f$ is Hamiltonian. As consequence, its Hamilton function $k + \varepsilon f$ is a first integral of $X_k + \varepsilon X_f$. In turn, the function $(\Phi_\varepsilon^Z)^*(k + \varepsilon f)$ is a first integral of the vector field $(\Phi_\varepsilon^Z)^*(X_k + \varepsilon X_f)$. This first integral provides the necessary confinement.

The only, real difference from the symplectic case is that the procedure cannot be iterated, and ‘exponentially long’ time scales are not reached. However, these considerations should make clear that the following result can be reached instead:

Proposition 17. *Consider the system of Hamiltonian $k(a) + \varepsilon f(a, \alpha)$ on $\mathcal{A} \times \mathbb{T}^n$, equipped with the almost-symplectic structure σ_{aa} . Assume that k and f are real analytic and that k is convex. Then, there exist positive constants A , T , c_1 and c_2 independent of ε and with*

$$1 < c_1 + c_2 < 2$$

such that, for ε sufficiently small, all motions $t \mapsto (a_t, \alpha_t)$ satisfy

$$\|a_t - a_0\| \leq A\varepsilon^{c_1} \quad \text{for} \quad |t| \leq T\varepsilon^{-c_2}. \quad (22)$$

We do not give here a proof of this result because, as we have already mentioned, the proof can be obtained with very minor modifications of, for instance, the proof given in [17] for the symplectic case. Specifically, besides some small differences in the construction and estimate of the normal form vector field (which is not the Hamiltonian vector field of the normal form Hamiltonian and thus needs to be treated on its own), the main difference is that in our case it is not necessary to iterate the construction of the normal form; this simplification leads to different estimates on the confinement of motions, which are however easily worked out.

Following the argument in [17] one obtains, as possible values of the two constants c_1 and c_2 , for instance, $c_1 = \frac{1}{8n}$ and $c_2 = \frac{3}{2}(1 - \frac{1}{4n})$. These values improve, even only slightly, on the apriori estimate $c_1 + c_2 = 1$. It is possible that better values of these constants, particularly of c_2 , might be found by carefully complementing the treatment with some

specificities of the problem at hand. We also note that, since in the standard Nekhoroshev theory real analyticity is needed only to obtain an exponentially long time scale (see e.g. [6]), the result of Proposition 17 remain valid (possibly with worse values of the constants c_1 and c_2) for smooth Hamiltonians. However, we leave these analyses for a possible future work because the technical arguments involved are rather extraneous to the purpose and the spirit of the present work.

Remark: The values of the constants c_1 and c_2 reported above can be obtained under the additional hypothesis that ω is uniformly bounded away from 0 in \mathcal{A} ; if not, slightly worse values can be found; see [17] for the treatment of this technical fact in the symplectic case.

4.3 On the case of strong Hamiltonian perturbations. If the perturbation f is strongly Hamiltonian then, at least under the hypotheses considered in Section 3.4, it is possible to study the reduced symplectic-Hamiltonian systems via the standard techniques of Hamiltonian perturbation theory. Thus KAM and Nekhoroshev theorem are valid for the reduced systems and can be lifted to the unreduced system by means of the reconstruction equation, see equation (14). Alternatively, however, one may apply the perturbation technique described in the previous Sections 4.1 and 4.2 to the unreduced system. At variance from the case of a perturbation that is only Hamiltonian, if the perturbation is strongly Hamiltonian, then the construction of the normal forms can be iterated, and the standard KAM and Nekhoroshev theorems of the symplectic case may be recovered.

This is due to the following fact:

Proposition 18. *If f is strongly Hamiltonian, then the normal form vector field (19), with z and g as in (20), is strongly Hamiltonian.*

Proof. First of all, we note that a function y is strongly Hamiltonian if and only if its harmonics y_ν are strongly Hamiltonian. In fact, $X_y = \sum_\nu (X_y)_\nu = \sum_\nu X_{y_\nu}$ and the vanishing of $i_{\sum_\nu X_{y_\nu}} d\sigma = \sum_\nu i_{X_{y_\nu}} d\sigma$ is equivalent to the vanishing of each $i_{X_{y_\nu}} d\sigma$.

If z is as in (20) then, for each ν , $z_\nu = \frac{f_\nu}{\sqrt{-1}\omega \cdot \nu}$ and

$$\sum_{k=1}^n C_{ijk} \frac{\partial z_\nu}{\partial \alpha_k} = \frac{1}{\sqrt{-1}\omega \cdot \nu} \sum_{k=1}^n C_{ijk} \frac{\partial f_\nu}{\partial \alpha_k}$$

which vanishes because f_ν is strongly Hamiltonian. This proves that each z_ν is strongly Hamiltonian, see (12), and hence z is strongly Hamiltonian.

Next, we note that if z and y are two strongly Hamiltonian functions, then, for any t , the function $(\Phi_t^{X_z})_* y$ is strongly Hamiltonian and its Hamiltonian vector field is $(\Phi_t^{X_z})_* X_y$. The proof of this fact is immediate because, restricted to its strongly Hamiltonian vector fields, an almost-symplectic structure behaves as a symplectic one.

It follows that $R_\varepsilon^1(X_f) = \Phi_\varepsilon^*(X_f) - X_f$ and, taking into account Lemma 1 as well, $R_\varepsilon^2(X_k) = \Phi_\varepsilon^*(X_k) - X_k - \varepsilon[Z, X_k]$ are strongly Hamiltonian. The proof is concluded by noting that, on account of what has been noticed above, $X_f^{\leq N}$ and $\Pi_\Lambda X_f^{\leq N}$ are strongly Hamiltonian, too. \square

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References

- [1] V.I. Arnold, V.V. Kozlov and A. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics. Dynamical Systems, III. Third Edition. Encyclopaedia Math. Sci.* **3**. (Springer, Berlin, 2006).
- [2] L. Bates and J. Śniatycki, *Nonholonomic reduction*. Rep. Math. Phys. **32** (1993), 99-115.
- [3] G. Benettin, G. Ferrari, L. Galgani and A. Giorgilli, *An extension of the Poincaré-Fermi theorem on the nonexistence of invariant manifolds in nearly integrable Hamiltonian systems*. Nuovo Cimento B (11) **72** (1982), 137-148.
- [4] G. Benettin and G. Gallavotti, *Stability of motions near resonances in quasi-integrable Hamiltonian systems*. J. Statist. Phys. **44** (1986), 293-338.
- [5] G. Blankenstein and A.J. van der Schaft, *Symmetry and reduction in implicit generalized hamiltonian systems*. Rep. Math. Phys. **47** (2001), 57-100.
- [6] A. Bounemora, *Nekhoroshev estimates for finitely differentiable quasi-convex Hamiltonians*. J. Diff. Eq. **249** (2010), 2905-2920.
- [7] F. Fassò, *Lie series method for vector fields and Hamiltonian perturbation theory*. Z. Angew. Math. Phys. **41** (1990), 843-864.
- [8] F. Fassò, *Hamiltonian perturbation theory on a manifold*. Celestial Mech. Dynam. Astronom. **62**, 43-69 (1995)
- [9] F. Fassò and N. Sansonetto, *Integrable almost-symplectic Hamiltonian systems*. J. Math. Phys., **48** (2007), 092902, 13 pp.
- [10] C.K. Fok, *Picard group of isotropic realizations of twisted Poisson manifolds*. arXiv:1511.03955
- [11] C. Godbillon, *Géométrie différentielle et mécanique analytique* (Hermann, Paris, 1969).
- [12] M. Guzzo, L. Chierchia and G. Benettin, *The Steep Nekhoroshev's Theorem*. ArXiv:1403.6776 (math-ph). In press in Comm. Math. Phys.
- [13] S. Lang, *Algebra*. Graduate Texts in Mathematics **211** (Springer, New York, 2002).
- [14] P. Libermann and C.-M. Marle, *Symplectic geometry and analytical mechanics*. Mathematics and its Applications **35** (D. Reidel Publishing Co., Dordrecht, 1987).
- [15] N.N. Nekhoroshev, *An exponential estimate of the time of stability of nearly integrable hamiltonian systems*. Russian Math. Surv. **32**, 1-65 (1977).
- [16] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Vol. 1 (Gauthier-Villars, Paris, 1892).
- [17] J. Pöschel, *Nekhoroshev estimates for quasi-convex Hamiltonian systems*. Math. Z. **213** (1993), 187-216.
- [18] N. Sansonetto and D. Sepe, *Twisted isotropic realizations of twisted Poisson manifolds*. J. Geom. Mech. **5** (2013), 233-256.

- [19] P. Ševera and A. Weinstein, *Poisson geometry with a 3-form background*, in “Noncommutative geometry and string theory” (Yokohama, 2001), Progr. Theoret. Phys. Suppl. **144** (2001), 145-154.
- [20] I. Vaisman, *Hamiltonian vector fields on almost symplectic manifolds*. J. Math. Phys. **54** (2013), 092902, 11 pp.
- [21] N.T. Zung *Action-angle variables on Dirac Manifolds*. arxiv:1204.3865