# Superintegrable Hamiltonian systems: Geometry and Perturbations

Francesco Fassò
*Università di Padova, Dipartimento di Matematica Pura e Applicata, Via G. Belzoni 7, 35131 Padova, Italy.* (E-mail: fasso@math.unipd.it)

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**Abstract.** Many and important integrable Hamiltonian systems are 'superintegrable', in the sense that there is an open subset of their $2d$–dimensional phase space in which all motions are linear on tori of dimension $n < d$. A thorough comprehension of these systems requires a description which goes beyond the standard notion of Liouville–Arnold integrability, that is, the existence of an invariant fibration by Lagrangian tori. Instead, the natural object to look at is formed by both the fibration by the (isotropic) invariant tori and by its (coisotropic) polar foliation, which together form what in symplectic geometry is called a 'dual pair', or 'bifoliation', or 'bifibration'. We review this geometric structure, relating it to the dynamical properties of superintegrable systems and pointing out its importance for a thorough understanding of these systems.

## 1. Introduction

**A.** Integrability of Hamiltonian systems is often identified with *complete integrability*, or *Liouville integrability*, that is, the existence of as many independent integrals of motion in involution as the dimension of the phase space. Under certain regularity and compactness conditions, the invariant geometric structure associated to Liouville integrability is a fibration by Lagrangian tori, on which motions are linear. This structure plays an important role in many questions, such as the quantization of the system and certain aspects of the study of small perturbations of it.

However, there are many and important systems which have more independent integrals of motion (with suitable properties) and correspondingly have motions linear on tori of smaller dimension. Systems of this type are called *superintegrable*, or *degenerate*, or *noncommutatively integrable*. Here are few examples:

- A point in a central force field: three degrees of freedom, four integrals of motion (energy and angular momentum), motions linear on two–dimensional tori.
- Kepler system, which has one further integral of motion (coming from the Laplace–Runge–Lenz vector) and periodic motions.

- The Euler top: three degrees of freedom, four integrals of motion (energy and spatial angular momentum), motions linear on tori of dimension two.

The fact that this short, and not exhaustive, list includes some of the 'classical' systems of mechanics indicates the importance of the subject.

The aim of this article is to review the geometric structure of superintegrable systems. As it turns out, the natural, intrinsic, invariant geometric object associated to superintegrability is not a Lagrangian fibration, but a finer and richer structure, which is formed by *both*

- the (isotropic) fibration by the invariant tori, and

- its (coisotropic) 'polar' foliation.

Together, they form what is well known in symplectic geometry under a variety of names: *dual pair*, *symplectically complete foliation*, *bifoliation*, *bifibration*.

If, as it often happens, both these foliations are fibrations, then the space of the leaves of the polar foliation is an affine manifold which has the same dimension of the invariant tori and plays the role of 'action manifold' of the system. Furthermore, the base of the fibration by the invariant tori is a Poisson manifold and the space of its symplectic leaves can be identified with the action manifold. There is therefore a clear distinction between the 'actions' and the other first integrals, which can be regarded as local coordinates on the symplectic leaves. This 'double' structure is obviously hidden in the Liouville case because any Lagrangian foliation coincides with its own polar. Its consideration provides however a thorough insight into superintegrable systems.

In general, of course, not all of the phase space of a (super)integrable system is fibered by invariant tori of a given dimension $n$. The orbits form in fact a foliation with singularities (equilibria, lower dimensional tori, hyperbolic sets, etc.). Nevertheless, we shall (tacitly) restrict our attention to the regular subset of the phase space, which is fibered by invariant tori of a given dimension: the idea is that the consideration of this regular structure has already several reasons of interest.

**B.** To our knowledge, a comprehensive study of superintegrability begins with the work of Nekhoroshev in 1972 [38]. (A good source for what was classically known is [12]). In [38] Nekhoroshev related superintegrability to properties of the integrals of motion, provided a generalization of the Liouville–Arnold theorem which describes the local structure of the fibration by the invariant tori, and began the study of the global structure of this fibration. In particular, he identified what became later known as 'monodromy' after Duistermaat's treatment of the Liouville case [17]. However, the role of the polar foliation is missing in Nekhoroshev's treatment. Nekhoroshev work passed largely unobserved. A few years later, Mischenko and Fomenko [36] gave a more Lie algebraically–oriented treatment which received more attention.

The characterization in terms of dual pairs is present e.g. in the work of Dazord and Delzant [16] and of Karasev and Maslov [29]. A complete study of the global geometry of the fibration by tori, which generalizes Duistermaat's treatment of the Lagrangian case, was given by Dazord and Delzant [16].

The importance of the consideration of the bifibrated structure for the comprehension of superintegrable systems of mechanical interest was pointed out in particular in [29, 20, 21, 22]. Nowdays there is a growing number of works which rest on the notion of superintegrability and on the corresponding geometric structure, see e.g. [10, 23, 24, 27, 11, 49, 18].

We note that there is also a vast literature on superintegrable systems which focuses on problems different from those considered here, such as the construction of potentials producing 'maximally superintegrable' systems (that is, systems with periodic dynamics), the connection with the spectrum of the Schrödinger operator, the separation of variables in Hamilton–Jacobi equation, etc., see e.g. [19, 30] and the very recent volume of proceedings [45].

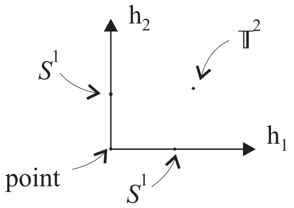
**C.** The article is organized as follows. In section 2 we preliminarily discuss an elementary but motivating example, where one can 'touch by hands' the bi–fibrated structure. Sections 3 and 4 are the core of the article: as a way of arriving at the dual pair structure of superintegrable systems, we start from the theorem on 'noncommutative integrability' by Mischenko and Fomenko, which is probably the best known among the various results in this field, and explain the symplectic geometrical meaning of its hypotheses. Section 5 is devoted to a few examples. Section 6 gives a very short account of perturbation theory for superintegrable systems—an important topic in which the geometry of these systems play an essential role and a *homage* to the title of this conference.

The present review is based on [21, 22].

## 2. An elementary example

**A. Two uncoupled oscillators.** We begin by considering a well known and elementary example, that is, a system of two uncoupled harmonic oscillators. This will help in pointing out some of the main facts about superintegrability. The Hamiltonian is

$$H(p_1, p_2, q_1, q_2) \;=\; \frac{\omega_1}{2}(p_1^2 + q_1^2) \;+\; \frac{\omega_2}{2}(p_2^2 + q_2^2)\,,$$



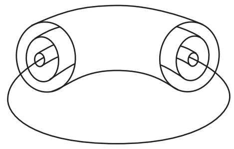




*Figure 1.* (a) Classification of the fibers of $(h_1, h_2)$, and (b) singularities of the Liouville tori.

where $(p, q)$ are canonical coordinates in $\mathbb{R}^4$ and the two frequencies $\omega_1$ and $\omega_2$ are *positive* constants. The energies $H_1 = \omega_1(p_1^2 + q_1^2)/2$ and $H_2 = \omega_2(p_2^2 + q_2^2)/2$ of the two oscillators are integrals of motion. Their common level set

$$M_{h_1,h_2} \;=\; \{(p_1, q_1) : p_1^2 + q_1^2 = 2h_1/\omega_1\} \times \{(p_2, q_2) : p_2^2 + q_2^2 = 2h_2/\omega_2\}$$

is a two–dimensional torus if $h_1$ and $h_2$ are both positive, a circle if one is positive and the other vanishes, a point if they are both zero, see Figure 1.a. Thus, a large subset of the phase space is foliated by invariant two–dimensional tori and this foliation has singularities, corresponding to motions where either one or both oscillators are at rest. Figure 1.b depicts the local structure of the foliation near the $S^1$–singular leaves.

The description of this system as Liouville–integrable focuses on the foliation by the two–dimensional tori. The subset where $h_1 > 0$ and $h_2 > 0$ is fibered[1] by them and one can there introduce action–angle coordinates $(a_1, a_2, \alpha_1, \alpha_2) \in \mathbb{R}^2 \times \mathbb{T}^2$ via the well known relations $p_j = \sqrt{2a_j} \cos \alpha_j$, $q_j = \sqrt{2a_j} \sin \alpha_j$. The Hamiltonian takes the form $\omega_1 a_1 + \omega_2 a_2$ and the equations of motion are $\dot{a}_1 = 0$, $\dot{a}_2 = 0$, $\dot{\alpha}_1 = \omega_1$ and $\dot{\alpha}_2 = \omega_2$. Of course, these coordinates are not defined on the singularities of the foliation.

The key question now is whether this invariant structure—the fibration by two–dimensional tori—reflects any intrinsic property of the system. This depends on the arithmetic properties of the two frequencies:

• If $\omega_1/\omega_2$ is irrational, then every trajectory fills densely an invariant two–dimensional torus $M_{h_1,h_2}$. Thus, there is no (continuous)

[1] In this work, by ‘fibration’ we mean a locally trivial fibration, see e.g. [1]. The distinction between fibration and foliation will appear constantly. We shall often (tacitly) invoke the important Ehresmann fibration theorem and some of its consequences, according to which *a surjective submersion whose fibers are compact and connected (or, more generally, have a constant number ofconnected components) is a fibration*, see [35].

integral of motion independent of $H_1$ and $H_2$ and the fibration by the two–dimensional tori is the finest fibration with fibers invariant under the flow—and as such, it is a geometric object naturally created by the dynamics.

• If instead $\omega_1/\omega_2$ is rational, then all motions are periodic. Correspondingly, the periodic orbits form a regular invariant structure (in fact, a fibration) which is *finer* than that by the Liouville tori. In fact, each Liouville torus is fibered by periodic orbits. For certain values of $\omega_1/\omega_2$, moreover, this fibration includes also one or both families of periodic orbits with either $h_1 = 0$ or $h_2 = 0$, and hence it is defined in a *larger* set than the fibration by Liouville tori (see below for the case $\omega_1 = \omega_2$). This situation reflects the existence of an additional independent integral of motion and is clear that it is this structure, and not that by the Liouville tori, which is, so to say, 'created by the dynamics'—and as such, should be taken into consideration.

The obvious question now is what is gained in comprehension, in the 'superintegrable' case $\omega_1/\omega_2 \in \mathbb{Q}$, from the consideration of the fibration by periodic orbits instead of that by the two–dimensional Liouville tori. And also, which is the relation between the two structures.

**B. The isotropic oscillator.** In order to elucidate these points we consider the simple case $\omega_1 = \omega_2$, in which the system can be regarded as an isotropic oscillator in the plane. Three independent first integrals are $\pi_1(q,p) = q_1q_2 + p_1p_2$, $\pi_2(q,p) = p_1q_2 - p_2q_1$, $\pi_3(q,p) = \frac{1}{2}(p_1^2 + q_1^2 - p_2^2 - q_2^2)$, see e.g. [29]. These functions are of course not independent of the Hamiltonian, but satisfy

$$\pi_1^2 + \pi_2^2 + \pi_3^2 = H^2\,. \tag{1}$$

It is easy to check that the map $\pi = (\pi_1, \pi_2, \pi_3) : \mathbb{R}^4 \to \mathbb{R}$ is a submersion at all points but at the origin. Moreover, its fibers are connected and hence coincide with the periodic orbits. Therefore, the periodic orbits are the fibers of the fibration $\pi = (\pi_1, \pi_2, \pi_3) : M \to B$ with total space $M = \mathbb{R}^4 \setminus \{0\}$ and base $B = \mathbb{R}^3 \setminus \{0\}$.

However, $M$ is also fibered by the level sets of $H$, which are three–dimensional spheres centered at the origin. Clearly, this is a trivial fibration with base $\mathbb{R}_+$, so that $M \approx \mathbb{R}_+ \times S^3$.

The key fact is now that, given the constancy of $H$, each of its level sets is in turn fibered by the periodic orbits. Each of these fibrations is described by the restriction of the map $\pi$ to the set $H = \text{const}$. Hence, according to equation (1), its base is a two–dimensional sphere. In fact, such a fibration is the well known Hopf fibration $S^3 \to S^2$, which is topologically nontrivial. (See e.g. [44] for its definition, which uses exactly the flow of the isotropic oscillator).

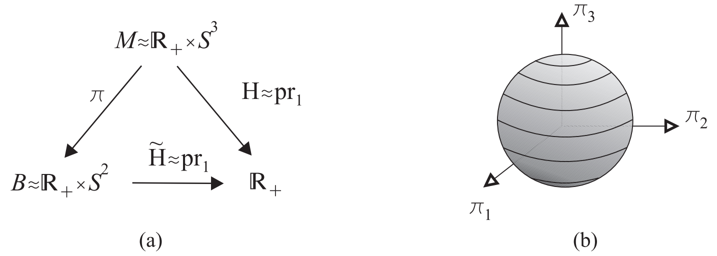



*Figure 2.* (a) Doubly–fibered structure of the phase space of the isotropic oscillator, and (b) foliation of the sphere $|\pi| = \text{const}$ producing the Liouville tori.

Another way of saying that the fibers of $H$ are fibered by the periodic orbits is that there is a map $\tilde{H} : B \to \mathbb{R}_+$, in fact a fibration, which is such that $H = \tilde{H} \circ \pi$, that is, its fibers are the spheres $|\pi| = \text{const}$ in $B$.

The overall structure is illustrated in Figure 2.a. Its meaning will be fully clarified when we consider the general case: the structure of super-integrable systems is an immediate generalization of this structure, the periodic orbits being replaced by the invariant tori, the Hamiltonian by the 'actions', the base $B$ by a Poisson manifold, the spheres $|\pi| = \text{const}$ by its symplectic leaves, the fibration $\tilde{H}$ by its symplectic foliation.

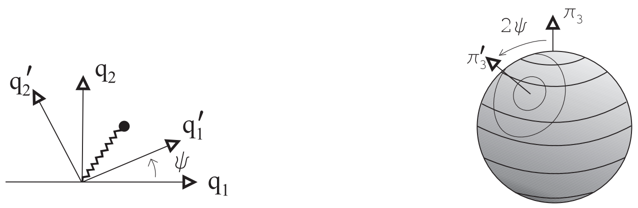
Note also that, in the present case, all the topological nontriviality of this structure comes from the 'internal structure' of the fibers of $H$, which is a Hopf fibration.

**C. Liouville tori.** The two–dimensional tori $M_{h_1,h_2}$ do not play any role in the previous picture, which however allows a clear understanding of their origin.

Expressing $\pi$ in terms of the action–angle coordinates one finds $\pi_1 = 2\sqrt{a_1 a_2}\cos(\alpha_2 - \alpha_1)$, $\pi_2 = 2\sqrt{a_1 a_2}\sin(\alpha_2 - \alpha_1)$ and $\pi_3 = a_1 - a_2$. Since $h_1 = \omega_1 a_1$ and $h_2 = \omega_2 a_2$, this shows that $M_{h_1,h_2}$ consists of all periodic orbits based on one of the parallels $\pi_3 = \text{const}$ of the sphere $|\pi| = h_1 + h_2$ in $B$. Thus, the Liouville tori are produced by grouping together the periodic orbits which sit on these circles.

As is clear, this procedure is highly not unique: one would obtain a *different* foliation by Liouville tori by choosing a different foliation into circles of these spheres. And moreover, since any foliation of $S^2$ into circles has singularities, any foliation by Liouville tori will necessarily have singularities of the type shown in Figure 1.b, while the fibration by the periodic orbits is there regular.

The question which remains to be answered is whether the considered foliation by Liouville tori has anything special. The answer is no and, in the case considered here of equal frequencies, there is a very simple argument showing it. As we have already remarked, the system




(a) (b)

*Figure 3.* Construction of different foliations by Liouville tori for the isotropic oscillator: (a) change of coordinates in configuration space, (b) foliations of the sphere $|\pi| = \text{const}$.

can be regarded as a point in the plane connected to the origin by a spring, see Figure 3. Since the system is invariant under rotations, we can freely choose the two axes $q_1$ and $q_2$ in the plane. A simple computation [24] shows that if we take new axes forming an angle $\psi$ with the previous ones and then repeat the construction of the map $\pi$, but in terms of the new coordinates, we arrive at Liouville tori $M'_{h_1,h_2}$ which are constructed by foliating the spheres $|\pi| = \text{const}$ into circles which form an angle $2\psi$ with the previous ones. The fact that there is no preferred choice of coordinates in the plane means that there is no preferred choice among these different families of Liouville tori.

It should appear clear from these considerations that the foliation by the Liouville tori is not a property of the system, but an artifact introduced by the observer. This is in fact the common situation with superintegrable systems, that we now consider from a general viewpoint.

## 3. Geometry of noncommutative integrable systems

**A. Mischenko–Fomenko's 'noncommutative' integrability.** We base our treatment on the analysis of a standard theorem about superintegrability, which is essentially Mischenko and Fomenko's theorem on 'noncommutative integrability' [36]. In all of this section $(M, \Omega)$ will be a symplectic manifold of dimension $2d$, $d \geq 1$, and $n \leq d$ will be a positive integer (the dimension of the invariant tori).

THEOREM 1. [38] [36] *Let $f = (f_1, \ldots, f_{2d-n}) : M \to f(M) \subset \mathbb{R}^{2d-n}$ be a submersion with compact and connected fibers (hence, a fibration). Assume that*

*MF1. There exist functions $P_{ij} : f(M) \to \mathbb{R}$ such that*

$$\{f_i, f_j\} \ = \ P_{ij} \circ f \,, \qquad i,j = 1, \ldots, 2d-n \,. \tag{2}$$

*MF2. The matrix $P$ with entries $P_{ij}$ has rank $2d-2n$ at all points of $f(M)$.*

*Then:*

*i. Every fiber of $f$ is diffeomorphic to $\mathbb{T}^n$.*

*ii. The fibration $f$ has local trivializations which are symplectic: precisely, every fiber of $f$ has a neighbourhood $U$ endowed with a diffeomorphism*

$$b\times\alpha \,:\, U\to\mathcal{B}\times\mathbb{T}^n\,,\qquad \mathcal{B}=b(U)\subset\mathbb{R}^{2d-n}\,,\tag{3}$$

*such that the level sets of $f$ coincide with the level sets of $b$ and, writing $b=(p_1,\dots,p_{d-n},\,q_1,\dots,q_{d-n},a_1,\dots,a_n)$,*

$$\Omega|_U \;=\; \sum_{k=1}^{d-n} dp_k\wedge dq_k+\sum_{j=1}^{n} da_j\wedge d\alpha_j\,.\tag{4}$$

This theorem reduces to the Liouville–Arnold theorem for $n=d$. We shall therefore consider only the 'superintegrable' case $n<d$.

The local coordinates $a$ and $\alpha$ will be called, respectively, (local) actions and (local) angles of the fibration $f$. The reason of this terminology is that, according to (4), the Hamiltonian vector field of $a_j$ is $\frac{\partial}{\partial\alpha_j}$ and, generally speaking, an 'action' of a toric fibration of a symplectic manifold is a function whose Hamiltonian vector field is tangent to the fibers and has $2\pi$–periodic flow. The novelty with respect to the Liouville–Arnold theorem is thus in the presence of the coordinates $(p,q)$, which are not action–angle pairs. Because of this, the diffeomorphism (3) is called (local) *generalized (or partial) action–angle coordinates* [38, 16].

If a Hamiltonian system has $2d-n$ first integrals $f_1,\dots,f_{2d-n}$ as in Theorem 1, then its phase space is fibered by the invariant tori $f=$ const. Moreover, since all coordinates $(a,p,q)$ are first integrals, the local representative $h$ of the Hamiltonian in any local system of generalized action–angle coordinates necessarily depends on the actions $a$ alone, $h=h(a)$. Therefore,

$$\dot\alpha \;=\; \frac{\partial h}{\partial a}(a)\tag{5}$$

and motions are quasi–periodic on the tori $f=$ const, with frequencies which depend on the actions alone.

Theorem 1 is only one of the possible characterizations of *noncommutative integrability,*[2] which is also called *superintegrability*, *degenerate integrability*, *generalized Liouville integrability*, etc. We shall

[2] Mischenko and Fomenko considered the special case in which the functions $f_1,\dots,f_{2d-n}$ form a Lie algebra $\mathcal{G}$ of dimension $2d-n$, that is $\{f_i,f_j\}=\sum_l c_{ijl}f_l$ for some constants $c_{ijl}$. When $n<d$ this algebra is necessarily not commutative.

later introduce a more geometric point of view. For the time being we shall therefore keep using the expression noncommutative integrability (NCI) in a rather vague way and we shall speak of *MF–integrability* to refer exactly to the situation of Theorem 1.

We shall not reproduce here the proof of Theorem 1, which is after all a slight modification of the well known proof of the Liouville–Arnold theorem (see Property B5 below). Instead, our aim is to elucidate its geometric content.

**B. Bifoliations.** To this end, we need to review a few basic notions from symplectic geometry. As above, let $(M, \Omega)$ be a $2d$–dimensional symplectic manifold.

If $N$ is a submanifold of $M$ and $x \in N$, the *symplectic complement* $(T_x N)^\perp$ of the tangent space $T_x N$ is the subspace of $T_x M$ constituted by all vectors which are symplectically orthogonal to $T_x N$:

$$(T_x N)^\perp = \{u \in T_x M : \Omega(u, v) = 0 \ \forall v \in T_x N\} .$$

If $N$ has dimension $n$, then $(T_x N)^\perp$ has dimension $2d - n$. Moreover, $((T_x N)^\perp)^\perp = T_x N$.

A submanifold of $M$ is said to be *isotropic* (resp. *coisotropic*) if its tangent spaces are contained in (resp. contain) their own symplectic complements. Isotropic (coisotropic) submanifolds have dimension $\leq d$ ($\geq d$). *Lagrangian* submanifolds are both isotropic and coisotropic.

Let $\mathcal{F}$ be a foliation of $M$. The *polar* of $\mathcal{F}$, if it exists, is the unique foliation $\mathcal{F}^\perp$ of $M$ with the property that the tangent spaces of its leaves are the symplectic orthogonals of the tangent spaces of the leaves of $\mathcal{F}$.

If the leaves of $\mathcal{F}$ have dimension $n$, then the leaves of $\mathcal{F}^\perp$ have dimension $2d - n$. Furthermore, $(\mathcal{F}^\perp)^\perp = \mathcal{F}$. If $\mathcal{F}$ is isotropic[3] then $\mathcal{F}^\perp$ is coisotropic; hence, each leaf of $\mathcal{F}^\perp$ is union of leaves of $\mathcal{F}$.

A foliation $\mathcal{F}$ which has a polar is called *symplectically complete* [16]. Here are a few examples:

- Any Lagrangian foliation, which coincides with its own polar.
- The orbits of a Hamiltonian vector field $X_H$: the polar is given by the (connected components of the) level sets of $H$. Indeed, if a vector field $Y$ is tangent to the level sets of $H$ then $\Omega(Y, X_H) = L_Y H = 0$.
- Every coisotropic foliation has a polar: this follows from Property B1 below.
- The orbits of a Hamiltonian action and the level sets of its momentum map are polar to each other, see e.g. [31, 33].

However, not every isotropic foliation has a polar, see e.g. [48].

[3] By saying that a foliation or a fibration is (co)isotropic we mean that its leaves or fibers are (co)isotropic.

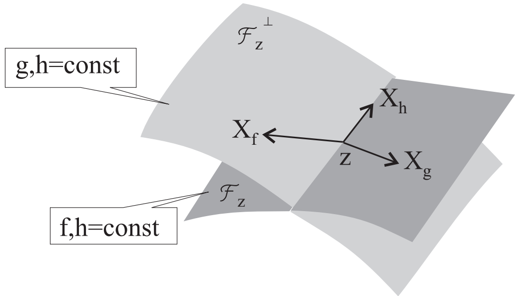



*Figure 4.* Illustrating the Lemma: $\mathcal{F}_z$ and $\mathcal{F}_z^{\perp}$ are the leaves of $\mathcal{F}$ and $\mathcal{F}^{\perp}$, respectively, through a point $z$, $f$ is a first integral of $\mathcal{F}$, $g$ of $\mathcal{F}^{\perp}$, $h$ of both.

If $\mathcal{F}$ is symplectically complete, then the pair $(\mathcal{F}, \mathcal{F}^{\perp})$ is called a *bifoliation* [29]. If, moreover, both $\mathcal{F}$ and $\mathcal{F}^{\perp}$ are fibrations,[4] let us denote them $i : M \to B$ and $c : M \to A$, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is also called a *bifibration* and will be denoted $A \overset{c}{\leftarrow} M \overset{i}{\rightarrow} B$. A bifibration is a particular case of *dual pair* [47]. For a general treatment of these objects see e.g. section 9 of [31], on which part of what follows is based.

**C. NCI and bifoliations.** A *first integral* of a foliation is a function (possibly defined only in an open subset) which is constant on the leaves of the foliation. If the foliation is a fibration, then its first integrals are the lifts of functions defined on the base. The following simple observation plays a key role in the sequel:

LEMMA *A function $f$ is a first integral of a foliation $\mathcal{F}$ of a symplectic manifold $M$ if and only if its Hamiltonian vector field $X_f$ is everywhere symplectically orthogonal to $\mathcal{F}$.*

*Proof.* $f$ is a first integral of $\mathcal{F}$ if and only if $L_Y f = 0$ for any vector field $Y$ tangent to the leaves of $\mathcal{F}$. But $\Omega(Y, X_f) = L_Y f$. ■

This elementary fact, which is illustrated in Figure 4, has some important consequences:

PROPERTY B1. *A foliation $\mathcal{F}$ is symplectically complete if and only if the Poisson brackets of any two first integrals of $\mathcal{F}$ is a first integral of $\mathcal{F}$.*

[4] By saying that a foliation is a fibration we mean that its leaves are the fibers of a fibration. Since the leaves of a foliation are connected, in the above definition of bifibration is hidden the condition that both fibrations have connected components.

*Proof.* If $\mathcal{F}$ is symplectically complete, then the distribution $\mathcal{F}^\perp$ consisting of the symplectic complements of the tangent spaces to the leaves of $\mathcal{F}$ is completely integrable. If $f$ and $g$ are two first integrals of $\mathcal{F}$, then by the Lemma $X_f, X_g \in F^\perp$. Hence, by Frobenious theorem, also $X_{\{f,g\}} = [X_f, X_g] \in F^\perp$. Invoking again the Lemma, one concludes that $\{f, g\}$ is a first integral of $\mathcal{F}$.

In order to prove the converse take local coordinates $z^1, \ldots, z^c$ transversal to the leaves of $\mathcal{F}$ (here $c = \mathrm{codim}\,(\mathcal{F})$). By the Lemma, $X_{z^1}, \ldots, X_{z^c}$ are tangent to $F^\perp$; being linearly independent, they form a local basis for $F^\perp$. Thus, to prove involutivity of $F^\perp$ we can just show that all brackets $[X_{z^i}, X_{z^j}] \in F^\perp$. By hypothesis, all functions $\{z^i, z^j\}$ are first integrals of $\mathcal{F}$. Hence, by the Lemma, $[X_{z^i}, X_{z^j}] = X_{\{z^i,z^j\}} \in F^\perp$. ■

The argument used in the second part of the proof of Property B1 shows that, given a fibration $f = (f_1, \ldots, f_k) : M \to f(M) \subset \mathbb{R}^{2d-n}$, it is sufficient to check that all Poisson brackets $\{f_i, f_j\}$ are first integrals to conclude that the Poisson bracket of any two first integrals is still a first integral. This clarifies the geometric meaning of condition MF1 of Theorem 1:

*The fibration by the invariant tori has a polar foliation.*

We should now understand which is the polar foliation and what are its properties.

**D. NCI and Poisson manifolds.** For this, we need a few basic properties of Poisson manifolds. See e.g. [31, 33]) for a general treatment.

A *Poisson manifold* $(P, \{\ ,\ \}_P)$ is a manifold $P$ equipped with Poisson brackets $\{\ ,\ \}_P$ for functions. By means of the Poisson brackets one can associate to every function $f : P \to \mathbb{R}$ a *Hamiltonian vector field* $X_f$, which is defined by the identity $L_{X_f} g = \{f, g\}_P$ for every function $g : P \to \mathbb{R}$.

The set of values of all Hamiltonian vector fields form a subspace $S_x$ of every tangent space $T_xP$, $x \in P$. It is a key fact that the distribution of subspaces $S_x$, $x \in P$, is completely integrable, so that it defines a foliation of $P$, and moreover that the leaves of this foliation carry a symplectic structure; for this reason, they are called the *symplectic leaves* of $P$.

The dimension of the symplectic leaf through a point $x \in P$ is called the *rank* of $P$ at the point $x$; if $(y^1, \ldots, y^m)$ are local coordinates around $x$ ($m = \dim P$), the rank of $P$ at $x$ equals the rank of the matrix $\{y^i, y^j\}_P(x)$.

The (local) *Casimirs* of a Poisson manifold are the functions (defined in open sets) which are constant on the symplectic leaves, or equivalently, they are the functions which Poisson–commute with every other function on $P$.

A map $\phi : P_1 \to P_2$ between two Poisson manifolds $(P_1, \{\ ,\ \}_1)$ and $(P_2, \{\ ,\ \}_2)$ is called a Poisson morphism if $\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$ for all functions $f, g$ defined in (open sets of) $P_2$.

We go back now to our symplectic manifold $(M, \Omega)$ and equip it with the Poisson structure induced by the symplectic form. We then have:[5]

PROPERTY B2. *Assume that a foliation $\mathcal{F}$ is a fibration $i : M \to B$. Then, $\mathcal{F}$ is symplectically complete if and only if there exists a Poisson structure on $B$ with respect to which $i$ is a Poisson morphism. Such a structure, if it exists, is unique.*

*Proof.* Any first integral of $\mathcal{F}$ is of the form $f \circ i$, for some function $f$ defined on an open subset of $B$. By Property B1, if $\mathcal{F}^\perp$ exists, then for any $U \subset B$ and any $f, g \in \mathcal{C}^\infty(U)$ the Poisson bracket $\{f \circ i, g \circ i\}_M$ is a first integral of $\mathcal{F}$. Hence, there exists a function $\mathcal{P}_{f,g}$ in $U$ such that $\mathcal{P}_{f,g} \circ i = \{f \circ i, g \circ i\}_M$. Since the leaves of $\mathcal{F}$ are connected, we can define $\{f, g\}_B = \mathcal{P}_{f,g}$. Conversely, the existence of a Poisson bracket $\{\ ,\ \}_B$ such that $\{f, g\}_B \circ i = \{f \circ i, g \circ i\}_M$ for all $f$ and $g$ implies that the Poisson bracket of any two first integrals of $\mathcal{F}$ is a first integral of $\mathcal{F}$; hence, the conclusion follows from B1. Uniqueness is obvious. ∎

Going back to NCI, we thus see that:

*The base of the fibration by the invariant tori is a Poisson manifold.*

In order to characterize this Poisson structure we investigate its Casimirs and its rank. Note that, by the Lemma, the lifts to $M$ of the Casimirs of $B$ are first integrals of both the fibration by the invariant tori and of its polar, see also Figure 4.

PROPERTY B3. *For a foliation $\mathcal{F}$ which is a fibration $i : M \to B$ and has a polar, the following four conditions are equivalent:*

i. *The leaves of $\mathcal{F}$ are isotropic.*
ii. *The rank of the induced Poisson structure of $B$ is everywhere equal to* $2 \dim B - \dim M$.
iii. *The leaves of $\mathcal{F}$ are generated by the Hamiltonian vector fields of the lifts to $M$ of the Casimirs of $B$.*
iv. *A function is a first integral of $\mathcal{F}^\perp$ if and only if it is the lift to $M$ of a Casimir of $B$.*

[5] This property and most of the following ones are valid under slightly weaker conditions than the foliations are fibrations, see e.g. [31]. However, this generality is irrelevant to our purposes.

*Proof.* The number $n_x$ of independent Casimirs of $B$ at a point $i(x)$ equals the number of independent germs of functions which are simultaneously first integrals of $\mathcal{F}$ and $\mathcal{F}^\perp$. Since the Hamiltonian vector fields of these functions are simultaneously tangent to $\mathcal{F}_x$ (the leaf of $\mathcal{F}$ through $x$) and to $\mathcal{F}_x^\perp$ (the leaf of $\mathcal{F}^\perp$ through $x$), this implies $n_x = \dim(\mathcal{F}_x \cap \mathcal{F}_x^\perp)$. Thus $n_x = \dim \mathcal{F}_x$ if and only if $\mathcal{F}_x \subset \mathcal{F}_x^\perp$. This proves that the first two statements are equivalent. In view of the Lemma, the remaining two are just restatements of the second. ∎

In the case of an MF–integrable system, $M$ has dimension $2d$ and $B$ has dimension $2d-n$. Hence, the isotropy of the invariant tori is equivalent to the fact that the rank of $B$ is $2d-2n$. Thus:

*Condition MF2 is equivalent to the isotropy of the invariant tori.*

**E. The action manifold.** The isotropy of the fibration by the invariant tori implies that its polar is coisotropic and thus each of its leaves is union of invariant tori, and this links the two structures. For simplicity, we consider only the case of a bifibration $A \xleftarrow{c} M \xrightarrow{i} B$. Note that if $B$ has dimension $2d-n$ then $A$ has dimension $n$.

PROPERTY B4. *Consider a bifibration $A \xleftarrow{c} M \xrightarrow{i} B$ and assume that the fibers of $i$ are isotropic. Then, there is a surjective submersion $s : B \to A$ which is such that*
- $c = s \circ i$.
- *The symplectic leaves of $B$ are the fibers of $c$.*

*Proof.* By Property B3.iv the fibers of $c$ are the level sets of the lifts to $M$ of the Casimirs of $B$. For $b \in B$ define $s(b) = c(m)$ where $m$ is any point in $c^{-1}(b)$. This map is a submersion because such are $i$ and $c$. ∎

Thus, for an isotropic–coisotropic bifibration $A \xleftarrow{c} M \xrightarrow{i} B$, the base $A$ of the coisotropic fibration coincides with the space of the leaves of the symplectic foliation of $B$. The overall structure of the bifibration is depicted in Figure 5.a.

**F. The fibers, and local coordinates.** We can now conclude this analysis with the following property, which implies Theorem 1 as a special case:

PROPERTY B5. *Assume that $i : M \to B$ is a symplectically complete fibration with compact, connected, isotropic fibers. Then*
*i. Its fibers are diffeomorphic to $\mathbb{T}^n$, $n = \dim M - \dim B$.*

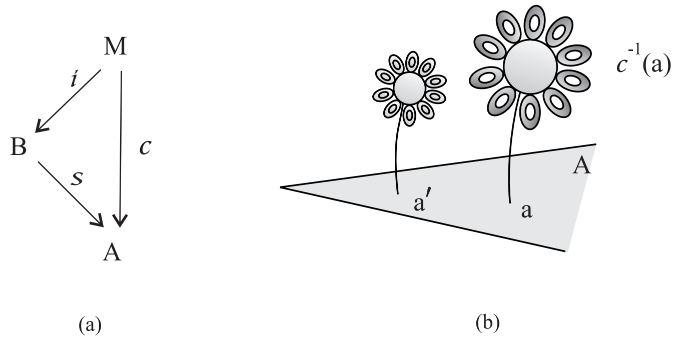



*Figure 5.* The bifibrated structure of the phase space: (a) The three fibrations. (b) Pictorial representation: the meadow is the action manifold $A$, the flowers the coisotropic fibers, their centers the symplectic leaves, the petals the invariant tori.

*ii.* *For each $b \in B$ there is a local trivialization*

$$(a,p,q,\alpha) : i^{-1}(U) \to V \times \mathbb{T}^n$$

*of $i$, where $U$ is a neighbourhood of $b$ in $B$ and $V$ an open set in $\mathbb{R}^{2d-n}$, which satisfies (4).*

*Sketch of the proof.* (i) Since $i$ is a Poisson morphism, the lifts to $M$ of the local Casimirs of B are in involution and hence their Hamiltonian vector fields commute. Therefore, according to Property B3.iii, each fiber of $i$ has $n$ everywhere linearly independent commuting vector fields. A well known argument, which is basic to the proof of the Liouville–Arnold theorem, then implies that the fibers, being compact and connected, are diffeomorphic to $\mathbb{T}^n$.

(ii) The proof of this fact is not very different from the proof of the analogous statement of the Lioville-Arnold theorem. The details can be found e.g. in [22], where an equivalent statement is given. ■

The local 'generalized action–angle coordinates' $(a,p,q,\alpha)$ are related to the bifibrated structure of $A \overset{c}{\leftarrow} M \overset{i}{\rightarrow} B$. The angles $\alpha$ are of course coordinates on the fibers of $i$ and (with a small abuse of notation) the first integrals $(a,p,q)$ can be regarded as local coordinates on the base $B$. Expression (4) of the symplectic structure implies that the $a$'s Poisson commute with all other first integrals of $i$. Hence, by Property B2, *the local actions $a_1, \ldots, a_n$ are local Casimirs of $B$*. In turn, they can also be regarded as local coordinates on the base $A$, which will therefore be called the *action manifold* of the bifibration. Finally, by property B4, $(p,q)$ *are local coordinates on the symplectic leaves of $B$*.

## 4. Superintegrability: geometry and dynamics

**A. Geometry.** If we shift the emphasis from the integrals of motion to the invariant geometric objects it is now very natural to give the following definition of superintegrablity:

DEFINITION. *A Hamiltonian vector field on a symplectic manifold $M$ is said to be* superintegrable *if it is tangent to the fibers of a symplectically complete fibration $i : M \to B$ whose fibers are isotropic, compact and connected.*

By the Lemma, this is equivalent to requiring that the Hamiltonian $H$ of the system is a function of the lifts to $M$ of the Casimirs of $B$. Equivalently, if the polar foliation is a fibration, $H$ is the lift to $M$ of a function $\tilde{H}$ on the action manifold: $H = \tilde{H} \circ c$.

As we have already remarked, usually not all of the phase space of a superintegrable system is fibered by invariant tori. It is therefore tacitly understood that in any practical situation the manifold $M$ is a submanifold of the phase space.

**B. Semilocal, semiglobal, global.** For a bifibration $A \xleftarrow{c} M \xrightarrow{i} B$ there are different notions of globality, which refer to the two fibrations. To distinguish between them we shall say that a property is *semilocal* if it holds in a neighbourhood of each fiber of $i$, *semiglobal* if it holds in a neighbourhood of each fiber of $c$, *global* if it holds in all of $M$. The reason for the term 'semiglobal' is that the fibers of the coisotropic fibration have their own internal structure, with related globality problems.

As a visual help for imagination, the doubly–fibered structure of the phase space of a superintegrable system can be represented as in Figure 5.b, from [21]. Each fiber of the coisotropic fibration is thought of as being a flower, whose center is the symplectic leaf and whose petals are the isotropic tori. Correspondingly, the action space is the meadow on which they grow. Note that the semilocal context pertains to (neighbourhoods of) individual petals while the semiglobal one pertains to (neighbourhoods of) entire flowers.

The distinction between semilocality and semiglobality appears in a number of questions. For instance, it is clear that any MF–integrable system is superintegrable in the sense of Definition 1. Conversely, a submersion $f$ as in Theorem 1 always exists semilocally: the lift to $M$ of any system of local coordinates on $B$ has all the desired properties: that the Poisson brackets of its components satisfy conditions MF1 and MF2 follows from Properties B1 and B3.ii.

Another issue where the distinction between the two local levels may help clarifying things is the relationship between superintegrability and Liouville integrability—a topic which has received a certain consideration, see e.g. [36, 11].

*Semilocally*, every superintegrable system has certainly the maximal number of independent commuting integrals: just take the coordinates $a$ and $p$ within the domain of every local system of generalized action–angle coordinates. If, as in the case of the isotropic oscillator of section 2, the $d-n$ coordinates $q$ can be taken to be angles, then such a domain is even fibered by Liouville tori.

*Semiglobally*, however, this interpretation can hardly be mantained. As in the case of the isotropic oscillator, taking into consideration the structure of even a single, but entire coisotropic fiber, makes evident that the Liouville tori are constructed in a rather arbitrary way: one takes the product of the isotropic tori and, so to say, of pieces of the symplectic leaves. The description of a superintegrable system as Liouville integrable thus hides the relevant geometric, invariant structures.

**C. Global generalized action–angle coordinates.** The existence of generalized action–angle coordinates is guaranteed only *semilocally*, see Property B5. One can of course construct an atlas for $M$ consisting of several of these coordinate systems, but it may happen that there is no such atlas made of a single chart. In fact, it often happens that it is even impossible to cover a single coisotropic fiber with a single system of generalized action–angle coordinates (which are therefore intrinsically semilocal).

A complete study of this problem can be found in [16]. Superintegrability gives this problem some new features and specificities compared to the case of completely integrable systems studied in [17]. For reasons of space we necessarily limit ourselves to a few remarks, and consider only the case of a bifibration. We begin by considering the transition functions between local systems of generalized action–angle coordinates:

PROPOSITION 1. [12, 38] *Consider a superintegrable system with bifibration $A \xleftarrow{c} M \xrightarrow{i} B$. Consider any two systems of local generalized action–angle coordinates $(a,p,q,\alpha)$ and $(a',p',q',\alpha')$ whose domains have connected nonempty intersection. Then*

$$\alpha' = Z\alpha + \mathcal{F}(a,p,q)\,, \qquad a' = Z^{-T}a + z \tag{6}$$

*for some matrix $Z \in SL_{\pm}(\mathbb{Z},n)$, some vector $z \in \mathbb{R}^n$ and some map $\mathcal{F}$, while $p'$ and $q'$ depend on $a$, $p$, $q$.*

*Proof.* Since $a$ and $a'$ are first integrals of the polar foliation, $a' = a'(a)$. Since $(a,p,q)$ and $(a',p',q')$ are first integrals of $i$, $p'$ and $q'$ do not depend on $\alpha$. Symplecticity implies $\sum_j da'_j \wedge d\alpha'_j + \sum_s dp'_s \wedge dq'_s = \sum_j da_j \wedge d\alpha_j + \sum_s dp_s \wedge dq_s$. Equating terms in $da_j \wedge d\alpha_l$ leads to $\frac{\partial \alpha'}{\partial \alpha} = [\frac{\partial a'}{\partial a}]^{-T}$. Thus, $\frac{\partial \alpha'}{\partial \alpha}$ is a function of $a$ alone and hence $\alpha' = Z\,\alpha + \mathcal{F}$ for some matrix $Z(a)$ and some vector $\mathcal{F}(p,q,a)$ which depend smoothly on $(p,q,a)$. Since $\alpha$ and $\alpha'$ are both coordinates on the torus, the matrix $Z(a) \in SL_\pm(n,\mathbb{Z})$ and hence is constant. Recalling that $a' = a'(a)$ we arrive at $a' = Z^{-T}a + z$ with some constant $z$. ■

Equations (6) show that there are several obstructions to the existence of an atlas with generalized action–angle coordinates with all transition functions of actions and angles equal to the identity (which is a little less than having a global system of these coordinates).

*1. Monodromy.* The matrices $Z$ appear in the transition functions of both angles and actions. On the one hand, the possibility of taking all these matrices equal to the identity reflects the fact that the fibration by the invariant tori is a principal $\mathbb{T}^n$–bundle. When this does not happen, the bifibration is said to have (nontrivial) monodromy.

On the other hand, the second group of equations (6) shows that the action manifold $A$ is an affine manifold: it possesses an atlas with affine transition functions which defines an affine (and hence locally flat) connection. Monodromy is precisely the obstruction to the global flatness of this connection [4].

For our purposes, it is important to note that *monodromy always vanishes if the action manifold is simply connected.* Therefore, if a subset $U$ of $A$ is simply connected, then $c^{-1}(U)$ is a principal $\mathbb{T}^n$–bundle and has an atlas made of local generalized action–angle coordinates with all matrices $Z$ equal to the identity. Note that this is true, in particular, for the individual coisotropic fibers.

Monodromy has been identified in several Liouville integrable systems, see e.g. [14], and plays an important role in their quantization, see e.g. [46, 15] and references therein. An example of a superintegrable system with monodromy is given in [18].

*2. Global section for the angles.* The presence of the functions $\mathcal{F}$ in the angles transition functions reflects an obstruction to the topological triviality of the fibration by the invariant tori, that is, the non–existence of a global section. This obstruction is met in very many superintegrable systems in an even stronger form: the global section does not even exist for the restriction of the fibration $i$ to each coisotropic fiber, see e.g. the isotropic oscillator of Section 2, the Euler top below, and ref. [27]. This obstruction is on the contrary rather exceptional among Liouville integrable systems, which lack the internal structure of the coisotropic fibers (to our knowledge, the only known example is the one given in [3]).

*3. The vectors $z$.* The presence of the vectors $z$ in the transition functions of the actions is more closely related to the symplectic structure: one can take all $z$ equal to zero if, e.g., the symplectic two–form of $M$ is exact.

*4. Coordinates on the symplectic leaves.* These inevitably vague considerations do not of course exhaust the problem. In particular, nothing has been said about the possibility of choosing in a global way the coordinates $(p, q)$. Clearly, this is never possible if the symplectic leaves are, e.g., compact. Together with the nontriviality of the fibration by tori of the coisotropic fibers, this is an obstruction to the possibility of constructing semiglobal system of generalized action–angle coordinates.

**D. Dynamics.** The doubly–fibered structure of a superintegrable system is of course linked to its dynamics. The dynamical meaning of the isotropic fibration is obvious, at least if all the integrals of motion have been taken into consideration i.e., more precisely, if the dimension $n$ of the invariant tori is the smallest as possible, so that they form the finest invariant fibration of (some part of) the phase space.[6] From a dynamical point of view, this is certainly the case if a dense subset of these tori is closure of orbits, that is, carry nonresonant motions.

In order to enlighten the dynamical meaning of the coisotropic fibration, we need to say something about the frequencies of motions. This will be important also for perturbation theory, because resonances depened on frequencies.

A simple way of defining frequencies is by means of local coordinates. Within a system of generalized action–angle coordinates, the frequencies are $\omega(a) = \frac{\partial h}{\partial a}(a)$, see (5). According to equations (6), in another system of generalized action–angle coordinates the frequencies of the same motion are $Z\omega(a)$. If we want to compare the frequencies of motions on the tori based on two different points $y, y' \in B$ we can join these two points by a curve in $B$, cover (any lift to $M$ of) this curve with a number of systems of generalized action–angle coordinates, and 'transport' the frequency vector along this chain, multipliying it by the appropriate matrix $Z$ at each change of chart. A little reflection will show that what we are using here is exactly the parallel transport of $A$. The question now is whether the result is independent of the chosen curve. Clearly, this happens if and only if the connection of $A$ is globally flat, that is, there is no monodromy.

[6] The distinction between fibration and foliation here is crucial: the finest invariant foliation of phase space is given by the orbits.

Therefore, if one restricts the analysis to a simply connected subset $U$ of $A$, then there is a way of comparing the frequencies of motions of different tori lying in $c^{-1}(U)$. Note that this always happens for the tori based on an individual coisotropic fiber. This is after all obvious, since any coisotropic fiber can be covered with an atlas with all matrices $Z$ equal to the identity and there is no doubt at all that, with respect to such an atlas, all tori have the same frequencies.

The conclusion is thus that *all motions inside the same coisotropic fiber have the same frequencies.* Thus, the coisotropic fibration has a double origin: geometrically, it is the polar to the invariant tori; dynamically, it is related to the frequencies of motions.

This conclusion can be made even stronger if Kolmogorosv's condition is satisfied, that is, in each chart, the frequency map is a local diffeomorphism:

$$\det \frac{\partial \omega}{\partial a}(a) \neq 0\,. \tag{7}$$

(This condition is chart–independent because the frequencies transform linearly). In such a case there is a one–to–one local correspondence between coisotropic fibers and frequencies. Consequently, the motion is nonresonant in all tori of a dense subset of coisotropic fibers, what implies the uniqueness of isotropic fibration. Note that another case in which the isotropic fibration is unique is when the frequencies are constant, but nonresonant.

## 5. Examples

We provide now a few examples of superintegrable systems and of their bifibrated structure.

*1. Maximally superintegrable systems.* Any Hamiltonian system whose orbits are all periodic and have continuous nonvanishing period is superintegrable. In fact, the continuity and the positiveness of the period ensure that the orbits are the fibers of a fibration [26] and their isotropy is granted by the fact that they are one–dimensional. The coisotropic foliation is given by (the connected components of) the level sets of the Hamiltonian, which is the only Casimir. The action manifold is thus a subset of $\mathbb{R}$. Kepler system (restricted to the subset of negative energy and nonzero angular momentum) and the $d$–dimensional isotropic oscillator (after the equilibrium has been removed) belong to this class. The bifibrated structure is easily worked out, see e.g. [29, 22] for some details.

*2. The point in a central force field.* The details depend of course on the choice of the potential $V(r)$, so for definitness we consider the case of $V(r) = -r^{-k}$, $k > 2$. Let us restrict the analysis to the subset $M$ of the phase space where the energy $H$ is negative and the angular momentum $m$ is nonzero, where all motions are bounded.

The energy–momentum map $(H, m)$ takes values in $B = \mathbb{R}_- \times (\mathbb{R}^3 \setminus \{0\})$. It is very easy to check that this map is a submersion and that the Hamiltonian and the three components $m_x$, $m_y$, $m_z$ of $m$ satisfy conditions MF1 and MF2 with $d = 3$ and $n = 2$. In fact

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & m_z & -m_y \\ 0 & -m_z & 0 & m_x \\ 0 & m_y & -m_x & 0 \end{pmatrix} \tag{8}$$

and this matrix has rank two wherever $m \neq 0$. (Global) Casimirs of $B$ are the Hamiltonian and $G := |m|^2$. Thus, the symplectic leaves of $B$ are the two–dimensional spheres $H = \text{const}$, $|m| = \text{const}$ which, for fixed energy, can be identified with the spheres of constant modulus of the angular momentum.

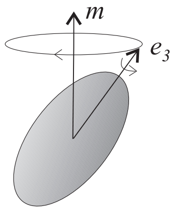
*C. The Euler top.* The Euler top (or Euler–Poinsot system) is the rigid body with a fixed point and no external torques acting on it. It is known from elementary treatments that this system, which has three degrees of freedom, has motions which are (generically) quasi–periodic with two frequencies, see e.g. [2]. The description of the motions and of the geometric structure of the resulting bifibration is a little bit easier for a dynamically symmetric body, for which two of the three moments of inertia relative to the fixed point are equal, so we restrict to this case. (For details and for the general case of a non–symmetric body see [21]).

The configuration of the system can be parametrized by the matrix $R \in \mathrm{SO}(3)$ which carries a given 'spatial' orthonormal frame $\{O; e_x, e_y, e_z\}$ into a given 'body' frame $\{O; e_1, e_2, e_3\}$. We assume that $e_1, e_2, e_3$ are principal axes of inertia of the body, with corresponding moments of inertia $I_1 = I_2$ and $I_3$. After left–trivialization of $T^*\mathrm{SO}(3)$, the phase space can be identified with $\mathrm{SO}(3) \times \mathbb{R}^3 \ni (R, m^b)$, where $m^b = (m_1, m_2, m_3)$ is the body angular momentum vector (that is, the triple of the components of the angular momentum vector $m$ relative to the body frame).

The system has four integrals of motion: the kinetic energy

$$H(R, m^b) = \frac{m_1^2 + m_2^2}{I_1} + \frac{m_3^2}{I_3}$$

and the three components $m_x, m_y, m_z$ of the *spatial* angular momentum vector $m^s := Rm^b$. In the dynamically symmetric case under consideration, also the component of $m$ along the body symmetry axis $e_3$ is a first



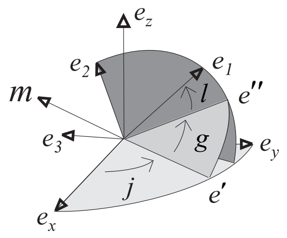




*Figure 6.* (a) Motions of the symmetric Euler top. (b) The local coordinates $(g, l, j)$, part of a local system of generalized action–angle coordinates; $e'$ is the line orthogonal to $e_z$ and $m$, $e''$ the line orthogonal to $m$ and $e_3$.

integral, that we denote either $m_3$ or $L$. Of course these five integrals are not all independent. To describe the fibration by the invariant tori it is convenient to use $L, m_x, m_y, m_z$ (this gives a submersion also on the points where $m_3 = 0$, while $(H, m_x, m_y, m_z)$ does not).

The classical description of motions, which is due to Euler and Poinsot, is obtained by exploiting these integrals, see e.g. [2]: the body rotates with constant angular velocity $(I_3^{-1} - I_1^{-1})m_3$ around its symmetry axis $e_3$, which in turn rotates in space with angular velocity $I_1^{-1}|m^s|$ around the direction of $m$, keeping a constant angle with it, see Figure 6.a, where one can 'see' the two–dimensional invariant tori.

*The Fibration by tori.* It is easy to recognize that the level sets of $(m_3, m^s)$ are all two–dimensional tori, but those with zero angular momentum (equilibria: the fiber is SO(3)) and those with angular momentum aligned with the symmetry axes $e_3$ (steady rotations around the body symmetry axes: the fibers are circles). Thus, if we restrict to the subset $M$ of the phase space were $m \neq 0$ and $m_3 \neq \pm|m|$, we obtain a fibration with fiber $\mathbb{T}^2$ given by

$$(L, m^s) : M \to B\,, \qquad B = \{(L, m^s) \in \mathbb{R} \times \mathbb{R}^3 \;:\; |L| < |m^s|\}\,.$$

*The structure of the base.* The matrix of the Poisson brackets of the functions $L, m_x, m_y, m_z$ has the form (8) and therefore has everywhere rank two in $M$. Two independent (global) Casimirs are $L := m_3$ and $G := |m^s|$. Hence, the symplectic leaves are the two–dimensional spheres $L = \text{const}$, $|m^s| = \text{const}$; for given $L$, they can be identified with the two–dimensional spheres of constant modulus of the angular momentum vector in space.

*The coisotropic foliation.* The lifts to $M$ of the two Casimirs of $B$ form a submersion

$$(L, G) \,:\, M \;\to\; A\,, \qquad A \;=\; \{(L, G) \in \mathbb{R}^2 : |L| < G\}$$

and it is immediate to verify that its fibers are all diffeomorphic to $\mathrm{SO}(3)\times S^1$ (the matrix $R$ is unrestricted while the vector $m$ belongs to the intersection of a sphere and of a plane). Thus, the polar foliation is a fibration with fiber $\mathrm{SO}(3)\times S^1$. Since the base $A$ is contractible, there is no monodromy. The two Casimirs $L$ and $G$ are in fact (global) actions, as can be verified by checking that the flows of their Hamiltonian vector fields are $2\pi$–periodic. Thus, the Hamiltonian is $K\circ c$ with

$$K(G,L) \;=\; I_1^{-1}G^2 + (I_3^{-1}-I_1^{-1})L^2\,. \tag{9}$$

*The internal structure of the coisotropic fibers.* It remains to be described the structure of the fibration by the invariant tori of the coisotropic fibers. Since the base of this fibration is a symplectic leaf, this is a fibration $\mathrm{SO}(3)\times S^1 \to S^2$ which is given by $(R,m^b)\mapsto (m_3, Rm^b)$. It is easy to see that this map contains a Hopf fibration $\mathrm{SO}(3)\to S^2$ and is thus nontrivial. Hence, the system does not possess global angles.

*Local generalized action–angle coordinates.* Local systems of generalized action–angle coordinates are provided for instance by the so called Andoyer–Deprit coordinates $(G,L,J,g,l,j)$, where $J$ is a component of the angular momentum vector in space, conventionally identified with the $z$–one, and $g,l,j$ are defined as in Figure 6.b, see e.g. [21] and references therein. These coordinates have singularities on the boundary of $M$ (that is where $|m_3| = |m|$ and the two–dimensional tori collapse to circles or equilibria) and at the poles $|m_z| = |m|$ of each symplectic leaf. In fact, $(J,j)$ are cylindric–like coordinates on the symplectic leaves. The angles are $g$ and $l$. If necessary, an atlas for $M$ can be constructed with two of these coordinate systems, relative to two spatial frames with non–parallel $z$–axis.

*4. Nekhoroshev superintegrability criterion.* In [38], Nekhoroshev considered a submersion $f := (f_1,\ldots,f_{2d-n}) : M \to \mathbb{R}^{2d-n}$ such that its first $n$ components are mutually in involution with all others:

$$\{f_j,f_s\} \;=\; 0\,, \qquad j=1,\ldots,n, \quad s=1,\ldots,2d-n\,.$$

By the Lemma and by Property B3, this condition implies that the foliations by the (connected components of the) level sets of $f$ and of $g=(f_1,\ldots,f_n)$ are polar to each other and that the former is isotropic.

Given a superintegrable system, functions $f_1,\ldots,f_{2d-n}$ as in Nekhoroshev's hypotheses always exist *semilocally*: just take $n$ Casimirs and a set of local coordinates on the symplectic leaves. However, it may happen that no set of such functions exists *semiglobally*: after all, the functions $f_{n+1},\ldots,f_{2d-n}$ are local coordinates on the symplectic leaves and there can be obstructions to their global existence.

*5. Symmetry and superintegrability.* Superintegrablity is often linked to the existence of a 'large' symmetry group, the fibration by the invariant tori being given either by the momentum map or by the energy–momentum map. We give a very quick look at this important topic, presuming some knowledge of group actions and momentum maps (see e.g. [31, 33]).

Assume there is a Hamiltonian action of a Lie group $G$ on a symplectic manifold $M$, with coadjoint equivariant momentum map $J : M \to \mathcal{G}^*$. As we have already remarked in Section 3.B, the level sets of $J$ form a symplectically complete foliation, with polar given by the group orbits. Therefore, if $J : M \to J(M)$ is a fibration with compact and connected fibers, what remains to be checked in order to conclude that its level sets are the invariant tori of a superintegrable system is isotropy. Since $J$ is a Poisson morphism from the symplectic manifold to $\mathcal{G}^*$, by Property B3.ii the isotropy of its level sets is a condition on the dimension of the coadjoint orbits $\mathcal{O}_\mu$. Precisely, as a simple computation shows, the fibers of $J$ are isotropic if and only if

$$\dim \mathcal{O}_\mu \;=\; 2d \;-\; 2n \qquad \forall \mu \in J(M)\,. \tag{10}$$

(This is equivalent to the zero–dimensionality of the reduced phase spaces, if they exist). In this situation, the symplectic leaves of the base $J(M)$ can be identified with the (connected components of the) coadjoint orbits. An example of this situation is offered by the dynamically symmetric Euler top, with symmetry group $\mathrm{SO}(3) \times S^1$.

If the Hamiltonian $H$ is independent of the momentum map $J$, one should consider the energy–momentum map $(H, J)$, which defines an invariant foliation which is finer than $J$. This case can be reduced to the previous one by observing that $(H, J)$ is the momentum map of the action of the direct product $\mathbb{R} \times G$ given by $(t, g).\, x = \Phi_t^H(g.\, x)$, where $\Phi_t^H$ is the map at time $t$ of the flow of the Hamiltonian vector field of $H$. Thus, if the energy momentum map is a fibration with compact connected fibers, then the isotropy condition becomes

$$\dim \mathcal{O}_\mu \;=\; 2 \;+\; 2 \dim J(M) \;-\; \dim M\,.$$

If the reduced phase spaces exist, then this condition is equivalent to the fact that they are two–dimensional.

*6. Other cases.* Even though superintegrability is rather general, and to some extent even typical of Hamiltonian systems, it does not exhaust all possible cases. There exist indeed Hamiltonian systems on non–exact symplectic manifolds which have all motions linear on coisotropic tori [40, 41, 42] as well as on tori of any dimension $1 < n < 2d$ which are neither isotropic (if $n \le d$) nor coisotropic (if $n \ge d$), see [10, 21, 25].

## 6. Perturbations of superintegrable systems

The bifibrated geometry of superintegrable systems plays an important role for the study of small (Hamiltonian) perturbations of them. Concluding this review, we thus give a very quick look at this topic. Since perturbation theory is rather technical, we shall necessarily restrict ourselves to lay down a few ideas.

We consider a superintegrable system with bifibration $A \xleftarrow{c} M \xrightarrow{i} B$ and Hamiltonian $K \circ c$ for some function $K : A \to \mathbb{R}$, and add to it a small perturbation $\epsilon F$, where $F : M \to \mathbb{R}$ is some function and $\epsilon$ a small (positive) parameter. The object of interest is the long–term dynamics of the perturbed system with Hamiltonian $K \circ c + \epsilon F$.

**A. Perturbation theory is semiglobal.** Hamiltonian perturbation theory is based on the construction of normal forms which are

- Adapted to the (local) resonance properties of the unperturbed motions.
- Defined in sets which, for the time–scale of interest, are invariant under the perturbed flow.

For a superintegrable system, resonances—like frequencies of motion—are a property of the coisotropic fibers. Therefore, the natural sets where normal forms should be constructed are of the form $c^{-1}(U)$, where $U$ is a subset of $A$ characterized by certain resonance properties.

A key observation, however, is that any perturbation result will show that the actions remain nearly constant for the time–scale of interest. Therefore, for $\epsilon \to 0$, the set $U$ can be taken as small as one wants—hence contractible. This has the important consequence that monodromy, even if present, can be ignored in the perturbation study.

On the other hand, it is in general[7] impossible to confine the other integrals of motion, namely the projection of motions on the symplectic leaves. Therefore, the normal forms should be constructed *semiglobally*, in sets of the form $c^{-1}(U)$, not just semilocally in neighbourhoods of the individual tori, as is done when working in coordinates.

The existence of semiglobal normal forms has been proved in [37, 13] for the maximal superintegrable case, in which the unperturbed dynamics is periodic and there are no resonances. The general case, which is significantly different because the treatment of resonances requires an essential use of the bifibrated structure, has been studied in [20] (see also [9], where however only averaging on periodic orbits is considered). As there shown, thanks to the fact that monodromy is not

[7] This might be possible for nonresonant motions (e.g., in the KAM case and, if the symplectic leaves are two–dimensional, also in the Nekhoroshev case) but in general not for all motions, see [20] for a detailed discussion.

seen at the semiglobal level, normal forms can always be constructed semiglobally—even if the isotropic fibration does not have a global section: this is important, because this semiglobal obstruction to the triviality of the isotropic fibration is quite frequent, if not even typical, among superintegrable systems.

**B. Averages and normal forms.** The possibility of defining averages and constructing normal forms on a bifibration with trivial mondodromy rests on the fact that each fiber of the coisotropic fibration is a $\mathbb{T}^n$–principal bundle. This ensures that there is a (semiglobal) Fourier series defined on it. The simplest way of looking at this matter is probably via local coordinates.

Consider the bifibration $U \xleftarrow{c} C_U \xrightarrow{i} S_U$ where $U \subset A$ is simply connected, $C_U := c^{-1}(U) \subset M$ and $S_U := \overset{-1}{s}(U) \subset B$. Choose an atlas with generalized action–angle coordinates for $C_U$ which has all matrices $Z$ equal to the identity. If $F : C_U \to \mathbb{R}$ is any function, then its local representative $f$ in any chart of this atlas can be expanded in Fourier series $f = \sum_{\nu \in \mathbb{Z}^n} f_\nu$, where $f_\nu(a,p,q,\alpha) = \hat{f}_\nu(a,p,q) e^{i\nu\cdot\alpha}$, $\hat{f}_\nu = (2\pi)^{-n} \int_{\mathbb{T}^n} f e^{-i\nu\cdot\alpha} d\alpha$ being the Fourier components. The local representative $f'$ of $F$ in another chart can also be expanded in Fourier series $f' = \sum_{\nu \in \mathbb{Z}^n} f'_\nu$. If the two chart domains intersect, then using the transition functions (6) it is immediate to verify that, for each $\nu \in \mathbb{Z}^n$, the two 'harmonics' $f_\nu$ and $f'_\nu$ match: that is, they are the local representatives of a function $F_\nu : C_U \to \mathbb{R}$. (Note that this does not happen for the Fourier components). Hence, there is a Fourier series $F = \sum_{\nu \in \mathbb{Z}^n} F_\nu$ for functions in $C_U$. Only the labelling $\nu$ of the 'harmonics' $F_\nu$ depends on the choice of the atlas.

Thanks to this fact, we can average functions $F : C_U \to \mathbb{R}$. Given any subset $L$ of $\mathbb{Z}^n$, the $L$–average of $F$ is

$$\Pi_L F \;:=\; \sum_{\nu \in L} F_\nu \;:\; C_U \to \mathbb{R}\,.$$

Here too, only the labelling by $L$ depends on the choice of the atlas. In particular:

- $\Pi_{\{0\}} F$ is the lift of a function defined on the base $S_U$ and its restriction to each torus is the average of $F$ over that torus.
- For a lattice $L$ of positive dimension, instead, $\Pi_L F$ is a function on $C_U$ which, on each torus, coincides with the average of $F$ over the subtorus determined by $L$.

From here on, the construction of semiglobal normal forms proceeds exactly as the standard semilocal construction using coordinates—just observing that all operations can be performed semiglobally [20]. For instance, the ultraviolet cutoff is defined in terms of Fourier series.

The symplectic diffeomorphisms leading to the normal forms can be constructed by means of the so called Lie method, that is, as the time–one map of the Hamiltonian flow of a suitable 'generating' function; if this function is semiglobally defined, and if the symplectic leaves are for instance compact so that no escape is possible within each coisotropic fiber, then this diffeomorphism is semiglobally defined, too. The generating function is the solution of a homological equation, which is formulated in terms of Poisson brackets, and can be represented by a Fourier series which, thanks to the fact that the frequencies of motions are constant on the coisotropic fibers, is also semiglobal.

Estimates, which control the small denominators (and the remainders) can be performed using local charts [20]. Ref. [9] even succeeds in doing estimates without the use of local coordinates on the symplectic leaves, under the hypothesis that the coisotropic fibers are the orbits of a compact Lie group.

For details, and for a discussion of the role of monodromy, see [20].

**C. KAM theory.** Typically, KAM theory proves the existence of a Cantor set of invariant tori of the perturbed system, which carry motions with highly nonresonant (e.g., diophantine) frequencies. A necessary condition for the KAM construction is a nondegenracy condition, which expresses the fact that the frequencies of the unperturbed system change 'as much as possible' from torus to torus. This is clearly incompatible with superintegrability, where all tori in the same coisotropic fibers have the same frequencies and the best one can require is that the frequency map has rank $n$, see (7). The standard way out is to try to use (part of) the perturbation to construct a nondegenerate Liouville integrable system to which the KAM construction will then be applied.

One begins by considering only the highly nonresonant coisotropic fibers and constructs a nonresonant semiglobal normal form

$$K \circ c + \epsilon N \circ i + \epsilon^2 F_2$$

in a small neighbourhood of each of them. Since the function $N$ is defined on the base $B$, by Property B2 its Hamiltonian vector field is tangent to the symplectic leaves. If $N$ defines a nondegenerate Liouville integrable system on each symplectic leaf, then (large parts of) the symplectic leaves are foliated into invariant tori of dimension $d - n$. In such a case, the truncated Hamiltonian $K \circ c + \epsilon N \circ i$ is Liouville integrable and nondegenerate and one can apply KAM theorem to it. (In fact, some further normalization may be necessary to make the perturbation small enough, see [34]). The conclusion is that there is a Cantor set of large measure of coisotropic fibers on each of which a Cantor set of large measure of invariant $d$–dimensional tori is created.

This structure has been studied in [28] and in [34] for perturbations of the Euler top. In this case the symplectic leaves are two–dimensional, so that integrability of the normal form is not an issue.

**D. Nekhoroshev theory.** Nekhoroshev theory [39, 43, 32, 5] has a double content. First of all, it provides a sharp confinement of the actions in all motions for a very long time–scales. Second, the (resonant) normal forms constructed within the proof give detailed informations on the motions: Nekhoroshev theory is to a large extent a theory of motions in resonance.

A semiglobal formulation of Nekhoroshev theorem for superintegrable systems was obtained in [20] by coupling the techniques of the standard semilocal proof with the semiglobal techniques described in subsection B. Another proof was given in [9] using Lochak's simultaneous approximation technique [32], which however is not suitable for a detailed analysis of motions in resonance because it avoids the construction of all resonant normal forms.

A possible statement is that if the unperturbed Hamiltonian $K \circ c$ has some convexity property (see below) and if the symplectic leaves are (e.g.) compact, then *any* motion $t \mapsto z_t$ of the perturbed system satisfies

$$|c(z_t) - c(z_0)| < \epsilon^{\beta} \qquad \text{for times} \quad |t| < T_\epsilon := \exp\left(\epsilon^{-\gamma}\right)$$

for some positive constants $\beta$ and $\gamma$, e.g. $\beta = \gamma = (2n)^{-1}$. 'Convexity' of $K : A \to \mathbb{R}$ means that the Hessian of its local representative in any local system of action coordinates is positive definite. (This condition is well defined because of the affine structure of $A$).

Based on Nekhoroshev theorem, one obtains the following picture of the perturbed motions, which is valid for the whole time–scale $T_\epsilon$:

- Since the actions stay nearly constant, the system remains very near the coisotropic fiber from which it starts.
- For the same reason, the angles advance with nearly uniform speed: the projection of the motion on the isotropic tori is approximately quasi–periodic, with nearly constant frequencies which are approximately equal to the unperturbed ones.

What remains to be described is the projection of the motion on the symplectic leaves. Since this motion is due to the perturbation $\epsilon F$, it is of course slow:

$$\tfrac{d}{dt} i(z_t) \; = \; \mathcal{O}(\epsilon)\,.$$

This slow motion may however become important on the very long time–scale $T_\epsilon$. A better understanding of it is possible, at least in simple cases, by analysing the (resonant) normal forms. We discuss this point with reference to a specific example, from [7].

*Figure 7.* Spatial motion of the direction of the angular momentum vector in a perturbed Euler top, from [6]. Left: three different nonresonant orbits. Right: a single orbit in the resonance $(1,-1)$ (relative to the actions $(G,L)$, see the expression (9) of the unperturbed Hamiltonian). All orbits are relative to $\epsilon = 10^{-5}$, $I_1 = I_2 = 2$, $I_3 = 1$. Only one hemishpere of the unit sphere is shown, in stereographic projection.

**E. The Euler top.** For a small perturbation of the Euler top, the previous results tell that, for a very long time–scale, the perturbed motion is approximately an Euler motion around the instantaneous direction of the angular momentum vector in space, which however moves (slowly) in space. As we now discuss, the properties of this slow motion, which are clearly important for applications, are deeply influenced by the resonance properties of the unperturbed motions. Since the invariant tori of the Euler top are two–dimensional, there are only two possibilities: either a coisotropic fiber is nonresonant or it is resonant with one–dimensional resonant lattice.

*Nonresonant coisoptropic fibers.* The nonresonant normal forms are of the type $K \circ c + \epsilon N_\epsilon \circ i + \exp(\epsilon^{-1/4}) F_\epsilon$. Neglecting the exponentially small remainder one is left with the Hamiltonian $K \circ c + \epsilon N_\epsilon \circ i$. Since $K \circ c$ is constant on the coisotropic fibers, the motion on the symplectic leaves is described by $N_\epsilon$. This is a Hamiltonian system with one degree of freedom. Hence, conservation of energy implies that, for times $|t| < T_\epsilon$, the direction of $m^s$ moves along the level curves of the restriction of $N_\epsilon$ to the symplectic leaves, which are almost all closed curves. The symplectic diffeomorphism leading to the normal form and the exponentially small remainder add a small noise around these essentially ordered motions.

*Resonant coisotropic fibers.* If the resonant lattice is $L$, then the normal form is $K \circ c + \epsilon N_\epsilon + \exp(\epsilon^{-1/4}) F_\epsilon$ where $N_\epsilon = \Pi_L N_\epsilon$. Neglecting the remainder, one is left with the Hamiltonian $K \circ c + \epsilon N_\epsilon$, which has two degrees of freedom. Typically, such a system is not integrable and might have chaotic dynamics. In particular, the projection of motions on the symplectic leaves need not stay close to the level curves of a regular function and could in principle erratically wander through extended, two–dimensional regions of the spheres.

These deeply different behaviours, which are present no matter how small is $\epsilon$, are actually found in numerical integrations of sample perturbations of the Euler top, see Figure 7 (from [6]). For details and extensions of these results see [7, 8] and references therein.

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