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NEKHOROSHEV–STABILITY OF L4 AND L5 IN THE SPATIAL RESTRICTED THREE–BODY PROBLEM

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We show that L_4 and L_5 in the spatial restricted circular three–body problem are Nekhoroshev–stable for all but a few values of the reduced mass up to the Routh critical value. This result is based on two extensions of previous results on Nekhoroshev–stability of elliptic equilibria, namely to the case of "directional quasi–convexity", a notion introduced here, and to a (non-convex) steep case. We verify that the hypotheses are satisfied for L_4 and L_5 by means of numerically constructed Birkhoff normal forms.

Dedicated to Professor Jürgen Moser on occasion of his 70^{th} birthday

1. Introduction

A. The Lagrangian equilateral points L_4 and L_5 of the restricted circular three–body problem are well known to be elliptic for all values of the reduced mass μ below Routh's critical mass

$$\mu_R = \frac{1}{2}\Big(1 - \frac{\sqrt{69}}{9}\Big) \approx 0.0385209 . \tag{1.1}$$

Their Lyapunov stability, however, is in general unknown: the Hamiltonian has a saddle point and is not a Lyapunov function. The problem has been completely answered only in the planar case, with the essential aid of KAM theory [16, 6, 18, 1, 2, 27, 20].

In this paper we consider the spatial case. Since the system has three degrees of freedom, KAM theory does not provide Lyapunov stability and Arnold diffusion is possible. Our aim is to study the so–called Nekhoroshev–stability of L_4 and L_5, namely stability over exponentially long times.

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Specifically, we ask whether any motion which starts sufficiently close to the equilibrium remains close to it for times which grow exponentially fast with the inverse of the initial distance: namely, whether for any small $\varepsilon > 0$

$$d(0) \leqslant \varepsilon \implies d(t) \leqslant \varepsilon^a \quad \text{for} \quad |t| \leqslant \exp \varepsilon^{-b} \tag{1.2}$$

for some positive constants a and b, d denoting some distance from the equilibrium. This problem has been studied so far only under some arithmetic conditions on the frequencies, which were typically assumed to satisfy a diophantine condition, see e. g. [12], while our approach is independent of these properties and aims to apply to all values of the reduced mass μ up to μ_R.

Nekhoroshev–stability of elliptic equilibria of analytic Hamiltonian systems without any arithmetic assumption on the frequencies has been proven only recently, under the hypotheses that the fourth order Birkhoff normal form of the Hamiltonian exists (which requires there are no resonances up to order four) and satisfies a convexity or more generally quasi–convexity assumption [9, 24, 14]. The precise formulation of these hypotheses, which are typical of Nekhoroshev theory, will be given in Section 2 below. However, as we shall see in Section 3, the Hamiltonian of the spatial restricted problem does not satisfy the quasi–convexity hypothesis at L_4 and L_5 for any value of the reduced mass below the critical value μ_R. Therefore, our study will rest in a crucial way on two extensions of the above results:

(i) The first extension replaces the quasi–convexity assumption with a weakened requirement which is specific for the case of an equilibrium point, and has no analogue in the general Nekhoroshev theorem for the stability of the actions in nearly integrable systems. This property will be called *directional quasi–convexity* because it consists in testing quasi–convexity not in the whole plane of fast drift, but only in its intersection with the action space of an elliptic equilibrium, that is, the "first octant" where all actions are nonnegative.

(ii) The second extension relaxes instead quasi–convexity to a simple "steepness" condition on the 3–jet of the sixth order Birkhoff normal form, much in the line of Nekhoroshev's original statement of the theorem. For simplicity we consider here only the application to the case of a system with three degrees of freedom; in this case it is necessary that the Birkhoff normal form can be constructed up to order eight at least.

Precise statements of these results will be given in Section 2. We shall not provide, however, detailed proofs (which will be anyhow reported in [15]). In fact the proof of (i) is a very minor modification of the method used in [9], and we think it is enough to indicate the necessary changes. The proof of (ii) uses instead the methods of [14], and we plan to publish it elsewhere in a more general framework.

Let us add here that, in our opinion, directional quasi–convexity is a natural hypothesis for the case of elliptic equilibria, which plays the same role that quasi–convexity plays in Nekhoroshev theorem, and allows one to obtain the best stability estimates. We expect that this hypothesis will play an important role also in the study of other systems, noticeably in the so–called Riemann ellipsoids studied in [10].

B. In Section 3 we apply the above results to L_4 and L_5. To this end, we need to construct the Birkhoff normal forms of order four and eight, depending on the case, and test that the hypotheses are satisfied. We do this for a number of values of μ, by numerically determining the coefficients of the Taylor series of the Hamiltonian and of the Birkhoff normal forms. Although we procede numerically, and hence we are not able to state a theorem about the Nekhoroshev–stability of L_4 and L_5, in our opinion the numerical results provide a clear evidence. (In order to obtain rigorous statements, one should construct symbolically the Birkhoff normal forms).

In order to apply result (i), we need the Birkhoff normal form of order four. As it turns out, the Birkhoff normal form of order four for L_4 and L_5 exists for all values of $\mu < \mu_R$ but at the $(1,2,0)$ and $(1,3,0)$ resonances, i. e. for

$$\mu \neq \mu_{(1,2,0)} \approx 0.0242939 \qquad \text{and} \qquad \mu \neq \mu_{(1,3,0)} \approx 0.0135160\,.$$

Moreover, as we shall see, the system turns out to be directionally quasi–convex for all $\mu < \mu_R$ but in

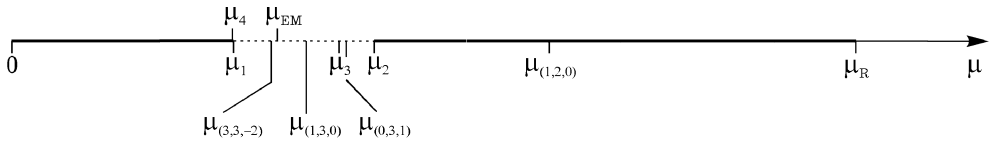

Fig. 1. Nekhoroshev–stability of L_4 and L_5, as a function of μ. Solid line: directionally quasi–convex case. Dotted line: steep case.

a relatively small interval $\mu_1 \leqslant \mu \leqslant \mu_2$, with

$$\mu_1 \approx 0.0109137\,, \qquad \mu_2 \approx 0.0163768\,. \tag{1.3}$$

Note that only one of the above two resonances falls inside the range of directional quasi–convexity $(0,\mu_1)\bigcup(\mu_2,\mu_R)$, see Figure 1. Therefore, by applying result (i), we will obtain the stability estimates (1.2), with $a = b = 1/3$, for all values of μ in this range but the resonant value $\mu_{(1,2,0)}$.

For $\mu \in [\mu_1,\mu_2]$ we need extension (ii). This requires testing the steepness condition on the sixth order Birkhoff normal form and checking the possibility of constructing the Birkhoff normal form of order eight. As we shall see, the latter can be constructed for all values of μ in the interval $[\mu_1,\mu_2]$ but for three isolated values, namely the already met $\mu_{(1,3,0)}$ and the two values

$$\mu_{(0,3,1)} \approx 0.0148525\,, \qquad \mu_{(3,3,-2)} \approx 0.0115649$$

which correspond to the resonances $(0,3,1)$ and $(3,3,-2)$, respectively. Concerning instead the steepness condition, this is satisfied for all $\mu \in [\mu_0,\mu_1]$ but for

$$\mu_3 \approx 0.0147808\,.$$

The three resonances and μ_3 must be excluded from our consideration. For all the remaining values of μ in the interval $[\mu_1,\mu_2]$ we obtain instead Nekhoroshev–stability, although the stability time is significantly worse than the previous one, namely we obtain $b = 1/20$. (The other constant instead improves, namely we obtain $a = 1$, a fact which will be commented later).

The whole situation is depicted in Figure 1. It is rather likely that at $\mu = \mu_3$ the Hamiltonian possesses some steepness property weaker than the one considered above in (ii), in which case one might study it on the basis of a further extension of Nekhoroshev's estimates. However, we will not pursue this goal here.

C. Concluding this Introduction it is worth observing that, although we may conclude that L_4 and L_5 are Nekhoroshev–stable for all but (possibly) five values of the reduced mass μ below the critical one, the estimated stability times are highly nonuniform in μ, depending strongly on whether the system is directionally quasi–convex or only steep. Moreover, a worsening of the stability estimates is expected for μ near μ_3 and near the resonances. In this connection, we should mention that there is a further value $\mu_4 \approx 0.0108854$, outside the interval $[\mu_1,\mu_2]$ but very close to μ_1, where the considered steepness condition is violated; its presence is expected to produce a worsening of the stability estimates for μ slightly larger than μ_1.

The problem clearly poses whether all these differences in the stability properties are real, or just the product of pessimistic estimates. This is in our opinion a very interesting question, and we shall come back on it in the Conclusions.

As a final remark, we notice that the reduced mass μ_{EM} of the Earth–Moon system ($\mu_{EM} \approx 0.0121506$) lies in the interval $[\mu_1,\mu_2]$ in which the system is not directionally quasi–convex, and moreover that it is rather close to $\mu_{(3,3,-2)}$; so, according to our analysis, somehow weaker stability properties might be found for L_4 and L_5 of the Earth–Moon system.

2. Nekhoroshev–stability of elliptic equilibria

A. Review of previous results. Consider an analytic Hamiltonian system with n degrees of freedom in a neighbourhood of an elliptic equilibrium with frequency vector $\omega = (\omega_1, \dots, \omega_n)$. If ω does not satisfy any resonance condition up to a certain order $N \geqslant 2$, namely if one has

$$\omega \cdot \nu \neq 0 \qquad \text{for all } \nu \in \mathbb{Z}^n \setminus \{0\}\,, \quad |\nu| \leqslant N$$

(where $|\nu| = \sum_j |\nu_j|$),[1] then one can put the Hamiltonian into Birkhoff normal form up to order N. This means that there exists a system of canonical coordinates (p, q) near the equilibrium in which the Hamiltonian has the form

$$h \;=\; k \circ I \;+\; f^{(N+1)}\,, \tag{2.4}$$

where $f^{(N+1)}$ is a convergent power series in (p, q) which begins with terms of degree $N+1$, $I = (I_1, \dots, I_n)$ are the n action functions

$$I_j(p,q) \;=\; \frac{p_j^2 + q_j^2}{2}\,, \tag{2.5}$$

and

$$k(I) \;=\; k_2(I) \;+\; \sum_{j=2}^{[N/2]} k_{2j}(I)\,, \qquad k_2(I) \;=\; \omega \cdot I\,; \tag{2.6}$$

$k_{2j}(I)$ is here a homogeneous polynomial of degree j in $I_1, \dots, I_n$, and [ ] denotes the integer part. Thus, $k_4(I)$ is a quadratic form, which we will always write

$$k_4(I) \;=\; \frac{1}{2} I \cdot AI\,,$$

A being an $n \times n$ symmetric matrix.[2]

Hamiltonian (2.4) is a small perturbation of the integrable Hamiltonian k and, as such, can be studied by the methods of Hamiltonian perturbation theory. We assume $N \geqslant 4$, so that k is anisochronous and might be nondegenerate (for the application of KAM theory) and/or have some quasi–convexity or steepness properties (for the application of Nekhoroshev theory). In his celebrated 1977 article [23], Nekhoroshev conjectured that, if the function $k(I)$ is "steep", then for any small $\varepsilon > 0$

$$|I(0)| \leqslant \varepsilon \implies |I(t)| \;\leqslant\; \varepsilon^a \quad \text{for} \quad |t| \leqslant \exp \varepsilon^{-b} \tag{2.7}$$

for some positive constants a and b, where | | denotes any norm. This result does not directly follow from Nekhoroshev theorem about the variation of the actions in nearly integrable systems, because of the singularity of the action–angle coordinates at the equilibrium. However, this technical difficulty has been recently overcome in [5] and in the already mentioned papers [9, 24, 14] where, as we now explain, the case of convex or quasi–convex k was addressed. (In fact, all the quoted references treated explicitly the convex case, but the extension to quasi–convexity is immediate).

We say that the function $k(I)$ as in (2.6) is

- *convex at $I = 0$*, if the quadratic form $k_4(I)$ is either positive or negative definite;

[1] More precisely, one has to exclude only those resonances for which there is a nonzero harmonic in the spectrum of a certain normal form Hamiltonian, see below.

[2] In the notation employed here, k is a polynomial in the n variables $(I_1, \dots, I_n)$, not a function on the phase space, as are instead h and the composition $k \circ I$. Nevertheless, for simplicity of notation, we shall frequently not distinguish between the polynomial k and the function $k \circ I$.

- *quasi–convex at* $I = 0$, if the restriction of the quadratic form $k_4(I)$ to the plane orthogonal to ω is either positive or negative definite; equivalently, if

$$\omega \cdot I \;=\; 0\,, \quad k_4(I) \;=\; 0 \quad\Longrightarrow\quad I = 0\,. \tag{2.8}$$

The results of [9] show that either of these hypotheses imply estimates (2.7) with $a = b = 1/n$, or else[3] with $a = 1/2$, $b = 1/(2n)$. At a certain extent, the approach of ref. [9] was oriented to keep N as small as possible, i. e. $N = 4$. Instead, ref. [24, 14] take advantage of the possibility of carrying the Birkhoff normalizations up to higher orders, and simultaneously obtain a better confinement of the actions, with $a = 1$; both references get $b = \frac{N-3}{4n}$ (see also [26]).

Before describing the extensions of these results which will be used for L_4 and L_5, it is worth discussing, at least shortly, the role of quasi–convexity in providing the confinement of the actions. To this end, we refer to Hamiltonian (2.4) with $N = 4$, so that $k(I) = \omega \cdot I + k_4(I)$. Within the proof of Nekhoroshev's theorem one first divides the action space into suitable "resonant blocks" characterized by different resonance properties and then constructs in each of them a canonical transformation which puts the Hamiltonian into a suitable normal form, adapted to the local resonant lattice Λ, up to terms of exponentially high order. The normal form Hamiltonian has the form

$$\omega \cdot I \;+\; k_4(I) \;+\; g_\Lambda \;+\; \text{exponentially small terms}\,, \tag{2.9}$$

where the "normalized" part g_Λ is small and has nonzero "harmonics" only in the lattice Λ (see Section 3B below for formal definitions). This implies that, in any motion of the truncated Hamiltonian $\omega \cdot I + k_4 + g_\Lambda$, the movement of the actions is flattened on the "plane of fast drift", i. e., on the hyperplane through the initial datum generated by the resonant lattice. Hence, the problem reduces to finding a mechanism of confinement of the actions along such a hyperplane.

In the simple situation in which the unperturbed Hamiltonian is quasi–convex, as was realized by Gallavotti [11, 4], the confinement is provided just by conservation of energy. Indeed, quasi–convexity exactly means that the restriction of the quadratic form $k_4(I)$ to the hyperplane orthogonal to the frequency vector (and thus to the plane of fast drift, which is contained in it) has a quadratic minimum or maximum at the point where the resonance is exact. This situation is illustrated in Figure 2a, which shows the level curves of the restriction of $k_4(I)$ to the fast drift plane; the figure refers to $n = 3$ and to a double resonance, so the fast drift plane is two–dimensional; in higher dimensions one has nevertheless an elliptic structure around the point of exact resonance. Taking into account the fact that $\omega \cdot I$ vanishes on the fast drift plane, and using the smallness of the other terms of the normal form Hamiltonian (2.9), it immediately follows that, in the quasi–convex case, energy conservation does provide confinement in the plane of fast drift, up to exponentially long times on which the effects of the remainder becomes important.

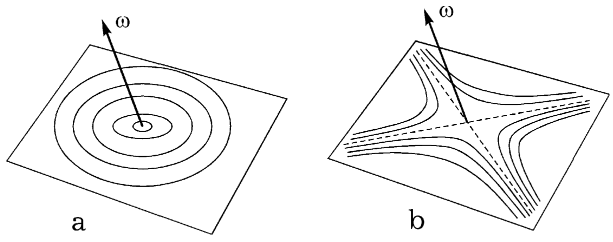

Fig. 2. (a) Quasi convexity and (b) lack of quasi–convexity.

In lack of quasi–convexity, instead, there is a hyperbolic (or mixed) structure in place of the elliptic one, see Figure 2b, and in particular there are lines in the plane of fast drift on which the quadratic form $k_4(I)$ vanishes (the two asymptots of the quadratic form, in dimension two). As is clear, in such a situation conservation of energy does not provide any confinement.

[3]More precisely, one finds $a = \frac{1+m}{n+m}$, $b = \frac{1}{n+m}$ for any $m \geqslant 0$. This fact extends also to Theorem 1 below, even though we do not explicitely state it there.

B. Directional quasi–convexity. The previous discussion should indicate rather clearly that the requirement of quasi–convexity is excessively restrictive in order to use energy conservation for confining motions near an elliptic equilibrium. In fact, since all actions are nonnegative for real motions, the action space of an elliptic equilibrium is the "first octant" $\{I \in \mathbb{R}^n \ : \ I_j \geqslant 0\}$: if a motion arrives at the border of the octant, where some actions vanish, it bounces back. Hence, it is necessary to control the actions only in those directions of the fast drift plane which point inside the first octant. This simple consideration, which is illustrated in Figure 3, leads to the following

Definition 1. We say that the Hamiltonian $k(I)$ as in (2.6) is *directionally quasi–convex at* $I = 0$ if the restriction of the quadratic form $k_4(I)$ to the plane orthogonal to ω does not vanish in the first octant:

$$\omega \cdot I \ = \ 0 \,, \quad k_4(I) \ = \ 0 \,, \quad I_1, \dots , I_n \geqslant 0 \quad \Longrightarrow \quad I = 0 \,. \tag{2.10}$$

The following statement should be rather evident on the basis of the mentioned result for the quasi–convex case and of the previous considerations:

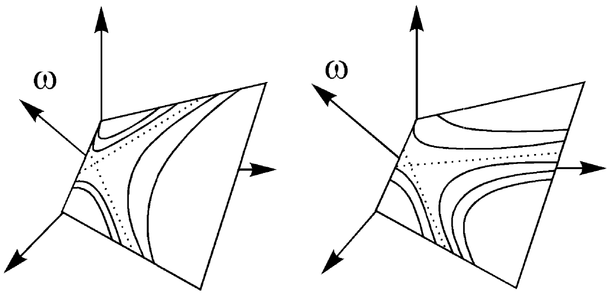

Fig. 3. (a) Directional quasi–convexity and (b) lack of directional quasi–convexity.

Theorem 1. *Assume that $N \geqslant 4$ and that $k(I)$ is directionally quasi–convex. Then, for ε sufficiently small, any motion with initial datum such that $|I(0)| \leqslant \varepsilon$ satisfies estimates (2.7) with $a = b = 1/n$ as well as with $a = 1/2$, $b = 1/(2n)$.*

Proof.

The proof of Theorem 1 differs only in one point from the proof of the analog result for the convex case reported in [9], namely in the use of the normal forms (2.9) for obtaining the confinement of the actions. Specifically, one needs to modify the proof of Lemma 4 of [9] so as to account (i) for quasi–convexity rather than convexity and (ii) for the fact that here we have only directional quasi–convexity. The first modification is obtained with the standard argument for the quasi–convex case, as e.g. in [25]: one observes that hypothesis (2.10) implies that there exist positive constants m and l such that

$$I_1, \dots , I_n \geqslant 0 \quad \Longrightarrow \quad \text{either} \quad |I \cdot AI| \geqslant m|I|^2 \quad \text{or} \quad |\omega \cdot I| \geqslant l|I|$$

(this is assured by continuity and by the compactness of the intersection of the first octant with the unit sphere; of course, these constants depend crucially on the angle formed by the asymptots of the quadratic form $k_4(I)$ with the first octant); one then uses either of these two inequalities, the former in conjuction with conservation of energy, so as to confine the motion of the (transformed) actions I'_t. The second modification is very simple: the only difference with respect to the usual procedure (which was used in particular in [9]), is that one should produce estimates for I'_t rather than for $I'_t - I'_0$, because that is the vector which belongs to the first octant.

Filling in all details, one easily verifies that Lemma 4 of ref. [9] carries over to the present case, even though with different constants in the estimates for $|I'_t|$. The proof of Theorem 1 is then concluded by proceeding exactly as in [9]; of course, some of the constants have to be recomputed, but it is immediate to check that the relevant constants a and b retain their values. (For the details, see [15]). ■

C. A steepness condition. As explained in the Introduction, we also need to weaken the quasi–convexity assumption to a steepness assumption. Steepness is a technical property, which has been shown by Nekhoroshev to be generic. Its precise definition, actually somehow complex, can be found in [23], and is not necessary to recall it here. Let us only mention that steepness depends on $3n-3$ constants, the most important of which are the $n-1$ "indices of steepness" $\alpha_1, \dots, \alpha_{n-1}$, which directly determine the exponents a and b of Nekhoroshev theorem: the smaller the α's, the better the exponents. The simplest case of steepness is quasi–convexity, which has the best possible values of the indices of steepness, all equal to one. (Directionally quasi–convex Hamiltonians are instead not steep).

The next steep case considered by Nekhoroshev, after quasi–convexity, is a 3–jet non–degeneracy condition:

Definition 2. A function $k(I) = \omega\cdot I + k_4(I) + k_6(I) + \dots$ as in (2.6) is said to be 3–jet nondegenerate at $I = 0$ if

$$\begin{aligned} \omega\cdot I \;=\; 0\,, \quad k_4(I) \;=\; 0\,, \\ k_6(I) \;=\; 0 \quad\Longrightarrow\quad I = 0\,. \end{aligned} \tag{2.11}$$

As noticed in [23], 3–jet nondegeneracy implies steepness, even though the corresponding indices of steepness are not evaluated there.

Here we focus the attention on the case of L_4 and L_5. Hence, we consider only the case of a system with three degrees of freedom and moreover, in order to simplify the proof and to obtain good values for the two exponents a and b, we further supplement the above condition with the requirement, satisfied by L_4 and L_5, that the restriction of the matrix A to the plane orthogonal to ω is nonsingular:

$$\omega\cdot I \;=\; 0\,, \quad AI \;=\; 0 \quad\Longrightarrow\quad I = 0\,. \tag{2.12}$$

It is not difficult to prove that for $n = 3$, under these two assumptions together, the steepness indices are $\alpha_1 = 2$, $\alpha_2 = 1$. Our result for this steep case is the following:

Theorem 2. *Consider the Hamiltonian (2.4) for $n = 3$, with $k(I)$ as in (2.6) and $N \geqslant 8$. Assume that k is 3–jet nondegenerate and that it satisfies (2.12). Then, estimates (2.7) hold for any ε small enough, with $a = 1$ and $b = \min\left(\frac{N-7}{20}, \frac{N+1}{36}\right)$.*

As already mentioned, the proof of this theorem will be given elsewhere, in a broader context (see also [15]). Let us just remark that the confinement of motions in the fast drift plane is not obtained by means of energy conservation, but by means of the "trapping mechanism" originally used by Nekhoroshev.

Remark 1. So far, Nekhoroshev theory has been investigated mostly in the quasi–convex case (see for instance [4, 17, 25]). In particular, the estimates for the constants entering Nekhoroshev theorem in the steep case have never been revisited after Nekhoroshev's original work, all of the effort having been devoted to the quasi–convex case. As far as we know, after Nekhoroshev's papers, the importance of going beyond the quasi–convex case and consider steepness properties was remarked in [13, 21], in connection with the dynamics of the asteroids in the Main Belt. The same problem was also met in the already quoted analysis of the Riemann ellipsoids [10]. These two cases, and now the case of the triangular Lagrangian equilibria, suggest that the steep case is also important and worth of attention.

3. The case of L_4 and L_5

A. The Hamiltonian. The Hamiltonian of the spatial restricted circular three–body problem, in baricentric rotating coordinates $\tilde Q = (\tilde Q_1, \tilde Q_2, \tilde Q_3)$ with the two primaries on the $\tilde Q_1$ axis, is

$$\tilde H(\tilde Q, \tilde P; \mu) \;=\; \frac{1}{2}\,\|\tilde P\|^2 - \tilde Q_1\tilde P_2 + \tilde Q_2\tilde P_1 - \frac{1-\mu}{\|\tilde Q - (-\mu,0,0)\|} - \frac{\mu}{\|\tilde Q - (1-\mu,0,0)\|}\,,$$

where $\tilde{P} = (\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)$ are the conjugate momenta, $\mu \leqslant 1/2$ is the reduced mass, and $\| \ \|$ denotes the Euclidean norm (see e. g. [19]). The equilibrium L_4, to which we restrict our considerations, is the point

$$(\tilde{Q}^{L_4}, \tilde{P}^{L_4}) \ = \ \Big(\frac{1-2\mu}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, 0, \frac{1-2\mu}{2}, 0\Big) \, .$$

In the translated coordinates $(Q,P) = \big(\tilde{Q} - \tilde{Q}^{L_4}, \tilde{P} - \tilde{P}^{L_4}\big)$ the Hamiltonian becomes

$$H(Q,P;\mu) \ = \ \frac{1}{2}\|P\|^2 \ - \ Q_1P_2 \ + \ Q_2P_1 \ - \ \frac{1-2\mu}{2}Q_1 \ - \ \frac{\sqrt{3}}{2}Q_2 \ - \ \frac{1-\mu}{\rho_+(Q)} \ - \ \frac{\mu}{\rho_-(Q)} \, ,$$

where $\rho_\pm(Q) \ = \ \big(\|Q\|^2 \pm Q_1 + \sqrt{3}Q_2 + 1\big)^{1/2}$.

The eigenvalues of the linearization at the equilibrium $Q = P = 0$ of the Hamiltonian vector field of H are $\pm i f_+(\mu)$, $\pm i f_-(\mu)$, and $\pm i$, where

$$f_\pm(\mu) \ = \ \sqrt{\frac{1 \pm f(\mu)}{2}} \, , \qquad f(\mu) \ = \ \sqrt{1 - 27\mu + 27\mu^2} \, .$$

All eigenvalues are purely imaginary if $f(\mu) > 0$, i. e., if $\mu < \mu_R$. Since for $0 < \mu < \mu_R$ the three frequencies $f_+(\mu)$, $f_-(\mu)$, and 1 are distinct, one can perform a linear canonical change of coordinates $X \mapsto x = M(\mu)X$, with $X = (Q,P)$ and $x = (q,p)$, which "diagonalizes" the quadratic terms of H. One can verify that the matrix $M(\mu)$ can be written as

$$M(\mu) = \begin{pmatrix} -3kf_+g_+ & (7+2f)g_- & 0 & (7-2f)g_+ & -3kf_-g_- & 0 \\ (2f-3)f_+g_+ & -3kg_- & 0 & -3kg_+ & -(2f+3)f_-g_- & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -4f_+g_+ & 3kf_+^2g_- & 0 & 3kf_-^2g_+ & -4f_-g_- & 0 \\ 0 & (4ff_-g_-)^{-1} & 0 & (4ff_+g_+)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ,$$

where $k \ = \ \sqrt{3}(1 - 2\mu)$ and $g_\pm \ = \ 1/\sqrt{ff_\pm(9k^2 \mp 10f - 1)}$. If we introduce the action functions $I_j = (p_j^2 + q_j^2)/2$, the quadratic part $k_2 \circ I$ of the Hamiltonian $H(M(\mu)x)$ is

$$k_2(I;\mu) \ = \ \omega(\mu) \cdot I \, , \qquad \omega(\mu) \ = \ \big(f_+(\mu), -f_-(\mu), 1\big) \, .$$

The construction of the Birkhoff normal forms will be done, as usual, by means of the complex coordinates

$$(z,w) \ = \ \Big(\frac{p - iq}{i\sqrt{2}} \, , \, \frac{p + iq}{\sqrt{2}}\Big)$$

(z are the coordinates and w the momenta). Thus, denoting by C the matrix such that $x = C\xi$, where $\xi = (z,w)$, we shall work with the Hamiltonian

$$h(\xi;\mu) \ = \ H\big(M(\mu)C\xi;\mu\big). \tag{3.13}$$

We now proceed as follows. For fixed μ, we evaluate the matrix $M(\mu)$ and numerically compute the coefficients of the Taylor expansion of the Hamiltonian $h(z,w;\mu)$; we then construct the Birkhoff normal forms. All computations have been done using the software package *Mathematica* on a Personal Computer.

B. Notations for Birkhoff normal forms. We shall use the following notations and terminology: we write

$$f(z,w) \ = \ \sum_{m,n \in \mathbb{N}^3} f_{nm} z^n w^m$$

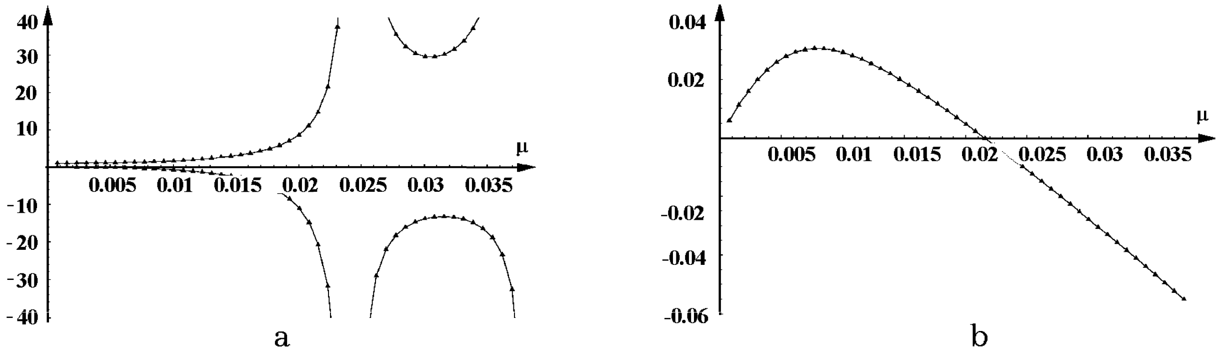

Fig. 4. Lack of convexity of L_4: (a) The minimum and maximum and (b) the intermediate eigenvalues of $A(\mu)$.

the Taylor expansion of a function f, where $\mathbb{N}$ is the set of all nonnegative integers. For any given integer vector $\nu \in \mathbb{Z}^3$, the ν–th *harmonic* of f is defined by

$$\langle f\rangle_\nu(z,w) \;=\; \sum_{\substack{m,n\in\mathbb{N}^3\\ n-m=\nu}} f_{nm} z^n w^m\,.$$

The *average* of f is its harmonic $\langle f\rangle_0$. The *spectrum* of f is $\mathfrak{f} = \{\nu\in\mathbb{Z}^3 : \langle f\rangle_\nu \neq 0\}$. We shall also make use of the linear operator $S(\,\cdot\,)$ defined by

$$S(f) \;=\; \sum_{\nu\in\mathfrak{f}\setminus\{0\}} \frac{\langle f\rangle_\nu}{i\omega(\mu)\cdot\nu}\,,$$

so as to write the solution of the "homological equation" $\{k_2,\chi\} = f - \langle f\rangle_0$ in the form $\chi = S(f)$.

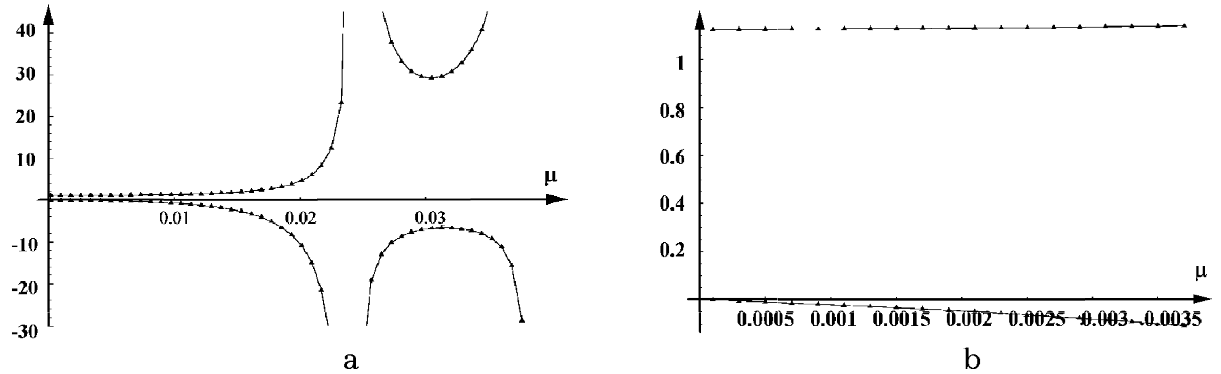

Fig. 5. Lack of quasi–convexity of L_4: (a) The eigenvalues of the restriction of $A(\mu)$ to the plane orthogonal to $\omega(\mu)$. (b) Enlargement of the interval of (a) where directional quasi–convexity is violated.

We shall construct the Birkhoff normal forms by means of the standard Lie method. We denote by Φ^χ the map at time one of the flow of a function χ; then, the Lie transform generated by χ of a function f is

$$f\circ\Phi^\chi \;=\; \sum_{j=0}^{\infty}\frac{1}{j!}L_\chi^j f\,,$$

where $L_\chi f = \{\chi, f\}$ denotes the Poisson brackets (with sign convention $\{f,g\} = \dfrac{\partial f}{\partial w}\dfrac{\partial g}{\partial z} - \dfrac{\partial g}{\partial w}\dfrac{\partial f}{\partial z}$).

For given $N = 4$ or 8, and for fixed μ, we write

$$h^{(2)} \;=\; k_2\circ I \;+\; h_3^{(2)} \;+\; \ldots \;+\; h_N^{(2)} + O(N+1)\,, \qquad k_2(I) = \omega\cdot I\,,$$

the Taylor expansion of h around the origin $z = w = 0$, with $h_l^{(2)}$ homogeneous polynomial in (z, w) of degree l. The Birkhoff normal forms $h^{(j)}$ of orders $j = 3, \ldots, N$ will be written

$$h^{(j)} \;=\; k_2 \circ I \;+\; \ldots \;+\; k_j \circ I \;+\; h_{j+1}^{(j)} \;+\; \ldots \;+\; h_N^{(j)} \;+\; O(N+1)\,,$$

with $h_l^{(j)}$ homogeneous polynomials in (z, w) of order l and k_l homogeneous polynomials in I of order $[l/2]$; $k_l = 0$ for odd l. The normal forms $h^{(j)}$ are constructed according to the iterative scheme

$$\chi_j \;=\; S(h_{j-1}^{(j)})\,, \qquad h^{(j)} \;=\; h^{(j-1)} \circ \Phi^{\chi_j}\,, \qquad j = 3\ldots, N\,.$$

At each step, only the terms of order N and lower are evaluated.

Note that the Birkhoff normal form of order j is here constructed by composing $j-2$ Lie transforms; closed or iterative formulas for the polynomials $h_l^{(j)}$ are easily worked out (see e. g. [7, 8]); however, since we are working with small N, we find more convenient to produce explicitly all the terms we need.

C. Fourth order Birkhoff normal form, convexity, quasi–convexity. When $N = 4$, the above scheme gives the Birkhoff normal form of order four, i. e. $h^{(4)} = h^{(2)} \circ \Phi^{\chi_3} \circ \Phi^{\chi_4}$; its expression is

$$h^{(4)} \;=\; k_2 \;+\; k_4 \;+\; O(5)\,,$$

with

$$k_4(I) \;=\; \langle h_4^{(3)} \rangle_0\,, \qquad h_4^{(3)} \;=\; h_4^{(2)} + \frac{1}{2} L_{\chi_3} h_3^{(2)}\,, \qquad \chi_3 \;=\; S(h_3^{(2)})\,. \tag{3.14}$$

To construct $h^{(4)}$, two Lie transforms are necessary, with generators χ_3 and

$$\chi_4 \;=\; S(h_4^{(3)})\,.$$

Thus, the fourth–order Birkhoff normal form exists for all those $\mu < \mu_R$ for which $\omega(\mu)$ does not have any resonance up to order three in the spectrum of $h_3^{(2)}$, and up to order four in the spectrum of $h_4^{(3)}$.

We now turn to the numerical results. For $0 < \mu < \mu_R$ the frequency vector $\omega(\mu)$ has exactly two resonances of order three, namely $(0, 2, 1)$ for $\mu \approx 0.0285955$ and $(1, 2, 0)$ for $\mu = \mu_{(1,2,0)}$. However, within the precision of our numerical computations we find $\langle h_3^{(2)} \rangle_{(0,2,1)} = 0$ (for all μ). Similarly, there are exactly two resonances of order four for $\omega(\mu)$, namely $(0, 3, 1)$ for $\mu \approx 0.0148525$ and $(1, 3, 0)$ for $\mu = \mu_{(1,3,0)}$, but $\langle h_4^{(3)} \rangle_{(0,3,1)} = 0$ for any μ. We thus conclude that the Birkhoff normal form of order four exists for all $0 < \mu < \mu_R$, but $\mu_{(1,2,0)}$ and $\mu_{(1,3,0)}$.

Let us write $k_4(I) = \frac{1}{2} I \cdot A(\mu) I$. Figure 4 reports the three eigenvalues of the matrix $A(\mu)$ as a function of μ: since two of them have everywhere different signs, the integrable part of $h^{(4)}$ is not convex for any $\mu < \mu_R$. (The fact that one of the two eigenvalues in Figure 4a is negative even for μ close to zero does not emerge clearly from the picture but is confirmed by a finer numerical analysis. Let us also note that the divergences at $\mu_{(1,2,0)}$ and at μ_R reflect the resonance and the loss of ellipticity, respectively; the resonance $(1, 3, 0)$ has no effect because, being of order four, it does not affect k_4 but only the terms of order five).

Let us now denote by $B(\mu)$ a 2×2 restriction of the matrix $A(\mu)$ to the plane $\Pi(\mu)$ orthogonal to $\omega(\mu)$. (Precisely, the matrix $B(\mu)$ is computed as follows: once the matrix $A(\mu)$ is evaluated, we rotate the coordinates I so as to carry the vector $\omega(\mu)$ onto the first coordinate axis, and then we take the appropriate block of the rotated matrix). Figure 5 reports the two eigenvalues of $B(\mu)$: since they have different signs, the system is not quasi–convex for any $\mu < \mu_R$.

Remark 2. As Figure 4 shows, the matrix $A(\mu)$ is nonsingular for all $\mu < \mu_R$ but for $\mu = \mu_S \approx 0.0215391$. Thus, by invoking KAM theorem for elliptic equilibria [3, 22] we may conclude that, for all values of $\mu < \mu_R$ but μ_S, $\mu_{(1,2,0)}$ and $\mu_{(1,3,0)}$, there is a set of large measure of quasi–periodic motions in any sufficiently small neighbourhood of L_4 and L_5.

D. Directional quasi–convexity. We now investigate directional quasi–convexity. Since, as is seen on Figure 5, the matrix $B(\mu)$ is everywhere nonsingular, there are exactly two lines in the plane $\Pi(\mu)$ on which the quadratic form $k_4(I)$ vanishes. Let $b_+(\mu) > 0 > b_-(\mu)$ be the two eigenvalues of $B(\mu)$ and let $w_\pm(\mu) \in \Pi(\mu)$ be the corresponding eigenvectors of unit norm. The two lines in $\Pi(\mu)$ on which $k_4(I)$ vanishes are parallel to the two vectors

$$u_\pm(\mu) \;=\; \sqrt{|b_-(\mu)|}\, w_+(\mu) \;\pm\; \sqrt{b_+(\mu)}\, w_-(\mu)\,.$$

The three components of the vectors $u_-(\mu)$ and $u_+(\mu)$ are reported in Figures 6a and 6b, respectively, as functions of μ: two components of each vector have different signs, so that the system is directionally quasi–convex, for all μ outside the interval $\mu_1 < \mu < \mu_2$, with μ_1 and μ_2 as in (1.3); the components of $u_-(\mu)$ for μ around such an interval, where directional quasi–convexity is violated, are also reported in the enlarged Figure 6c.

In conclusion, as already announced in the Introduction, we can apply Theorem 1 for all values of $\mu < \mu_1$ and $\mu > \mu_2$, but $\mu_{(1,2,0)}$.

E. Steepness. For $\mu_1 < \mu < \mu_2$ we resort to Theorem 2. To this end, we first check the possibility of constructing the Birkhoff normal form of order eight, namely

$$h^{(8)} \;=\; \omega \cdot I \;+\; k_4 \;+\; k_6 \;+\; k_8 \;+\; O(9)\,.$$

For this, we need computing the six generating functions

$$\chi_j \;=\; S(h_j^{(j-1)})\,, \qquad j = 3, \ldots, 8\,,$$

where, as one readily verifies, $h_4^{(3)}$ and k_4 are as in (3.14) and

$$\begin{aligned}
h_5^{(4)} \;&=\; h_5^{(2)} + L_3 h_4^{(2)} + \frac{1}{3} L_3^2 h_3^{(2)}, \\
h_6^{(5)} \;&=\; h_6^{(3)} + \frac{1}{2} L_4 (h_4^{(3)} + k_4)\,, \\
&\quad \text{with} \quad h_6^{(3)} \;=\; h_6^{(2)} + L_3 h_5^{(2)} + \frac{1}{2} L_3^2 h_4^{(2)} + \frac{1}{8} L_3^3 h_3^{(2)}, \\
h_7^{(6)} \;&=\; h_7^{(2)} + L_3 h_6^{(2)} + \frac{1}{2} L_3^2 h_5^{(2)} + \frac{1}{3!} L_3^3 h_4^{(2)} + \frac{1}{30} L_3^4 h_3^{(2)} + L_4 h_5^{(4)} + L_5 k_4, \\
h_8^{(7)} \;&=\; h_8^{(2)} + L_3 h_7^{(2)} + \frac{1}{2} L_3^2 h_6^{(2)} + \frac{1}{3!} L_3^3 h_5^{(2)} + \frac{1}{4!} L_3^4 h_4^{(2)} + \frac{1}{144} L_3^5 h_3^{(2)} + \\
&\quad + L_4 h_6^{(3)} + \frac{1}{3} L_4^2 (h_4^{(3)} + \frac{1}{2} k_4) + \frac{1}{2} L_5 h_5^{(4)} + L_6 k_4
\end{aligned}$$

(L_j stays here for L_{χ_j}). Moreover,

$$k_6 \;=\; \langle h_6^{(5)} \rangle_0\,,$$

while the exact expression of k_8 is of no interest here (only $h_8^{(7)}$ matters for the resonances).

For each $j = 3, \ldots, 8$ we must exclude all values of μ for which there is a resonance of order j in the spectrum of $h_j^{(j-1)}$. One verifies that, for $\mu < \mu_R$, there are 48 distinct resonances of $\omega(\mu)$ of order $\leqslant 8$, but only five of them are encountered for $\mu_1 < \mu < \mu_2$, namely:

$$\begin{aligned}
(0,3,1) \quad &\text{at} \quad \mu = \mu_{(0,3,1)} \approx 0.0148525, \\
(1,3,0) \quad &\text{at} \quad \mu = \mu_{(1,3,0)} \approx 0.0135160, \\
(2,3,-1) \quad &\text{at} \quad \mu = \mu_{(2,3,-1)} \approx 0.0124467, \\
(1,6,1) \quad &\text{at} \quad \mu = \mu_{(1,6,1)} \approx 0.0141441, \\
(3,3,-2) \quad &\text{at} \quad \mu = \mu_{(3,3,-2)} \approx 0.0115649\,.
\end{aligned}$$

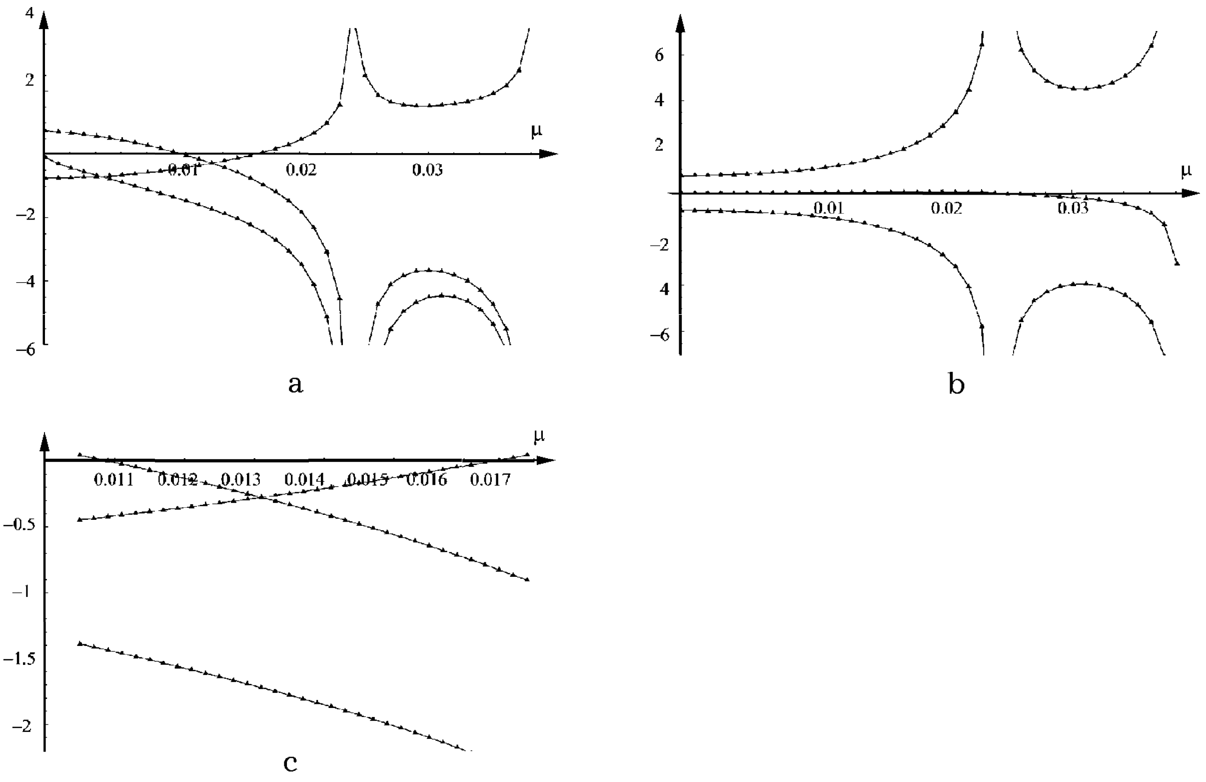

Fig. 6. Directional quasi–convexity of L_4: (a) and (b) the components of the two asymptots of the restriction of $A(\mu)$ to the plane orthogonal to $\omega(\mu)$; (c) enlargement of the interval from Figure (a) where directional quasi–convexity is violated.

However, $h_4^{(3)}$ and $h_6^{(5)}$ do not have the harmonics $(0,3,1)$ and $(2,3,-1)$ respectively, while $h_8^{(7)}$ does not have the harmonic $(1,6,1)$ (for any μ); $h_8^{(7)}$ has instead the harmonics $(0,6,2)$ (equivalent to $(0,3,1)$) and $(3,3,-2)$. In conclusion, only the three resonant values $\mu_{(1,3,0)}$, $\mu_{(0,3,1)}$ and $\mu_{(3,3,-2)}$ must be excluded.

In order to apply Theorem 2, we first note that the fourth–order Birkhoff normal form satisfies condition (2.12) for all μ under consideration, since the eigenvalues of the previously introduced matrix $B(\mu)$ are everywhere nonzero (see Figure 5). Thus, we only need to check that the sixth–order Birkhoff normal form is 3–jet nondegenerate at $I = 0$. On account of equation (2.11), this amounts to verify that $k_6(I)$ does not vanish on the two asymptots $u_\pm$ of the restriction of k_4 to the plane $\Pi(\mu)$ orthogonal to ω. Figure 7 reports (as is graphically more convenient) the two quantities $\sqrt{|k_6(u_\pm)|}$ for $\mu_1 \leqslant \mu \leqslant \mu_2$: one of them is always nonzero, while the other vanishes at just one point in the interval $[\mu_1, \mu_2]$, namely the value denoted μ_3 in the Introduction. As mentioned in the Introduction, such a quantity also vanishes at the point $\mu_4 \approx 0.0108854$, which is however outside the interval $[\mu_1, \mu_2]$. Thus, Theorem 2 can be applied for all μ in the interval (μ_1, μ_2) but μ_3, $\mu_{(3,3,-2)}$, $\mu_{(1,3,0)}$, and $\mu_{(0,3,1)}$.

4. Conclusions

A. The use of either Theorem 1 or Theorem 2, depending on the geometric properties of h, allowed us to obtain definite results about L_4 and L_5 for all values of μ up to μ_R, but a few isolated values. The situation, however, is still far from being conclusive, and a number of problems remain open and warrant future consideration.

First of all, the two theorems are rather different, and it is disappointing that we could not unify them into a single proposition. As a matter of fact, Theorem 1 is based on a rather simple geometric

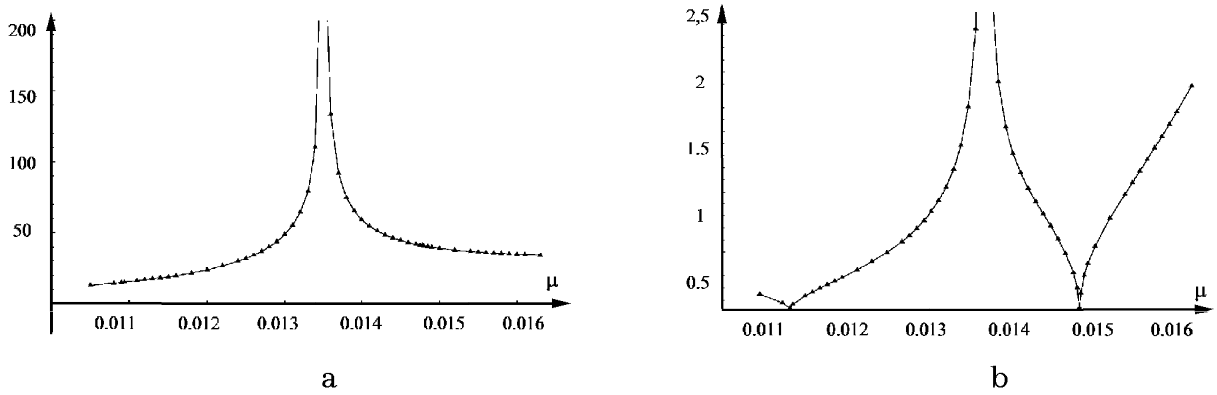

Fig. 7. Steepness of L_4: the two quantities $\sqrt{|k_6(u_\pm(\mu))|}$.

construction (after [9]) in which for each value of ε (and thus of the cut–off) one looks only at the resonance properties of the frequency ω of the equilibrium point; correspondingly, the neighbourhood of the equilibrium point under study is entirely contained into a single resonant region. The advantage is that, in the case of directional quasi–convexity, a single resonant normal form contains the whole hyperbolic structure represented in Figure 3a, and energy conservation can be invoked. As a result one can successfully exploit the directional quasi–convexity of the system, and good estimates on the stability time are found already for small N. A minor disadvantage is that (as explained in [9]) the estimates on the stability time do not improve for larger N, and moreover the confinement of the actions near the equilibrium, which is determined by the constant a, is somehow poor. A major disadvantage is that the scheme of proof does not work in the steep case: indeed the use of a single resonant region prevents the use of the “trapping” confinement mechanism proposed by Nekhoroshev, which appears to be essential in the steep case.

Theorem 2, instead, is based on a more sophisticated geometric construction (after [14]) and on the use of the resonance properties of the frequency $\omega + AI_0$ at the initial point, rather than the frequency ω at the equilibrium point. The scheme of proof is closer to Nekhoroshev's original one, so it works in the steep case, and moreover one can profit of the possibly better arithmetic properties of ω (absence of resonances up to a larger N) to improve the estimates of the constant b. The counterpart is that the estimate on b is poor unless N is very large, and foremost, that this method does not seem to be suitable to exploit directional quasi–convexity.

Whether the two schemes can be in some way unified, and in particular, whether it is possible to obtain estimates improving with N under the assumption of directional quasi–convexity, is a question that, at the moment, we are not able to answer. We hope to be able to come back on this point in the future.

B. A second comment that we wish to make on our results is that, in order to understand the long time behavior of a system in the neighborhood of an elliptic equilibrium point, the geometric properties of the Hamiltonian are not less important than the analytic and the arithmetic properties. Of course this is not a problem specific to L_4 and L_5, nor to the elliptic equilibria, but seems to be a very essential feature of Nekhoroshev's theory, which is not commonly stressed because, as already remarked, much of the effort has been devoted, up to now, to the convex or quasi–convex case. In fact, Nekhoroshev himself explicitely conjectured [23] that the steepness indices influence the stability properties of a system in an important way (the smaller the indices, the better the stability), and suggested to test this conjecture by numerical methods.

Now, L_4 and L_5 seem to us particularly promising for such a test. Indeed, the system is physically interesting, has the minimal number of degrees of freedom, and its steepness properties change with the parameter μ. As a matter of fact, even though the system is not steep in the range of directional quasi–convexity, the stability times are there the same as if it was quasi–convex and thus had $\alpha_1 = \alpha_2 = 1$.

On the contrary, the steepness indices α_1 and α_2 are equal to two and one, respectively, for $\mu_1 < \mu < \mu_2$ but for $\mu = \mu_3$, and are larger, if any, for $\mu = \mu_3$. As is well known, measuring numerically the rate of diffusion in general is difficult: nevertheless, if the conjecture is correct, it should be possible to detect, in some way, the difference in behavior due to different steepness properties. We plan to deserve some attention to this analysis in a near future.

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УСТОЙЧИВОСТЬ ПО НЕХОРОШЕВУ L4 И L5 В ПРОСТРАНСТВЕННОЙ ОГРАНИЧЕННОЙ ЗАДАЧЕ ТРЕХ ТЕЛ

Поступила в редакцию 7 октября 1998 г.

Показано, что L_4 и L_5 в пространственной ограниченной круговой задаче трех тел устойчивы по Нехорошеву для почти всех значений приведенных масс до критического значения Раусса. Данный результат основан на двух расширениях предыдущих результатов, касающихся устойчивости по Нехорошеву эллиптических равновесий, а именно, случае «направленной квази-выпуклости» (понятие, введенное здесь) и (невыпуклом) случае крутизны. При помощи численного построения нормальных форм Биркгоффа проверяются выдвинутые предположения для L_4 и L_5.