The reaction–annihilator distribution and the nonholonomic Noether theorem for lifted actions

Dedicated to Richard Cushman on the occasion of his sixtyfifth birthday

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Abstract

We consider nonholonomic systems with linear, time–independent constraints subject to positional conservative active forces. We identify a distribution on the configuration manifold, that we call the reaction–annihilator distribution $\mathcal{R}^\circ$, the fibers of which are the annihilators of the set of all values taken by the reaction forces on the fibers of the constraint distribution. We show that this distribution, which can be effectively computed in specific cases, plays a central role in the study of first integrals linear in the velocities of this class of nonholonomic systems. In particular we prove that, if the Lagrangian is invariant under (the lift of) a group action in the configuration manifold, then an infinitesimal generator of this action has a conserved momentum if and only if it is a section of the distribution $\mathcal{R}^\circ$. Since the fibers of $\mathcal{R}^\circ$ contain those of the constraint distribution, this version of the nonholonomic Noether theorem accounts for more conserved momenta than what was known so far. Some examples are given.

Keywords: Nonholonomic systems, First integrals, First integrals linear in the velocities, Symmetries of nonholonomic systems, Reaction forces, Noether theorem, Gauge momenta.

1 Introduction

An obstruction to extending Noether theorem to nonholonomic systems is related to the fact that, in general, not all the components of the momentum map of a lifted action which preserves the Lagrangian are conserved quantities. In this article we consider nonholonomic systems with linear, time–independent constraints subject to positional conservative forces, so that the Lagrangian is time independent and has the form kinetic energy minus potential energy. If the Lagrangian is invariant under (the tangent lift of) a group action on the configuration manifold, then a standard formulation of a 'nonholonomic Noether theorem' states that those infinitesimal generators of the group action which are sections of the constraint distribution $\mathcal{D}$, namely the 'horizontal

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symmetries', have a conserved momentum [2, 20, 3, 8, 11, 14, 21, 7, 15]. Horizontality of the infinitesimal generator is however only a sufficient condition. A necessary and sufficient condition follows from the work of [13, 24]: a component of the momentum map is conserved if and only if the tangent lift of the orthogonal projection of the infinitesimal generator onto the constraint distribution preserves the Lagrangian. (Here 'preserves' means 'infinitesimally preserves on the constraint manifold' and the orthogonality is relative to the kinetic energy metric). The aim of this paper is that of providing a necessary and sufficient condition on the infinitesimal generator, rather than on its projection onto the constraint distribution, and to show that, in nonholonomic systems, the class of conserved components of the momentum map is not restricted to horizontal symmetries but can be (even significantly) larger than that.

Specifically, we shall prove the following version of a 'nonoholonomic Noether theorem for lifted actions': *if the Lagrangian is invariant under a lifted group action, then an infinitesimal generator has a conserved momentum if and only if it is a section of a certain distribution* $\mathcal{R}^\circ$ on the configuration manifold. The fibers of the distribution $\mathcal{R}^\circ$ are the annihilators of the sets of all values taken by the reaction forces on the fibers of the constraint distribution, namely, on all possible velocities compatible with the constraints. For this reason, we call $\mathcal{R}^\circ$ the *reaction–annihilator distribution*.

As it will appear from the forthcoming treatment, the distribution $\mathcal{R}^\circ$ plays a key role in the study of first integrals linear in the velocities of linear nonholonomic systems. In order to understand the origin of this distribution, recall that d'Alembert principle postulates that the reaction forces take values in the annihilator $\mathcal{D}^\circ$ of the constraint distribution $\mathcal{D}$. However, all that matters for the existence of a conserved quantity of the equations of motion are the values that the reaction forces assume when evaluated *on the velocities compatible with the constraints*, that is, on the fibers of $\mathcal{D}$ (see the proof of Proposition 2). These values form in general only a *subset* of the annihilator of $\mathcal{D}$, that we call the 'reaction set'. Since the reaction forces are known functions of position and velocities, there is an explicit expression for the reaction set.

If the reaction set has positive codimension in $\mathcal{D}^\circ$, then the fibers of its annihilator $\mathcal{R}^\circ$ are larger than those of $\mathcal{D}$, and may host more infinitesimal generators of the group action than $\mathcal{D}$. This is why the consideration of the reaction–annihilator distribution may produce more conserved components of the momentum map than what has been known so far. We shall illustrate this mechanism on a number of sample cases in Section 5, but for illustrative purposes we note here that the most striking example is that of a nonholonomic system for which the reaction forces identically vanish (for example a sphere which rolls without slipping on a horizontal plane). In this case, the fibers of $\mathcal{R}^\circ$ consist of the entire tangent spaces to the configuration manifold and the momentum map of any lifted action which leaves the Lagrangian invariant is conserved. However, the fibers of $\mathcal{R}^\circ$ may be strictly larger than those of $\mathcal{D}$ also in other, less obvious cases. Noticeably, this mechanism can in principle produce conserved momenta in cases which, like Chaplygin systems [19, 8, 10, 12], have no horizontal symmetries at all.

There is also another fact which leads to the emergence of the distribution $\mathcal{R}^\circ$. As is well known, and we shall recall in Propostion 1, the generator of a first integral of a nonholonomic system which is linear in the velocities is only determined up to its component orthogonal (in the kinetic energy metric) to the fibers of the constraint distribution [17, 18, 13, 24]. Therefore, it is possible to consider generators which are sections of any distribution whose fibers contain those of $\mathcal{D}$. From this point of view, the distribution $\mathcal{R}^\circ$ emerges as the answer to the natural question "which generators of a first integral linear in the velocities preserve the Lagrangian?". This is the content of Proposition 2, which plays a key role in the entire argument.[1] Among its consequences are the use of the distribution $\mathcal{R}^\circ$ to characterize the first integrals of a holonomic system which persist under the addition of nonholonomic constraints (Corollary 2) and the above mentioned

[1]Proposition 2 as well can be regarded as a 'nonholonomic Noether theorem'. In fact, in the literature, Noether theorem is presented either as a statement on the conservation of momentum maps of symmetry groups, or as a statement on vector fields which generate conserved momenta.

version of the nonoholonomic Noether theorem for lifted actions (Corollary 3).

Here is an outline of the content of this paper. Section 2 is devoted to the reaction–annihilator distribution. Section 3 to first integrals linear in the velocities. Section 4 to the nonholonomic Noether theorem. Section 5 to some examples. A short section of Conclusions follows.

2 The reaction–annihilator distribution

In this Section we introduce the central object of this article, the reaction–annihilator distribution $\mathcal{R}^\circ$. We use the Lagrangian description of nonholonomic systems since it is fully adequate for the treatment of lifted actions, to which we restrict our analysis. For simplicity, we resort to a coordinate description wherever possible and adequate. For general references on nonholonomic systems see [22, 9, 12, 7].

As a starting point, consider a holonomic mechanical system with n-dimensional configuration manifold Q and smooth Lagrangian $L = T - V$, with kinetic energy $T(q,\dot q) = \frac{1}{2}\dot q \cdot A(q)\dot q$ and potential energy $V(q)$. Here, $(q,\dot q)$ are bundle coordinates on TQ and the kinetic matrix A is symmetric and positive definite. A linear nonholonomic constraint of rank r, where $1 \le r < n$, is a non–integrable distribution $\mathcal{D}$ on Q of constant rank r. This distribution, which is called the *constraint distribution*, can be locally defined by annihilation of $k = n - r$ linearly independent differential 1–forms on Q. Using local coordinates, the fibers $\mathcal{D}_q$ of the constraint distribution can be described as the kernel of a $k \times n$ matrix $S(q)$, which depends smoothly on q and has rank k everywhere, namely

$$\mathcal{D}_q \;=\; \{\dot q \in T_qQ \;:\; S(q)\dot q = 0\,\}\,. \tag{2.1}$$

The constraint distribution can be also thought of as a sub–bundle D of TQ of dimension $2n - k$, which is called the *constraint manifold*.

As usual, we assume the validity of d'Alembert's principle on 'ideal' or 'perfect' constraints, namely, that the reaction forces annihilate (an appropriate jet extension of) $\mathcal{D}$. As is well known, this leads to a dynamical system on the constraint manifold $D \subset TQ$. The derivation of the equations of motion is performed with a standard technique, which is based on the introduction of Lagrange multipliers. Eliminating the multipliers give the reaction forces as a function

$$R \,:\, D \;\to\; D^\circ \;:=\; \bigcup_{q\in Q} \mathcal{D}_q^\circ\,,$$

where $\mathcal{D}_q^\circ \subset T_q^*Q$ is the annihilator of the fiber $\mathcal{D}_q$. In local coordinates the resulting equations have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = R\,. \tag{2.2}$$

Their restriction to D defines a vector field on D which gives the dynamics of the nonholonomic system. We call this vector field the *nonholonomic system* (L, Q, D).

The expression of the map $R : D \to D^\circ$ can be easily worked out in bundle coordinates $(q, \dot q)$, see [2]: if the constraint distribution is described as in (2.1), then

$$R(q,\dot q) \;=\; S(q)^T\big[S(q)A(q)^{-1}S(q)^T\big]^{-1}\Big[S(q)A(q)^{-1}\big(\beta(q,\dot q)+V'(q)\big)-\gamma(q,\dot q)\Big] \tag{2.3}$$

where $\beta(q,\dot q) \in \mathbb{R}^n$, $V'(q) \in \mathbb{R}^n$ and $\gamma(q,\dot q) \in \mathbb{R}^k$ have components

$$\beta_i(q,\dot q) \;=\; \left(\frac{\partial A_{ij}}{\partial q_h}(q) - \frac{1}{2}\frac{\partial A_{jh}}{\partial q_i}(q)\right)\dot q_j \dot q_h\,, \qquad V_i'(q) \;=\; \frac{\partial V}{\partial q_i}(q)\,, \qquad \gamma_a(q,\dot q) \;=\; \frac{\partial S_{aj}}{\partial q_h}(q)\dot q_j \dot q_h$$

$(i = 1, \dots, n,\, a = 1, \dots, k)$.

By construction, the restriction of the map R to each fiber $\mathcal{D}_q$ of the constraint distribution takes values in the annihilator $\mathcal{D}_q^\circ$. However, the image under R of each such fiber, namely the set

$$\mathcal{R}_q \;:=\; \bigcup_{\dot q\in\mathcal{D}_q} R(q,\dot q)\,,$$

may be only a proper subset of $\mathcal{D}_q^\circ$. Correspondingly, the image under R of the constraint manifold D may be a proper subset of the annihilator of the constraint manifold:

$$\mathcal{R} \;:=\; \bigcup_{q\in Q} \mathcal{R}_q \;\subseteq\; D^\circ\,.$$

We call $\mathcal{R}$ the *reaction set.* (This name is not standard; in the literature, it is D° which is sometimes called 'reaction bundle'.)

Since the restriction of R to each fiber of $\mathcal{D}$ is a nonlinear map, the sets $\mathcal{R}_q$ need not be linear or affine subspaces of T_q^*Q. Nevertheless, the annhilators $\mathcal{R}_q^\circ \subset T_qQ$ of these sets are linear spaces and are thus the fibers of a distribution $\mathcal{R}^\circ$ on Q, possibly of non–constant rank and non–smooth. Since the space $\mathcal{R}_q^\circ$ contains all tangent vectors $\dot q\in T_qQ$ which annihilate all possible values of the reaction forces on constraint motions through q, we call $\mathcal{R}^\circ$ the *reaction–annihilator distribution.* (Another possible name might be 'zero–reaction–work distribution')

Clearly

$$\mathcal{R}_q^\circ \supseteq \mathcal{D}_q\,, \qquad q\in Q\,,$$

with equality if and only if $\mathcal{R}_q$ contains an open subset of $\mathcal{D}_q^\circ$. A simple dimension count indicates that $\mathcal{D}_q$ is always a proper subspace of $\mathcal{R}_q^\circ$ if $\dim\mathcal{D}_q < \mathrm{codim}\mathcal{D}_q$, which happens whenever $n\geq 4$ and $k>n/2$. An extreme case is when the reaction forces vanish identically, as for instance for a homogeneous ball rolling on a plane, when $\mathcal{R}^\circ = TQ$.

As we shall see, and has already been explained in the Introduction, the fact that $\mathcal{D}$ is a (possibly proper) subset of $\mathcal{R}^\circ$ is relevant to a number of questions related to first integrals. Of course, the determination of $\mathcal{R}^\circ$ is more involved than that of $\mathcal{D}$, but the first step towards it, namely, the determination of the reaction force R (or of some equivalent quantity), has to be done in order to write down the equations of motion. The determination of R as given in (2.3) can be easily automated using any symbolic computation software, even though the resulting expression may be rather cumbersome.

3 First integrals linear in the velocities

From now on, we use the following notation. If ξ^Q is a smooth vector field on Q, we denote by ξ^{TQ} its tangent lift, namely the vector field on TQ which, in bundle coordinates $(q,\dot q)$, is given by

$$\xi^{TQ} \;=\; \sum_i \xi^Q_{q_i}\partial_{q_i} + \sum_i \dot q_j \frac{\partial \xi^Q_{q_i}}{\partial q_j}\partial_{\dot q_i}\,.$$

Moreover, thinking of the Lagrangian L as given, we denote by $p(q,\dot q) := \frac{\partial L}{\partial \dot q}(q,\dot q)$ the momenta. If $\mathcal{E}$ is a distribution on Q, then $\Gamma(\mathcal{E})$ denotes the space of sections of $\mathcal{E}$.

A first integral of a nonholonomic system (L,Q,D) is a smooth function $F: D\to\mathbb{R}$ which is constant along the solutions of (2.2). Using bundle coordinates and denoting by a dot the inner product in $\mathbb{R}^n$, a first integral which is linear in the velocities can be written as $F(q,\dot q) = \xi^Q(q)\cdot p(q,\dot q)$ for some smooth vector field ξ^Q on Q, that we call a *generator* of F; we also say that F *is generated* by ξ^Q. In the sequel, we will say for short *linear first integral* instead of first integral linear in the velocities. There are several studies of the conditions under which a function which is linear in the velocities is a first integral of a nonholonomic system, see particularly [2, 17, 18]. Here we adopt a somewhat different point of view, focussing on the role of the reaction–annihilator distribution $\mathcal{R}^\circ$.

The first fact to be mentioned is that, because of the restriction to the constraint manifold D, the generator of a linear first integral is never unique, even though it can be chosen in a unique way by the requirement that it is a section of $\mathcal{D}$. To our knowledge, this fact was proven by Iliev and Semerdzhiev [17, 18], but see also [2, 13] and, for a generalization to nonlinear first integrals and nonlinear constraints, [24]. In order to state this fact let us denote by $\mathcal{A}$ the Riemannian metric on Q given by the kinetic energy of the holonomic system and by $\mathcal{D}^\perp$ the distribution whose fibers $\mathcal{D}_q^\perp$ are the $\mathcal{A}$–orthogonal complements of the fibers $\mathcal{D}_q$ of $\mathcal{D}$. Moreover, given a distribution $\mathcal{E}$ on Q, we denote by $\mathcal{E}\cap\mathcal{D}^\perp$ the distribution with fibers $\mathcal{E}_q\cap\mathcal{D}_q^\perp$. The result of [17, 18] can be rephrased, and formalized, as follows:

Proposition 1 *Let F be a linear first integral of a nonholonomic system (L,Q,D). Consider any regular distribution $\mathcal{E}$ on Q with fibers $\mathcal{E}_q$ such that*

$$\mathcal{D}_q \subseteq \mathcal{E}_q$$

for all $q\in Q$. Then, there is a smooth generator $\xi^Q_{\mathcal{E}}\in\Gamma(\mathcal{E})$ of F and the set of all generators of F which are sections of $\mathcal{E}$ is $\xi^Q_{\mathcal{E}}+\Gamma(\mathcal{E}\cap\mathcal{D}^\perp)$. In particular, F has a unique generator in $\Gamma(\mathcal{D})$.

Proof. Let ξ^Q be a generator of F. Define $\xi^Q_{\mathcal{E}}$ as the $\mathcal{A}$–orthogonal projection of ξ^Q onto $\mathcal{E}$. Since $\mathcal{E}$ is a smooth distribution, $\xi^Q_{\mathcal{E}}$ is a smooth section of $\mathcal{E}$. That $\xi^Q_{\mathcal{E}}$ is a generator of F follows from observing that $(\xi^Q-\xi^Q_{\mathcal{E}})\cdot p|_D=(\xi^Q-\xi^Q_{\mathcal{E}})\cdot A\dot q|_D=0$ because, for each $q\in Q$, $\xi^Q(q)-\xi^Q_{\mathcal{E}}(q)$ is $\mathcal{A}$–orthogonal to any $\dot q\in\mathcal{E}_q$ and hence to any $\dot q\in\mathcal{D}_q$. This same observation proves that a section of $\mathcal{E}$ is a generator of F if and only if it is of the form $\xi^Q_{\mathcal{E}}+v$ with v a section of $\mathcal{E}$ and of $\mathcal{D}^\perp$. ■

Generators in $\Gamma(\mathcal{D})$ have been extensively considered in connection with symmetries, see [2, 20, 3, 8, 24, 11, 14, 21, 15]. In this article we point out that, even though there is always a generator in $\Gamma(\mathcal{D})$, there may be reasons to consider generators which are sections of $\mathcal{R}^\circ$.

The next proposition characterizes generators in $\mathcal{R}^\circ$:

Proposition 2 *Given a nonholonomic system (L,Q,D) and a smooth vector field ξ^Q on Q, any two of the following three conditions imply the third:*

C1. ξ^Q *is a section of* $\mathcal{R}^\circ$

C2. $\xi^{TQ}(L)=0$ *in* D

C3. ξ^Q *is the generator of a linear first integral of* (L,Q,D).

Proof. Write $F=p\cdot\xi^Q$. Denoting by a dot the derivative along the flow of equations (2.2), we have $\dot F=\dot p\cdot\xi^Q+p\cdot\dot\xi^Q=\big(\frac{\partial L}{\partial q}+R\big)\cdot\xi^Q+\frac{\partial L}{\partial\dot q}\cdot\xi^{TQ}_{\dot q}$, namely $\dot F = \xi^{TQ}(L)+R\cdot\xi^Q$. Thus, at each point $q\in Q$, the vanishing in all of $\mathcal{D}_q$ of any two among $\dot F$, $\xi^{TQ}(L)$ and $R\cdot\xi^Q$ implies the vanishing in all of $\mathcal{D}_q$ of the third. Since $\mathcal{R}_q=R(q,\mathcal{D}_q)$, the vanishing of $R(q,\dot q)\cdot\xi^Q(q)$ on $\mathcal{D}_q$ amounts to $\xi_Q(q)\in\mathcal{R}_q^\circ$. ■

This Proposition is the core of our analysis. It implies that, among all the generators of a linear first integral, those which are sections of $\mathcal{R}^\circ$ are exactly those which satisfy the invariance condition C2. It also implies that *any section of $\mathcal{R}^\circ$ which satisfies condition C2 is the generator of a linear first integral of* (L,Q,D). This statement is a generalization of a standard, very well known result which has been obtained several times and is often regarded as a nonholonomic version of Noether theorem: Any smooth section ξ^Q of $\mathcal{D}$ which satisfies condition C2 is the generator of a linear first integral of (L,Q,D) [2, 20, 3, 13, 8, 11, 14, 21, 15]. In this regard we note that any linear first integral has a generator with this property, a fact which also follows from [13, 24]. In fact, by Proposition 1, any linear first integral has a generator which is a section of $\mathcal{D}$; since this vector field satisfies conditions C1 and C3, Proposition 2 implies the following

Corollary 1 *Let F be any linear first integral of a nonholonomic system (L, Q, D). Then, the generator of F which is a section of $\mathcal{D}$ satisfies condition C2.*

The consideration of generators in $\Gamma(\mathcal{R}^\circ)$ becomes particularly significant when linear first integrals of the *holonomic* system (L, Q) are concerned. Since the generator of any linear first integral of a holonomic system is uniquely determined, the characterization of those among these integrals which are also first integrals of a nonholonomic system (L, Q, D) leads to the appearence of the reaction–annihilator distribution $\mathcal{R}^\circ$. Proposition 2 has indeed the following consequence:

Corollary 2 *Consider a nonholonomic system (L, Q, D) and assume that $F = \xi^Q \cdot p$ is a first integral of the holonomic system (L, Q). Then, $F|_D$ is a first integral of (L, Q, D) if and only if ξ^Q is a section of $\mathcal{R}^\circ$.*

Proof. F is a first integral of the holonomic system (L, Q) if and only if $\xi^{TQ}(L) = 0$. Under this hypothesis, C1 and C3 are equivalent. ■

An analytic characterization of the conditions under which a linear first integral of the holonomic system is a first integral of the nonholonomic one was given by Iliev [17], without however considering the role of the distribution $\mathcal{R}^\circ$. Note that, in the context of Corollary 2, $F|_D$ also has a generator in $\Gamma(\mathcal{D})$, which is obtained by projection onto D, even when the generator ξ^Q of F belongs to $\Gamma(\mathcal{R}^\circ)$ but not to $\Gamma(\mathcal{D})$.

Example: Chaplygin's sphere is a non–homogeneous sphere with the center of mass located in the geometric center and three distinct moments of inertia relative to the center, which is constrained to rotate without slipping on a horizontal plane, see particularly [15] and references therein. Since the nonholonomic reaction force acts on the point of the sphere which is in contact with the plane, the angular momentum M relative to the contact point is a conserved quantity (even though the point itself moves). This is an instance of a 'Chaplygin integral' or 'generalized angular momentum integral' [22]. The restriction to the constraint manifold of the components of M are first integrals which are linear in the velocities. The vertical component of M is a first integral of the holonomic system, so by Corollary 2 its (uniquely determined) generator is a section of $\mathcal{R}^\circ$. In fact, a computation (using e.g. Euler angles as local coordinates on SO(3) to write down the Lagrangian, the constraints and the reaction forces) shows that it is a section of $\mathcal{D}$. However, the two horizontal components of M are not first integrals of the holonomic system. Computing their generators from the expression of the angular momentum on TQ gives vector fields which, as can be verified with a somewhat lengthy computation, are neither sections of $\mathcal{D}$ nor of $\mathcal{R}^\circ$. Hence, according to Proposition 2, they do no infinitesimally preserve the Lagrangian in D. Nevertheless, by Propositions 1 and 2, generators which infinitesimally preserve the Lagrangian in D can be obtained by projection onto $\mathcal{D}$ (or onto $\mathcal{R}^\circ$, or onto any distribution whose fibers are contained between those of $\mathcal{D}$ and those of $\mathcal{R}^\circ$).

4 The nonholonomic Noether theorem for lifted actions

We apply now the results of the previous section to nonholonomic systems with symmetry, specifically to the conservation of the components of the momentum map of a lifted action.

Assume that a Lie group G acts on Q and denote by η^Q the infinitesimal generator corresponding to a Lie algebra vector $\eta \in \mathfrak{g}$. If L is a G–invariant Lagrangian, that is $\eta^{TQ}(L) = 0$ for all $\eta \in \mathfrak{g}$, then the momentum map $J : TQ \to \mathfrak{g}^*$ is a conserved quantity of the holonomic system (L, Q). Equivalently, for any $\eta \in \mathfrak{g}$, the momentum

$$J_\eta := \eta^Q \cdot p$$

is a first integral of (L, Q). (The momentum map is in fact defined by $\langle J(q, \dot{q}), \eta \rangle = J_\eta(q, \dot{q})$ for all $\eta \in \mathfrak{g}$, see [4, 1]). Hence, Corollary 2 gives

Corollary 3 (The nonholonomic Noether theorem for lifted actions) *Assume that a Lie group G acts on Q and that the Lagrangian L is G–invariant. Then, for any $\eta \in \mathfrak{g}$, $J_\eta|_D$ is a first integral of (L, Q, D) if and only if $\eta^Q \in \Gamma(\mathcal{R}^\circ)$.*

As we already noticed in the Introduction, this characterization of the Lie algebra elements which produce conserved momenta generalizes a very well known sufficient condition: if L is G–invariant and $\eta^Q \in \Gamma(\mathcal{D})$ then $J_\eta|_D$ is a first integral of (L, Q, D) [3, 8, 11, 14, 21]. If $\eta^Q \in \Gamma(\mathcal{D})$, then η and η^Q are often called 'horizontal symmetries'. We thus introduce the following terminology, where we also relax the invariance condition on L to infinitesimal invariance on D:

Definition 1 *Consider a nonholonomic system (L, Q, D), a Lie group G which acts on Q and a vector $\eta \in \mathfrak{g}$.*

i. If $\eta^Q \in \Gamma(\mathcal{R}^\circ)$, then η and η^Q are called $\mathcal{R}^\circ$–symmetries. *A* horizontal symmetry *is a $\mathcal{R}^\circ$–symmetry η such that $\eta^Q \in \Gamma(\mathcal{D})$.*

ii. An $\mathcal{R}^\circ$–momentum *is a linear first integral of (L, Q, D) generated by an $\mathcal{R}^\circ$–symmetry. A* horizontal momentum *is an $\mathcal{R}^\circ$–momentum generated by a horizontal symmetry.*

The fact that, if $\eta \in \mathfrak{g}$ is an $\mathcal{R}^\circ$–symmetry, then the $\mathcal{R}^\circ$–momentum $J_\eta|_D$ is a first integral of the nonholonomic system follows from Proposition 2. But in fact, Proposition 2 implies a stronger statement: *Given a nonholomic system (L, Q, D), a group G which acts on Q and a vector $\eta \in \mathfrak{g}$, any two of the following conditions imply the third: $\eta^{TQ}(L)|_D = 0$, η^Q is an $\mathcal{R}^\circ$–symmetry, $J_\eta|_D$ is a first integral.*

The advantage of the formulation of Corollary 3 over the traditional one is that, since the fibers of $\mathcal{R}^\circ$ can properly contain those of $\mathcal{D}$, a nonholonomic system with a G–invariant Lagrangian may have more $\mathcal{R}^\circ$–momenta than horizontal momenta. We will give some examples in the next Section. Note however that horizontal symmetries mantain an interest, because the 'horizontality' of $\eta \in \mathfrak{g}$ is a necessary condition for the flow of η^Q to leave the constraint manifold D invariant and hence allow reduction of the nonholonomic system (see [6, 8, 24, 11, 14, 21, 12]). Thus, when establishing which infinitesimal generators of a given group action allow reduction one should focus on horizontal symmetries, but when judging which infinitesimal generators produce conserved momenta one should consider $\mathcal{R}^\circ$–symmetries.

Remark: The term '$\mathcal{R}^\circ$–momentum' is in a way redundant. By Corollary 3, an $\mathcal{R}^\circ$–momentum is nothing else than a 'conserved component of the momentum map' or 'conserved momentum'. Nevertheless, we think it might be preferable to use here this term for greater clarity.

5 Examples

We provide now a few simple examples of systems which have $\mathcal{R}^\circ$–momenta which are not horizontal momenta. The distribution $\mathcal{R}^\circ$ could be easily determined in all cases, but its determination is not always necessary, particularly on account of Corollary 2.

A. Hamiltonian subsystems and the sphere rolling on a plane. There are certain nonholonomic systems for which the reaction forces identically vanish and the equations of motion are the restriction to the constraint manifold D of the Lagrange equations for the holonomic system. In these cases, D is an invariant submanifold of the holonomic Lagrange equations, the nonholonomic system is a subsystem of the holonomic one, and the reaction–annihilator distribution $\mathcal{R}^\circ$ coincides with the tangent bundle TQ. In particular, the nonholonomic system has all the first integrals of the holonomic system (even though some of them might be trivial, being constant on D). Even though very special, this case is encountered sometimes.

A simple case is that of a homogeneous sphere which rolls without slipping on a horizontal plane under no active force but gravity. The derivation of the equations of motion is elementary and shows that the reaction forces vanish identically, see e.g. [23, 22]. Therefore, the nonholonomic system is a subsystem of the holonomic system in which the sphere is constrained to touch the plane. Thus, the sphere rotates with constant angular velocity, its center of mass moves at constant speed on a straight line and the system has five first integrals which are linear in the velocities—two components of the linear momentum parallel to the plane and three components of the angular momentum relative to the center of the sphere. Since the components parallel to the plane of the angular momentum and of the linear momentum are related by the no–slipping constraint, only three of these five first integrals of the nonholonomic system are functionally independent, say the two components p_x and p_y of the linear momentum parallel to the plane and the vertical component m_z of the angular momentum. As in the case of the Chaplygin sphere of Section 3, the generator of m_z is a section of $\mathcal{D}$. (The angular momentum of the Chaplygin sphere was relative to the contact point, not to the center of the sphere, but the vertical component is the same). Instead, the generators of p_x and p_y are obviously not sections of $\mathcal{D}$, because translating the sphere does not respect the constraint. However, they are sections of $\mathcal{R}^\circ = TQ$.

Let us now introduce a symmetry group. The holonomic system is invariant under translation of the center of mass and under spatial rotations of the sphere. (There is also another SO(3)–invariance, namely particle relabelling, which fully accounts for the integrability of the system, but we do not consider it here). The momentum map of this $\mathbb{R}^2 \times \mathrm{SO}(3)$ action has the linear momentum and the angular momentum as its components. Their restriction to D gives the three independent first integrals of the nonholonomic system. Among them, m_z is a horizontal momentum. The other two, p_x and p_y, are $\mathcal{R}^\circ$–momenta but not horizontal momenta.

Remark: The fact that this nonholonomic system has the same linear first integrals of the associated holonomic system was analytically proven in [17] without reference to the reaction forces and to the fact that the nonholonomic system is a subsystem of the holonomic one.

B. The vertical coin. Let us now consider a disk which is constrained to roll without slipping on a horizontal plane while standing vertically, see e.g. [8]. At first, assume there are no active forces (except gravity, which plays no role). The holonomic system (the disk stands vertically and touches the plane) has configuration manifold $Q = \mathbb{R}^2 \times S^1 \times S^1 \ni (x, y, \varphi, \theta)$, where $(x, y) \in \mathbb{R}^2$ are cartesian coordinates of the point of contact C, φ is the angle between the x–axis and the projection of the disk on the plane, and θ is the angle between a fixed radius of the disk and the vertical. The Lagrangian is given by the kinetic energy which, assuming that the disk has unit mass, is $T = \frac{1}{2}\big(\dot{x}^2 + \dot{y}^2\big) + \frac{1}{2}J\dot{\varphi}^2 + \frac{1}{2}I\dot{\theta}^2$, where J and I are the pertinent moments of inertia. All four velocities are first integrals of the holonomic system.

The nonholonomic constraint of rolling without slipping consists of the fact that the point of contact has zero velocity. Thus the rank–two constraint distribution has fibers

$$\mathcal{D}_{(x,y,\varphi,\theta)} \;=\; \mathrm{span}_{\mathbb{R}}\big\{ \cos\varphi\,\partial_x + \sin\varphi\,\partial_y + \partial_\theta\,,\ \partial_\varphi \big\}\,.$$

The constraint manifold D is six–dimensional and can be globally parametrized with the coordinates $(x, y, \varphi, \theta, \dot{\varphi}, \dot{\theta})$. The equations of motion are

$$\dot{x} = \dot{\theta}\cos\varphi\,, \quad \dot{y} = \dot{\theta}\sin\varphi\,, \quad \ddot{\varphi} = 0\,, \quad \ddot{\theta} = 0\,.$$

Thus, the velocities $\dot{\varphi}$ and $\dot{\theta}$ are first integrals. Consequently, Corollary 2 ensures that the generators ∂_φ and ∂_θ of $p_\varphi = J\dot{\varphi}$ and $p_\theta = I\dot{\theta}$ are sections of $\mathcal{R}^\circ$. However, only one of them, namely ∂_φ, is a section of $\mathcal{D}$.

Let us now introduce a symmetry group. The Lagrangian T is invariant under an obvious action of $\mathrm{SE}(2) \times S^1$, where the first factor is rotation–translations of the disk and the second factor is rotations of the disk around its axis, namely translations of θ, see [8]. As remarked in [8], neither ∂_φ nor ∂_θ are horizontal symmetries. In fact, ∂_φ is a section of $\mathcal{D}$ but is not an infinitesimal generator of the action, while ∂_θ is an infinitesimal generator of the action but is not a section of $\mathcal{D}$. However, since ∂_θ is a section of $\mathcal{R}^\circ$, $p_\theta|_D$ is an $\mathcal{R}^\circ$–momentum.

Remarks: (i) Incidentally, the fact that ∂_φ and ∂_θ are sections of $\mathcal{R}^\circ$ is easy to verify: the equations of motion show that the reaction force is $R(x, y, \varphi, \theta, \dot\varphi, \dot\theta) = \dot\varphi\dot\theta(-\cos\varphi\,, \sin\varphi\,, 0\,, 0)$. Hence $\mathcal{R}^\circ_{(x,y,\varphi,\theta)} = \mathrm{span}_{\mathbb{R}}\{\cos\varphi\,\partial_x + \sin\varphi\,\partial_y\,,\ \partial_\varphi\,,\ \partial_\theta\}$. Also, note that $\mathcal{R}^\circ$ has rank three while $\mathcal{D}$ has rank 2.

(ii) Even though ∂_φ is not a horizontal symmetry for the above action of $\mathrm{SE}(2) \times S^1$, it is a horizontal symmetry for the abelian action of $\mathbb{R}^2 \times S^1 \times S^1 \ni (\lambda, \mu, \alpha, \beta)$ given by $(x, y, \varphi, \theta) \mapsto (x + \lambda, y + \mu, \varphi + \alpha, \theta + \beta)$.

C. A symmetrically torqued vertical coin. We apply now a conservative force field to the disk of the previous example, with potential energy $V(\varphi)$. The Lagrangian is $L = T - V$, where the kinetic energy T is as above. The equations of motion become

$$\dot x = \dot\theta\cos\varphi\,, \quad \dot y = \dot\theta\sin\varphi\,, \quad \ddot\theta = 0\,, \quad \ddot\varphi = -V'(\varphi)\,.$$

Thus, $p_\theta|_D$ is still a first integral. As we know from the previous example, ∂_θ is not a section of $\mathcal{D}$. However, it is a section of $\mathcal{R}^\circ$ because p_θ is also a first integral of the holonomic system with Lagrangian $L = T - V$.

The Lagrangian $L = T - V$ is invariant under the action of $G = \mathbb{R}^2 \times S^1 \ni (\lambda, \mu, \beta)$ given by $(x, y, \varphi, \theta) \mapsto (x + \lambda, y + \mu, \varphi, \theta + \beta)$. The first integral $p_\theta|_D$ is an $\mathcal{R}^\circ$–momentum but not a horizontal momentum for this action.

D. The same torqued vertical coin, but a symmetry subgroup. An interesting class of examples in which $\mathcal{R}^\circ$–symmetries are likely to play an important role are those where the constraint distribution has trivial intersection with the tangent spaces to the group orbits, particularly Chaplygin systems [19, 8, 12]. Systems of this class do not possess horizontal momenta, but may have $\mathcal{R}^\circ$–momenta.

A very simple example is given by the torqued vertical disk of subsection C together with the action of the (sub)group $S^1 \ni \beta$ given by $(x, y, \varphi, \theta) \mapsto (x, y, \varphi, \theta + \beta)$. The tangent spaces to the group orbits intersect the fibers of the constraint distributions in the zero vector. Hence, $p_\theta|_D$ is is still a $\mathcal{R}^\circ$–momentum.

6 Conclusions

In this article we have pointed out the role of a distribution on the configuration space, the reaction–annihilator distribution $\mathcal{R}^\circ$, in the study of first integrals linear in the velocities of nonholonomic systems with linear, time–independent constraints and natural Lagrangian given by kinetic energy minus potential energy. This point of view has led to a new formulation of the nonholonomic Noether theorem for lifted actions, where the infinitesimal generators of the group action which produce conserved momenta have been shown to be exactly those which are sections of $\mathcal{R}^\circ$.

As illustrated in a few simple examples, the passage from horizontal symmetries to sections of $\mathcal{R}^\circ$ leads to an actual enlargement in the class of conserved momenta of nonholonomic systems. Moreover, the clarification of the theoretical framework obtained with the introduction of the reaction–annihilator distribution seems to us to be substantial. Proposition 2 shows that it is only for vector fields which are sections of $\mathcal{R}^\circ$ that the property of infinitesimally preserving the Lagrangian in D is equivalent to generating a linear first integral.

Various questions remain to be studied further. First, one should explore more (and more complex) examples. Second, to our knowledge the properties of the reaction set $\mathcal{R}$ have never been studied and it might be of some interest to have some information on its structure. Finally, various generalizations are necessary. Among them, the generalization to Lagrangians with potential depending linearly in the velocities and to affine costraints should be a standard matter. The generalization to non–lifted actions might be based on the work of [24].

But foremost, what has been excluded from the present analysis is the consideration of first integrals linear in the velocities which are related in a 'gauge–like' way to the group action [5] (see also [21, 25] for related considerations). On the one hand, several of the $\mathcal{R}^\circ$–momenta of the examples of Section 5 can be viewed as gauge–momenta as well. On the other hand, the consideration of the reaction–annihilator distribution might open up new perspectives on this very interesting class of first integrals. We shall therefore return on this topic in a forthcoming paper [16].

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