An elemental overview of the nonholonomic Noether theorem

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1 Aim of the paper

The existence of first integrals of a dynamical system plays an important role in its study and influences, among other things, its integrability. For holonomic mechanical systems there is a well understood link between symmetries and existence of first integrals, which in the Lagrangian formulation is described by Noether theorem. The existence and properties of a link symmetries–first integrals for nonholonomic systems has been extensively studied over the last thirty years, and the first studies go back at least to the 1950's. Various 'nonholonomic Noether theorems' have been published, with different emphasis and at different levels of generalization. Related concepts such as 'gauge momenta' and the 'momentum equation' have appeared. Often, these studies have been done within sophisticated geometrical settings.

The purpose of this short article is to provide an overview of this subject which is as elementary as possible. To this end, we focus on the main ideas and illustrate them on the simplest possible case, that of lifted actions and natural Lagrangians, and we limit ourselves to just mention extensions to more general cases. We will restrict our consideration to the mere existence of first integrals related to symmetries; nothing will be said about the related and important topic of reduction under a symmetry group.

For general treatments of nonholonomic mechanics, from different perspectives and at different levels, see e.g. [30, 14, 7, 9]. However, our treatment does not require much more than standard knowledge from classical mechanics.

2 Noether theorem

Noether theorem of Lagrangian mechanics is a very particular case of the first of two very general theorems stated by Emmy Noether in her 1918's celebrated article [31], which is devoted to the conservation laws of variational problems that are invariant under the action of a Lie group (of either finite or infinite dimension). The depth of these theorems lies largely in their applications to the calculus of variations and to field and gauge theories. A modern formulation of Noether theorems can be found e.g. in [26]; for historical and critical insight see [10, 11, 27].

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The application to Lagrangian mechanics was not among Noether's main motivations, and is covered by just a short mention in [31]. Nevertheless, it is the simplest case of this particular case of Noether's first theorem that is nowdays usually identified as 'the Noether theorem'. A possible statement of it is that if the Lagrangian $L(q,\dot q)$ of a system with configuration space Q is invariant under the lift to TQ of a free action of a k–dimensional Lie group on Q, then the system possesses k independent first integrals which are linear in the conjugate momenta. In modern terms, this amounts to the conservation of the momentum map of a lifted action which leaves the Lagrangian invariant, see e.g. [4, 1].

Noether stated her results assuming the invariance of the Lagrangian under a Lie group action, but in the proofs she used, where possible, only the infinitesimal invariance of the Lagrangian. This dualism still permeates the literature on classical mechanics, where 'Noether theorem' is frequently regarded as a statement on Lagrangians which are invariant under lifted one–parameter (local) groups, or even as a condition on vector fields on the configuration space, which characterizes when one such vector field generates a first integral. A generalization of the latter approach to non–lifted actions is met frequently in the Hamiltonian setting, where the well known statement that a function f is a first integral of a system with Hamiltonian h if and only if the Hamiltonian vector field X_f of f preserves the Hamiltonian h (that is, the Lie derivative $X_f(h)=0$, or equivalently the Poisson bracket $\{f,h\}=0$) is sometimes regarded as a Hamiltonian version of Noether theorem, see e.g. [4].

The same variety of points of view is met in the statements of 'nonholonomic Noether theorems' which have been developed so far. Properly speaking, Noether theorem does not apply to mechanical systems with nonholonomic constraints, which are not variational (that is, Lagrangian) systems. The conceptual problem underlying the search of a 'nonholonomic Noether theorem' is thus that of understanding how much of the relationship between symmetry and conservation laws of the holonomic case carries over to the nonholonomic one. For instance, if a nonholonomic system is invariant (in an appropriate sense: see the Remarks in section 9) under a lifted action, which components of the momentum map are first integrals? The first results in this direction of which we are aware go back to the 1950's, see below. Since then, a number of variants of 'nonholonomic Noether theorem' have appeared, with different levels of generalities, and that, as we shall see, can be roughly classified in two main families, which will be described in sections 6-7 and 8-9, respectively.

3 Nonholonomic systems

As a starting point, we consider a holonomic mechanical system with n-dimensional configuration manifold Q and smooth Lagrangian $L=T-V$, with kinetic energy $T(q,\dot q)=\frac{1}{2}\dot q\cdot A(q)\dot q$ and potential energy $V(q)$. Here, $(q,\dot q)$ are bundle coordinates on TQ, the kinetic matrix $A(q)$ is symmetric and positive definite at each point $q\in Q$, and the dot denotes the Euclidean scalar product of $\mathbb{R}^n$. In order to keep the complexity of the notation to a minimum we resort wherever possible to a coordinate description and identify points and coordinates, thus writing $q\in Q$, etc.

We add now the nonholonomic constraint that, at each point $q\in Q$, the velocities of the system belong to a linear subspace $\mathcal{D}_q$ of the tangent space T_qQ. The subspaces $\mathcal{D}_q$ are the fibers of a distribution $\mathcal{D}$, that we suppose to be smooth and of constant rank r, $0<r<n$. Nonholonomy of the constraints amounts to the nonintegrability of the distribution $\mathcal{D}$, which is called the *constraint distribution*. Clearly, $\mathcal{D}$ may be as well regarded as a submanifold $D\subset TQ$ of dimension $n+r$, that will be called the *constraint submanifold*.

We assume that the nonholonomic constraint is 'ideal', that is, it satisfies d'Alembert principle. This implies that the reaction force $R(q,\dot q)$ that the nonholonomc constraint exerts when the system is in the configuration $q\in Q$ with velocity $\dot q\in\mathcal{D}_q$ annihilates the fiber $\mathcal{D}_q$, that is

$$R(q,\dot q)\cdot v \;=\; 0 \qquad \forall q\in Q\,,\quad \forall \dot q\,,v\in\mathcal{D}_q \tag{1}$$

(a more pecise statement of d'Alembert principle will be given in Section 8). In other words, $R(q,\dot q) \in \mathcal{D}_q^\circ$ for all $\dot q \in \mathcal{D}_q$.[1] It is well known that, under this hypothesis, the restriction to the constrained manifold D of Lagrange equations with the reaction force,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} \;=\; R_i\,, \qquad i=1,\dots,n\,, \tag{2}$$

define a dynamical system on D. We call this dynamical system the *nonholonomic system* (L,Q,D) and, when we need to stress its interpretation as a vector field on D, we use the symbol $X_{L,Q,D}$ to denote it.

It is important to note that the reaction force R is a known function of $(q,\dot q)\in D$. In order to give its expression we describe the fibers of the constraint distribution as the kernel of a $k\times n$ matrix $S(q)$, which depends smoothly on q and has rank k everywhere, namely

$$\mathcal{D}_q \;=\; \{\dot q \in T_qQ \,:\, S(q)\dot q = 0\,\}\,. \tag{3}$$

Let, moreover, $\Pi^{A^{-1}}_{\mathcal{D}^\circ}(q)$ be the $A(q)^{-1}$–orthogonal projector onto the annihilator of $\mathcal{D}_q = \ker S(q)$, that is

$$\Pi^{A^{-1}}_{\mathcal{D}^\circ} \;=\; S^T\big(SA^{-1}S^T\big)^{-1}SA^{-1}\,,$$

and define the vectors $\beta(q,\dot q)\in\mathbb{R}^n$, $\gamma(q,\dot q)\in\mathbb{R}^k$ and $V'(q)\in\mathbb{R}^n$ as having components

$$\beta_i \;=\; \sum_{j,h}\left(\frac{\partial A_{ij}}{\partial q_h} - \frac12\frac{\partial A_{jh}}{\partial q_i}\right)\dot q_j\dot q_h\,, \qquad \gamma_a \;=\; \sum_{j,h}\frac{\partial S_{aj}}{\partial q_h}\dot q_j\dot q_h\,, \qquad V'_i \;=\; \frac{\partial V}{\partial q_i}$$

$(i,j,h=1,\dots,n,\ a=1,\dots,k)$. Then,

$$R \;=\; \Pi^{A^{-1}}_{\mathcal{D}^\circ}(\beta+V') - S^T\big(SA^{-1}S^T\big)^{-1}\gamma\,. \tag{4}$$

The proof is elementary, and can be achieved by writing Lagrange equations with the undetermined multipliers, and eliminating them. Form (3)–(4) of the equations of motion can be found e.g. in [2]; more frequently, the equations of motion are written using local coordinates on D.

4 Linear first integrals

A *first integral* of the nonholonomic system (L,Q,D) is a function $F: D\to\mathbb{R}$ which is constant along the solution of equations (2), that is

$$X_{L,Q,D}(F) \;=\; 0 \qquad \text{in all of } D\,.$$

Here, following [33], we assume that the function F is defined on the constraint manifold D, not on the entire TQ, because the nonholonomic system is a dynamical system on D. (Usually, this distinction is not made and first integrals are regarded as defined in TQ; this is possible, given that every function on D can be extended to TQ, but has the disadvantage that the extension is not unique. Considering first integrals as functions on the constraint manifold is particularly useful in our context).

We will say that a function F on D is *linear in the momenta*, or simply that it is a *linear function on* D, if it is the restriction to D of a function which is linear in the conjugate momenta, that is,

$$F \;=\; Z\cdot p|_D$$

[1] In coordinates, the annihilator E° of a subspace E coincides with the Euclidean orthogonal complement to E, but we prefer reserve the term 'orthogonal' for the orthogonalities induced by the kinetic matrix and its inverse, see below.

where

$$p(q,\dot q) \;=\; \frac{\partial L}{\partial q}(q,\dot q) \;=\; A(q)\dot q$$

is the (co)vector of the conjugate momenta and Z is a vector field on Q, which will be called a *generator* of F. A *linear first integral* of (L,D,Q) is a linear function on D which is also a first integral.

The restriction to D has the consequence that the generator of a linear function on D is never unique, even though it can be chosen in a unique way by the requirement that it is a section of $\mathcal{D}$. This simple fact plays a central role in the formulation of the first family of 'nonholonomic Noether theorems', and is thus stated in Lemma 1 below. To this end, let $\mathcal{A}$ be the Riemannian metric on Q given by the kinetic energy of the holonomic system and $\mathcal{D}^{\perp_{\mathcal{A}}}$ the distribution whose fibers $\mathcal{D}_q^{\perp_{\mathcal{A}}}$ are the $\mathcal{A}$–orthogonal complements of the fibers $\mathcal{D}_q$ of $\mathcal{D}$. Moreover, $\Gamma(\mathcal{E})$ denotes the space of sections of a distribution $\mathcal{E}$ on Q.

Lemma 1. [24, 25] *A linear function on D has a unique horizontal generator $Z_{\mathcal{D}}$, and the set of its generators is $Z_{\mathcal{D}}+\Gamma(\mathcal{D}^{\perp_{\mathcal{A}}})$.*

Proof. Let F be a linear function on D and Z a generator of F. Define $Z_{\mathcal{D}}$ as its $\mathcal{A}$–orthogonal projection onto $\mathcal{D}$. $Z_{\mathcal{D}}$ is a generator of F because $(Z-Z_{\mathcal{D}})\cdot p|_D=(Z-Z_{\mathcal{D}})\cdot A\dot q|_D=0$ given that, at each $q\in Q$, $Z(q)-Z_{\mathcal{D}}(q)$ is $\mathcal{A}$–orthogonal to all $\dot q\in\mathcal{D}_q$. The same observation proves that a vector field is a generator of F if and only if it is of the form $Z_{\mathcal{D}}+v$ with v a section of $\mathcal{D}^{\perp_{\mathcal{A}}}$. The uniqueness of $Z_{\mathcal{D}}$ follows. ■

The oldest references we could find where this fact is correctly stated are [24, 25]. Previously, Agostinelli [2] had erroneously stated that the generator must be horizontal, rather than that there is a horizontal generator.

In the sequel, as is customary, we will say that sections of $\mathcal{D}$ are *horizontal* vector fields.

5 Two points of view

At the core of the extensions of Noether theorem to nonholonomic systems lie the following questions:

- If $\Psi: G\times Q\to Q$ is an action of a Lie group G on the configuration manifold Q whose lift to TQ leaves the Lagrangian invariant, which components of the momentum map are first integrals of the nonholonomic system (L,Q,D)?
- Under which condition does a vector field Z on Q generate a linear first integral $Z\cdot p|_D$ of a nonholonomic system (L,Q,D)?

Various answers to both questions have been regarded as 'nonholonomic Noether theorems'. Since the components of the momentum map of a lifted action are linear in the momenta, we begin our investigation from the latter question.

Let us write $Z=\sum_i Z_i\partial_{q_i}$ with functions $Z_i: Q\to\mathbb{R}$ and let Z^{TQ} be the tangent lift of Z, that is, the vector field on TQ given by

$$Z^{TQ}(q,\dot q) \;=\; \sum_i Z_{q_i}(q)\partial_{q_i}+\sum_i \dot q_j\frac{\partial Z_{q_i}}{\partial q_j}(q)\partial_{\dot q_i}\,.$$

Computing the time derivative of $Z\cdot p$ along a solution $t\mapsto q^t$ of the equations of motion (2) gives $\frac{d}{dt}(Z\cdot p)=Z\cdot\frac{d}{dt}\big(\frac{\partial L}{\partial\dot q}\big)+\sum_{ij}\dot q_i^t\frac{\partial Z_j}{\partial q_i}\frac{\partial L}{\partial\dot q_j}=Z^{TQ}(L)+Z\cdot R$. Thus, we obtain the *balance equation*

$$X_{L,Q,D}\big(Z\cdot p|_D\big) \;=\; Z^{TQ}(L)|_D+Z\cdot R|_D\,. \tag{5}$$

Thus:

Proposition 1. *A linear function on D is a first integral of (L,Q,D) if and only if it has a generator Z such that*

$$Z^{TQ}(L) + Z\cdot R \;=\; 0 \qquad \text{in } D\,. \tag{6}$$

This is completely elementary. The question is, however, which consequences can be drawn—and have been drawn—from condition (6), namely, under which conditions on its generator(s) is a linear function a first integral? As we shall see, there are essentially two types of answers: one in terms of the unique horizontal generator, the other in terms of a larger (and in a sense, the largest) class of generators. Sections 6 and 7 are devoted to the former approach, which has received most of the attention, and sections 8 and 9 are devoted to the latter, which is more recent.

6 Nonholonomic Noether theorem 1: horizontal generators

To our knowledge, the first attempt to characterize linear first integrals through their horizontal generators appears in a 1956's paper by Agostinelli [2], even though with some errors. A correct formulation appears in a paper by Iliev from the 1970's.

Theorem 1. [25] *A linear function on D is a first integral of a nonholonomic system (L,Q,D) if and only if its horizontal generator $Z_{\mathcal{D}}$ is such that*

$$Z_{\mathcal{D}}^{TQ}(L)|_D \;=\; 0\,. \tag{7}$$

The proof follows immediately from the balance equation (5) and the fact that $R(q,\dot q)\in\mathcal{D}_q^\circ$. Note that, since the Lagrangian is assumed to be of the form $L=T-V$ with kinetic energy T which is quadratic in the velocities and potential energy V which is independent of the velocities, condition $Z_{\mathcal{D}}^{TQ}(L)|_D=0$ is equivalent to the two conditions

$$Z_{\mathcal{D}}(V)=0\,, \qquad\qquad Z_{\mathcal{D}}^{TQ}(T)|_D \;=\; 0\,.$$

This is the way Iliev and Agostinelli state this result (even though Agostinelli erroneusly requires the vanishing of $Z_{\mathcal{D}}^{TQ}(T)$ in all of TQ). Their work seems to have passed largely unobserved.

However, more general results have been independently rediscovered since then: characterizations of *all* first integrals (not only those linear in the momenta) and to *any* Lagrangian (not only kinetic minus potential) have been given in [6, 17, 20] and have been further generalized to nonlinear constraints in [15, 33]. We shortly describe here the formulation of [33] which, as we have already mentioned, is the only one we are aware of that regards first integrals as defined on the constraint manifold.

We note preliminarily that Theorem 1 can obviously be regarded as a characterization of the vector fields Z on Q which generate linear first integrals. If $\Pi_{\mathcal{D}}^{\mathcal{A}}$ denotes the $\mathcal{A}$–orthogonal projector onto the fibers of $\mathcal{D}$, then $\Pi_{\mathcal{D}}^{\mathcal{A}}Z$ is the horizontal generator of $Z\cdot p|_D$ and Theorem 1 can be restated as: *a vector field Z on Q generates a linear first integral of (L,Q,D) if and only if*

$$(\Pi_{\mathcal{D}}^{\mathcal{A}}Z)^{TQ}(L)|_D = 0\,. \tag{8}$$

In the spirit of the Hamiltonian version of Noether theorem mentioned in Section 2, an extension of this condition to nonlinear first integrals can be obtained by passing to the Hamiltonian formulation, either on T^*Q or on TQ, and replacing the lift of the projection of the generator Z with an appropriate projection of the Hamiltonian vector field of the nonlinear first integral. This is, roughly speaking, the statement of [33].

More precisely, since the system and the first integral are defined only on the points of the constraint manifold D, one should consider the restriction to D of a certain projection (that we do not specify here) of the Hamiltonian vector field of an extension of the first integral off D; the resulting vector field is called 'nonholonomic Hamiltonian vector field' [33]. Then, [33] proves that *a function on D is a first integral of the nonholonomic system if and only if its nonholonomic Hamiltonian vector field preserves the energy on D* [33].

We refer to the cited references for all the details, particularly those regarding the definition of the projection involved, and to [18, 16] for a reformulation in terms of 'distributional' vector fields. We limit ourselves to remark that this result is equivalent to condition (8) in the case of linear first integrals and linear constraints, because, as it is easy to control, the nonholonomic Hamiltonian vector field of $Z \cdot p$ coincides in that case with the restriction to D of the lifted vector field $(\Pi^A_{\mathcal{D}} Z)^{TQ}$.

7 Nonholonomic Noether theorem 2: horizontal actions

We consider now Noether theorem from the other point of view, that is, in terms of group actions.

Consider an action $\Psi : G \times Q \to Q$, $(g, q) \mapsto \Psi_g(q)$ and denote by Z_η the infinitesimal generator corresponding to an element η of the Lie algebra $\mathfrak{g}$ of G. The action Ψ lifts to an action Ψ^{TQ} of G on TQ which in bundle coordinates is

$$\Psi^{TQ}_g(q, \dot q) \;=\; \Big(\Psi_g(q)\,,\; \frac{\partial \Psi_g}{\partial q}(q)\dot q\Big)\,.$$

For any $\eta \in \mathfrak{g}$, the infinitesimal generator of the lifted action corresponding to η coincides with the tangent lift Z^{TQ}_η of Z_η and the η–component of the momentum map of the lifted action is

$$J_\eta \;=\; Z_\eta \cdot p$$

(the momentum map $J : TQ \to \mathfrak{g}^\star$ satisfies $\langle J, \eta\rangle = J_\eta$, see e.g. [4, 1]). If the Lagrangian L is invariant under the lifted action, namely $L \circ \Psi^{TQ}_g = L$ for all $g \in G$, then it is preserved by the infinitesimal generators of this action, that is, $Z^{TQ}_\eta(L) = 0$ for all $\eta \in \mathfrak{g}$. Under this condition, Proposition 1 shows that, given $\eta \in \mathfrak{g}$, $J_\eta|_D$ is a first integral of the nonholonomic system (L, Q, D) if and only if

$$R \cdot Z_\eta \;=\; 0 \qquad \text{on } D\,. \tag{9}$$

According to the restatement of Theorem 1, this condition is equivalent to

$$(\Pi^A_{\mathcal{D}} Z_\eta)^{TQ}(L)|_D \;=\; 0\,. \tag{10}$$

The simplest consequence of condition (9) is that, *if L is Ψ^{TQ}–invariant, then any horizontal infinitesimal generator of the action Ψ has a conserved momentum.* To our knowledge, this statement was first given by Kozlov and Kolesnikov in [28]; since then, it has appeared a number of times in the literature; an inevitably non–exhaustive list include [3, 8, 13, 15, 14, 16]. This statement has been sometimes identified as the 'nonholonomic Noether theorem' but, for reasons which will become clear in the sequel, we will refer to it as to the 'horizontal' nonholonomic Noether theorem. Note that it is immediate to generalize this result to the conservation of the momentum map of groups which act horizontally on Q—that is, the tangent spaces to the group orbits lie in the fibers of the constraint distribution (see e.g. [29]).

Clearly, horizontality of an infinitesimal generator is only a sufficient condition for it to have a conserved momentum. Condition (10) gives instead a characterization of the infinitesimal generators with this property, but through a property of their horizontal component. Is it possible to give a more direct characterization?

8 The role of the reaction forces

The approach described so far has the two features of focusing on the horizontal generators and of disregarding the reaction forces, which in fact do not appear in any statement. These two features are linked to each other. As we have remarked already several times, if Z^{TQ} preserves the Lagrangian on the constraint manifold D, then the balance equation (5) shows that $Z \cdot p|_D$ is a first integral if and ony if at each point $q \in Q$,

$$Z(q) \cdot R(q, \dot q) \ = \ 0 \qquad \forall \dot q \in \mathcal{D}_q \,. \tag{11}$$

Since, as we already remarked, d'Alembert principle ensures that the reaction force $R(q, \dot q)$ annihilates $\mathcal{D}_q$, the horizontality of Z is obviously a sufficient condition for (9).

The reason why horizontality is not necessary is that condition (11) has to be tested only for velocities which belong to $\mathcal{D}_q$, not for all velocities in T_qQ. In order to gain some insight into this question, we recall the exact assumptions in d'Alembert principle. Given a configuration manifold Q and a nonholonomic constraint with constraint distribution $\mathcal{D}$, d'Alembert principle consists of the assumption that the set of reaction forces that the constraint can exert when the system is in the configuration $q \in Q$ consists of *exactly all* vectors of the annihilator $\mathcal{D}_q^\circ$. The reason for requiring that the constraint is capable of exerting all these reactions is that, if suitable external potentials are applied and suitable initial conditions are selected, then the horizontal component of the 'lost forces' may be arbitrary horizontal vectors.

However, for a given system, and particularly for a given potential energy V, typically only a subset of these reaction forces is actually exerted. This is clearly seen on expression (4) of the reaction force,

$$R \ = \ \Pi_{\mathcal{D}^\circ}^{A^{-1}} (\beta + V') - S^T (S A^{-1} S^T)^{-1} \gamma$$

because if we let the potential energy V vary among all functions on Q then, at each point $q \in Q$, the quantity $\Pi_{\mathcal{D}^\circ}^{A^{-1}} V'(q)$ span the entire range $\mathcal{D}_q^\circ$ of the projector $\Pi_{\mathcal{D}^\circ}^{A^{-1}}$. But for fixed V and A, the map

$$R(q, \,\cdot\,) : \mathcal{D}_q \ \to \ \mathcal{D}_q^\circ \,, \qquad \dot q \mapsto R(q, \dot q) \,,$$

need not be surjective and the image under $R(q, \,\cdot\,)$ of $\mathcal{D}_q$, namely the set

$$\mathcal{R}_q \ := \ R(q, \mathcal{D}_q) \ = \ \bigcup_{\dot q \in \mathcal{D}_q} R(q, \dot q) \,,$$

may be a proper subset of $\mathcal{D}_q^\circ$.

Since the restriction of R to each fiber of $\mathcal{D}$ is a nonlinear map, the sets $\mathcal{R}_q$ need not be linear or affine subspaces of T_q^*Q. Nevertheless, the annhilators $\mathcal{R}_q^\circ \subset T_qQ$ of these sets are linear spaces and are thus the fibers of a distribution $\mathcal{R}^\circ$ on Q, possibly of non–constant rank and non–smooth. Since the space $\mathcal{R}_q^\circ$ contains all tangent vectors $\dot q \in T_qQ$ which annihilate all possible values of the reaction forces on constraint motions through q, $\mathcal{R}^\circ$ was called the *reaction–annihilator distribution* [22]. Clearly

$$\mathcal{R}_q^\circ \supseteq \mathcal{D}_q \,, \qquad q \in Q \,.$$

For further analysis of the reaction–annihilator distribution, and some examples, see [22].

9 Nonholonomic Noether theorem 3: generators in $\mathcal{R}^\circ$

The analysis of the previous section suggests that the generators of linear first integrals of a nonholonomic system which have a natural 'mechanical' role are not the sections of $\mathcal{D}$, but the sections of $\mathcal{R}^\circ$. In fact:

Theorem 2. [22] *Given a nonholonomic system (L,Q,D) and a smooth vector field Z on Q, any two of the following three conditions imply the third:*

C1. *Z is a section of $\mathcal{R}^\circ$*

C2. $Z^{TQ}(L)|_D = 0$

C3. *$Z \cdot p|_D$ is a first integral of (L,Q,D).*

Proof. Write $F = p \cdot Z|_D$. The balance equation (5) gives $\dot F = Z^{TQ}(L) + R \cdot Z$. Thus, at each point $q \in Q$, the vanishing in all of $\mathcal{D}_q$ of any two quantities among $\dot F$, $Z^{TQ}(L)$ and $R \cdot Z$ implies the vanishing in all of $\mathcal{D}_q$ of the third. Since $\mathcal{R}_q = R(q, \mathcal{D}_q)$, the vanishing of $R(q,\dot q) \cdot Z(q)$ in all of $\mathcal{D}_q$ amounts to $Z(q) \in \mathcal{R}^\circ_q$. ■

This result clarifies that, among all the generators of a linear first integral, those which are sections of $\mathcal{R}^\circ$ (condition C1) are exactly those whose tangent lift preserves the Lagrangian in the constraint manifold (condition C2). The class of these generators is not empty, given that it always contains the generator which is a section of $\mathcal{D}$. From this point of view, however, the unique horizontal generator has no special role. In fact, Theorem 1 is a particular case of the following consequence of Theorem 2:

Corollary 1. [22] *A linear function on D is a first integral of (L,Q,D) if and only if the lift of one (and then of any) of its generators which are sections of $\mathcal{R}^\circ$ satisfies condition C2 of Theorem 2.*

Assume now that the fibers of the reaction–annihilator distribution are strictly larger than those of the constraint distribution, $\mathcal{R}^\circ_q \supset \mathcal{D}_q$ for all $q \in Q$. In view of Lemma 1, the consideration of generators which are sections of $\mathcal{R}^\circ$, rather than sections of $\mathcal{D}$, cannot enlarge in any way the class of functions which are linear first integrals. However, in such a case, if a symmetry group of the Lagrangian is given, then the consideration of the distribution $\mathcal{R}^\circ$ exactly identifies the class of conserved components of the momentum map, thus providing a version of Noether theorem which goes beyond the 'horizontal' one:

Theorem 3. [22] *Consider an action Ψ of a Lie group G on Q and assume that the Lagrangian L is Ψ^{TQ}–invariant. Then, for any $\eta \in \mathfrak{g}$, $J_\eta|_D$ is a first integral of (L,Q,D) if and only if Z_η is a section of $\mathcal{R}^\circ$.*

Since the infinitesimal generators of the group action are tangent to the group orbits, conserved components of the momentum map are due to those infinitesimal generators which are sections of the distribution with fibers $\mathcal{G}_q \cap \mathcal{R}^\circ_q$, not only of the distribution with fibers $\mathcal{G}_q \cap \mathcal{D}_q$. (Here, $\mathcal{G}$ is the distribution on Q whose fibers are the tangent spaces to the orbits of the action Ψ). If the fibers of $\mathcal{R}^\circ$ properly contain those of $\mathcal{D}$, it is thus possible that a nonholonomic system with an invariant Lagrangian has conserved components of the momentum map which are not due to horizontal infinitesimal generators.

This mechanism may be particularly effective in cases such as the Chaplygin systems and, more generally, the 'purely kinematics' case [8, 15, 14, 12], for which $\mathcal{D}_q \cap \mathcal{G}_q = \{0\}$ while $\mathcal{R}^\circ_q \cap \mathcal{G}_q$ may be non trivial. Some examples are given in [22, 21].

Remarks: (1) The symmetry properties of a nonholonomic system which are relevant to the existence of first integrals are only those of the Lagrangian, not those of the constraint distribution. This fact has been remarked several times, see e.g. [2, 29]. However, the invariance of the constraint manifold under the lifted action is necessary if reduction is concerned (see [6, 8, 33, 13, 19, 29, 14]). Thus, when establishing which infinitesimal generators of a given group action allow reduction one should focus on those which are horizontal, but when judging which infinitesimal generators produce conserved momenta one should consider those which are sections of $\mathcal{R}^\circ$.

(ii) The analogue of the distribution $\mathcal{R}^\circ$ for non–lifted actions on TQ has not yet been studied. For a related construction in the framework of Dirac manifolds see [32].

10 The gauge mechanism

Soon after the investigations of what we called here the horizontal Noether theorem began, it was noticed that several sample nonholonomic systems with Lagrangian invariant under a lifted action have linear first integrals which are not components of the momentum map. In [5] it was pointed out that the unique horizontal generator Z of some of these linear first integrals is tangent to the group orbit. Therefore, even though Z is not an infinitesimal generator of the action, at each point $q \in Q$ it coincides with an infinitesimal generator of the action, but this generator changes from point to point:

$$Z(q) \;=\; Z_{\tilde\eta(q)}$$

for some map $\tilde\eta : Q \to \mathfrak{g}$. For this reason, this fact was interpreted in [5] as indicating that the first integral is related to the group action in a gauge–like way.

In fact, [5] viewed this idea more as a technique to construct first integrals: one starts with a pointwise linear combination of infinitesimal generators, $Z = \sum_i f_i Z_{\eta_i}$ for a basis $\eta_1, \dots, \eta_k$ of the Lie algebra of the group, and tries to determine the functions $f_i : Q \to \mathbb{R}$ so that Z is horizontal and $Z^{TQ}(L)|_D = 0$. Then, by Theorem 1, Z generates a linear first integral, which equals $\sum_i f_i J_{\eta_i}$. A variant of this method consists in considering linear combinations of the lifts $Z^{TQ}_{\eta_1}, \dots, Z^{TQ}_{\eta_k}$ of the infinitesimal generators. Partially similar ideas appeared e.g. in [15, 29, 23, 14, 35].

Note that, based on Theorem 2, this 'gauge' method obviously extends to sections of $\mathcal{R}^\circ$, see [21]. Whether the gauge method should be considered as a fundamental mechanism which links symmetries and conservation laws is presently unclear.

Remark: As was shown in [21], this gauge mechanism is closely related (equivalent, if e.g. the action is locally free) to the so–called 'momentum equation' and 'nonholonomic momentum map' [8, 7, 12, 13, 14, 15, 34, 16, 32]. This method, too, generalizes to sections of $\mathcal{R}^\circ$. For further details see [21].

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