Periodic flows, rank–two Poisson structures, and nonholonomic mechanics

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Abstract

It has been recently observed that certain (reduced) nonholonomic systems are Hamiltonian with respect to a rank–two Poisson structure. We link the existence of these structures to a dynamical property of the (reduced) system: its periodicity, with positive period depending continuously on the initial data. Moreover, we show that there are in fact infinitely many such Poisson structures and we classify them. We illustrate the situation on the sample case of a heavy ball rolling on a surface of revolution.

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1 Introduction and statement of results

A. Introduction. The search for invariant geometric structures in nonholonomic mechanics dates back to the early days of the subject. Invariant measures and, typically after the introduction of a Chaplygin's 'reducing multiplier' [13], symplectic structures are encountered in reduced systems. In general, however, nonholonomic mechanical systems possess only an almost–Hamiltonian structure, that is, the vector field is Hamiltonian with respect to a nondegenerate (but not necessarily closed) two–form [4, 12, 25]. A crucial difference with respect to the standard Hamiltonian case resides in the fact that such a two–form is not invariant under the flow, an instance which is rich of consequences (e.g., no period–energy relation for periodic flows, no natural analogue of the action variables for quasi–periodic flows [19, 35], etc.)

It therefore appeared of a certain interest when Borisov, Mamaev and Kilin recently discovered that a number of classical nonholonomic systems with symmetry give rise to reduced systems which are Hamiltonian with respect to a Poisson structure of rank two [7, 8, 9]. More precisely, these Poisson tensors were there produced for suitably time–rescaled systems but, as was pointed out in [33], the time rescaling is not necessary. In the quoted references, rank–two Poisson structures are proven to exist for a variety of cases in which a heavy body rolls on a surface, including Routh sphere, the disk rolling on a plane, and the sphere rolling inside a convex surface of revolution. (See e.g. [34, 32, 11, 14, 22, 37] for background on these systems). In all these cases, the reduced systems have some common features: their phase space is four–dimensional, they possess three independent integrals of motion, and their dynamics is periodic (at least in some open dense subset of the phase

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space). One of the integrals of motion (the energy) is the Hamiltonian of the reduced system and the other two emerge as Casimirs of the Poisson structure. A detailed study of the form of these Poisson tensors in a number of cases has been given in [33].

This situation deserves an explanation. One may wonder whether these Poisson structures have any natural origin—e.g., are they unique? Can their consideration shed some light on the properties of the systems? Can they be linked to any geometric properties of the unreduced systems—e.g., is the unreduced system a Poisson system, too, perhaps with a Poisson structure which projects onto the reduced one [9]?

In this paper we give some answers to these questions—in negative. In fact, we shall show that rank–two Poisson structures for these reduced systems are *far not unique* and that their existence can be linked to a very strong *dynamical* property of the reduced systems: the fact that the flow is periodic, with period which is a continuous function of the initial data.

Roughly speaking, the geometric reason is that if all the orbits are periodic, then there is plenty of freedom in grouping them together so as to form two–dimensional surfaces which foliate the manifold, carry a (leafwise) symplectic structure, and on which the flow is Hamiltonian. The condition that the rank of the Poisson structure is two is crucial: since any area form on a two–dimensional surface is closed, it is easy to endow the leaves of a two–dimensional foliation with a symplectic structure. For instance, if the system is Hamiltonian with respect to an almost symplectic structure one can simply use its restriction to the leaves of the considered foliation. However, as we shall show, there is in fact much more freedom.

On the other hand, as we shall shortly discuss in the Conclusions, the situation might be completely different for systems with quasi–periodic dynamics. No global Poisson structures is expected in such cases.

B. Fibrating–periodic flows. We give now precise statements of our results. For shortness, we will tacitly assume that all objects we are dealing with are smooth—we will in fact stress smoothness only in those few cases in which there might be ambiguity.

We consider a vector field X on a manifold M with only periodic orbits. At first, we also assume that the vector field is everywhere nonzero, so that the (minimal) period of all the orbits is positive. This can be achieved simply removing from phase space the zeros of X.

Our analysis rests in a crucial way on the hypothesis that the orbits of X are the fibers of a locally trivial fibration of M. If this happens we say that the vector field has *fibrating–periodic flow.* We shall discuss this condition in Section 2. Specifically, Theorem 4 characterizes the fibrating property in a variety of ways, two of which are in our opinion of particular interest since they indicate that the fibrating property is not infrequent, among vector fields with periodic flows, and provide practical ways of verifying it:

- The continuity of the period as a function of the initial data.
- The fact that the periodic orbits are the connected components of the level sets of some submersion.

(The reason why we prefer referring to the fibrating property rather than to the continuity of the period, which is dynamically more intuitive, is that this way will simplify the organization of Section 2).

C. Existence of rank–two Poisson structures. Our first result about the existence of Poisson structures is that any vector field with fibrating–periodic flow admits at least one Poisson structure which makes it Hamiltonian with any preassigned Hamiltonian H. The only restrictions on the function H are, obviously, that it must be a first integral of X and, since X is never zero, must have no critical points:

Theorem 1 *Assume that a vector field X on a manifold M of dimension ≥ 2 has fibrating–periodic flow and possesses a first integral $H : M \to \mathbb{R}$ with no critical points. Then there is a rank–two Poisson tensor Λ on M such that $\Lambda^\sharp(dH) = X$.*

This theorem is proven in Section 3. In the case of the nonholonomic systems we are interested in, the natural choice for the Hamiltonian H is the reduced energy. In view of the almost–Hamiltonian structure of the reduced system, the reduced energy has in fact no critical points if restricted to the subset of the phase space where X is nonzero.

D. Classification. Next, we consider the uniqueness question: once the Hamiltonian H has been fixed, how much freedom is there in the choice of a rank–two Poisson tensor Λ which makes X Hamiltonian with that Hamiltonian? Note that if such a structure exists, then necessarily there is a foliation of M which plays the role of the symplectic foliation of Λ. Hence its leaves are two–dimensional, invariant under the flow of X, and transversal to the level sets of H. As it turns out, these obvious necessary conditions are also sufficient for the existence of a rank–two Poisson structure, and in fact completely determine it:

Theorem 2 *Let X be a vector field on a manifold M of dimension $n \geq 2$ with fibrating–periodic flow. Assume that there exist:*
 i. A first integral $H : M \to \mathbb{R}$ of X which has no critical points.
 ii. A foliation Σ of M with leaves of dimension two, invariant under the flow of X and transversal to the level sets of H.
Then there is a unique rank–two Poisson tensor Λ in M with symplectic foliation Σ and such that $\Lambda^\sharp(dH) = X$.

Theorem 2 is proven in Section 3, where we also shortly discuss its relationship with Theorem 1. Note that Theorem 2 assumes the existence of both the function H and the foliation Σ. Therefore, in view of Theorem 2, Theorem 1 states that, once H is given, there exists at least one foliation Σ of M which has the properties needed to produce a Poisson tensor. However, such a foliation is not unique. This fact will appear clear from the proof of Theorem 1 and is also demonstrated with an elementary but significative example:

Example: Let X be a vector field with fibrating–periodic flow. Assume that X has $n-1$ global first integrals $(H, J_1, \ldots, J_{n-2})$, which are everywhere functionally independent in M.[1] Thus, H has no critical points and the level sets of the map $J = (J_1, \ldots, J_{n-2})$ are the leaves of a foliation Σ_J of M which satisfies condition ii. of Theorem 2. Therefore, there is a rank–two Poisson structure Λ_J which is such that $\Lambda_J^\sharp(dH) = X$ and has symplectic foliation Σ_J. Considering instead the map $J' = (J_1 + H, J_2, \ldots, J_{n-2})$ produces a *different* foliation Σ' of M which satisfies condition ii. and hence a *different* Poisson structure $\Lambda_{J'}$ such that $\Lambda_{J'}^\sharp(dH) = X$. As we shall see, this situation is encountered in the nonholonomic system of Section 4.

Remarks: (i) Theorems 1 and 2 globalize various local and 'semilocal' known results. ('Semilocal' means 'defined in an X–invariant neighbourhood', that is, a small neighbourhood of a periodic orbit).

(i.a) The existence of (infinitely many) *local* rank–two Poisson structures adapted to *any* two–dimensional foliation is well known. Specifically, if $\mathcal{F}$ is a foliation with two–dimensional leaves of a manifold M, then any point of M has a neighbourhood U equipped with a family of rank–two Poisson structures which are multiple of each other and all have the restriction of $\mathcal{F}$ to U as symplectic foliation [21]. It is easy to see that the assignment of the vector field X and of the first integral H as in Theorem 2 fixes uniquely such a multiplicative constant.

(i.b) While completing this work, we discovered that a similar, but intrinsically semilocal result had appeared recently in [20]. Specifically it is there proven that, under certain conditions, a system with quasi–periodic flow on k–dimensional tori is Hamiltonian with respect to Poisson structures of rank $2k$ which are defined in neighbourhoods of the invariant tori. In the periodic case, this is essentially a semilocal version of Theorem 2. We stress that globality can be achieved

[1] This condition also implies that, if X has periodic flow, then it is fibrating; see Theorem 4.

in Theorem 2 because of the periodicity of the flow: a similar result is not true, in general, for quasi–periodic flows (see the Conclusions).

(ii) If $\dim M = 2$, Theorem 1 ensures the existence of a (global) symplectic structure on M which makes the vector field Hamiltonian. The existence of symplectic structures, detected on a case–by–case basis, is displayed in the study of several two–dimensional systems with periodic flows arising in various contexts (see e.g. [14, 24]).

E. Nonholonomic mechanics, extensions, and open problems. As an illustrative example from nonholonomic mechanics, in Section 4 we consider the case of the heavy ball which rolls without sliding inside a convex surface of revolution. As we have already mentioned, a rank–two Poisson structure has been found in [8, 33] for an $\mathrm{SO}(3) \times S^1$ reduction of this system. On the other hand, the reduced system is known to have periodic flow [22, 23, 37]. We shall therefore show that the reduced flow is fibrating–periodic. Thus, Theorems 1 and 2 apply and link the Poisson structure of [8, 33] to the periodicity. Moreover, we shall also emphasize the non–uniqueness of Poisson structures.

To be precise, however, the setting of Theorems 1 and 2 is still insufficient for a full explanation of the existence of the Poisson tensor exhibited in [8, 33] for this reduced system. The fact is that such a Poisson tensor exists in the entire (regularly) reduced manifold, including the equilibria, which are instead excluded by our hypothesis that the vector field does not vanish. In order to fulfill this gap we show that, under certain rather natural hypotheses, the Poisson structures of Theorem 2 extend to equilibria. Roughly, the idea is that, if the equilibria form a codimension–two submanifold and are transversally elliptic, then the two–dimensional surfaces interpolating the periodic orbits and forming the symplectic leaves can be smoothly extended to include them too:

Theorem 3 *Let X be a vector field on a manifold M of dimension $n \geq 2$. Assume that there exists a first integral $H : M \to \mathbb{R}$ of X and a foliation Σ of M with leaves of dimension two, invariant under the flow of X, and transversal to the level sets of H. Assume moreover that*

i. The zeros of X form a codimension–two submanifold $\mathcal{E}$ of M which intersects transversally the leaves of Σ.

ii. In $M \setminus \mathcal{E}$, X is fibrating–periodic and H has no critical points.

iii. For each leaf F of Σ and for each point $p \in F \cap \Sigma$, $H|_F$ has a Morse critical point at p and the linearization of $X|_F$ has purely imaginary nonzero eigenvalues at p.

Then there is a unique rank–two Poisson tensor $\widetilde{\Lambda}$ in M which is such that $\widetilde{\Lambda}^\sharp(X) = dH$ and which has Σ as symplectic foliation.

The proof of this Theorem is a simple application of the Morse–Bott lemma and is given at the end of Section 3. At the end of Section 4 we shall verify that this theorem applies to the case of the sphere in the convex surface of revolution.

We would like to remark that the presence of a codimension–two transversally elliptic equilibrium set as in Theorem 3 does not enforce the uniqueness of the Poisson tensor $\widetilde{\Lambda}$. In fact, the key requirement for the validity of Theorem 3 is that the leaves of Σ, which are formed by interpolating the periodic orbits, should intersect transversally $\mathcal{E}$—and it is clear that this can ordinarily be made in a number of ways.

In the Conclusions we shall shortly discuss some related questions and generalizations, such as the (non) existence of Poisson structures of higher rank and the (non) existence of Poisson structures for quasi–periodic flows.

2 Fibrating–periodic flows

A. Semilocal description. As mentioned, our analysis deals with a vector field X on an n–dimensional manifold M which has 'fibrating–periodic flow': X is everywhere nonzero, its orbits are periodic and are the fibers of a fibration

$$\pi \,:\, M \;\to\; B$$

with fiber S^1, where B is a manifold of dimension $(n-1)$. By 'fibration' we mean a locally trivial smooth fibration: each point of B has a neighbourhood U and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times S^1$ such that $\pi \circ \Psi^{-1} : U \times S^1$ is the projection on the factor U, see e.g. [1].

Part of our study will be based on the use of local trivializations. Thus, we begin by stating some elementary properties of fibrating–periodic vector fields:

Lemma 1 *Assume that X has fibrating–periodic flow. Then*

i. For each point $m \in M$ there exists a neighbourhood V of the orbit through m and a diffeomorphism $(u, \varphi) : V \to \mathcal{U} \times S^1$, where $\mathcal{U} \subset \mathbb{R}^{n-1}$, such that

$$X_{|_V} \;=\; (\omega \circ u)\,\frac{\partial}{\partial \varphi}$$

for some smooth function $\omega : \mathcal{U} \to \mathbb{R}$. (Any such diffeomorphism will be called a semilocal system of trivializing coordinates*.)*

ii. If (u, φ) and (u', φ') are two semilocal systems of trivializing coordinates, then in the intersection of their domains V and V' (if nonempty)

$$u' = u'(u)\,, \qquad \varphi' = \pm\,\varphi + \mathcal{F}(u)$$

for some smooth function $\mathcal{F} : V \cap V' \to \mathbb{R}$.

Proof. (i) Consider a local trivialization $\Psi : \pi^{-1}(U) \to U \times S^1$ with U containing $\pi(m)$. Restricting U if necessary, equip it with coordinates $\tilde{u} : U \to \mathcal{U} \subset \mathbb{R}^{n-1}$. Then $(\tilde{u}, \mathrm{id}) \circ \Psi$ is a diffeomorphism from $\pi^{-1}(U)$ onto $\mathcal{U} \times S^1$, that we denote (u, x). Since X is tangent to the fibers of π and never vanishes, $X = \tilde{\omega}(u, x)\frac{\partial}{\partial x}$ for some nowhere vanishing smooth function $\tilde{\omega}$. Thus, the function $\omega(u) := 2\pi\big[\int_0^{2\pi} \tilde{\omega}(u, x)^{-1} dx\big]^{-1}$ is smooth. Hence, if we define

$$\varphi(u, x) \;:=\; \int_0^x \frac{\omega(u)}{\tilde{\omega}(u, x')}\, dx'$$

we obtain that $(u, x) \mapsto (u, \varphi(u, x))$ is a diffeomorphism of $\mathcal{U} \times S^1$ onto itself which is such that $X = \omega(u)\frac{\partial}{\partial \varphi}$.

(ii) Clearly $u' = u'(u)$ because they are first integrals of X. From $\dot{\varphi} = \omega(u)$ and $\dot{\varphi}' = \omega'(u')$ it follows that the function $\varphi'(u, \varphi)$ satisfies $\frac{\partial \varphi'}{\partial \varphi}(u, \varphi)\omega(u) = \omega'(u'(u))$. Hence $\frac{\partial \varphi'}{\partial \varphi}$ is independent of φ. Thus, for fixed u, $\varphi \mapsto \varphi'(u, \varphi)$ is an affine diffeomorphism of S^1. ■

The local coordinates $u = (u_1, \ldots, u_{n-1})$ are (semilocal) first integrals of X. Conversely, any set of $n-1$ (semilocal) independent first integrals of X can clearly be used as u–coordinates in a set of semilocal trivializing coordinates.

B. Global characterizations. We now characterize the fibrating property of a periodic flow in a number of ways, some of which will be used below. In our opinion, some of these characterizations are of some interest in themselves and show that the fibrating property is quite common among periodic flows.

Theorem 4 *Let X be a smooth vector field on an n–dimensional manifold M which has periodic flow, with everywhere nonzero period. Then the following conditions are equivalent:*

i. *X has fibrating–periodic flow.*

ii. *The orbits of X are the connected components of the level sets of a submersion $f : M \to N$, where N is an $(n-1)$–dimensional manifold.*

iii. *In a neighbourhood of each orbit of X there exist $n-1$ everywhere functionally independent smooth first integrals of X.*

iv. *The flow of X admits a slice*[2] *in a neighbourhood of each point of M.*

v. *The period of the flow of X is a continuous function of the initial data.*

vi. *The period of the flow of X is a smooth function of the initial data.*

vii. *The orbit space of X is a smooth manifold and the projection map defines an S^1–principal bundle.*

viii. *In a neighbourhood of each orbit of X there exist semilocal trivializing coordinates.*

Proof. *i.* $\Longrightarrow$ *ii.* Obvious: $f = \pi$ and $N = B$.

ii. $\Longrightarrow$ *iii.* Since f is a submersion, for each $m \in M$ there exists a codimension–one submanifold $\mathcal{S}$ of M which contains m and is such that $f|_{\mathcal{S}} : \mathcal{S} \to f(\mathcal{S})$ is a diffeomorphism. Restricting $\mathcal{S}$ if necessary, we can assume that $f(\mathcal{S})$ has global coordinates $y_1, \ldots, y_{n-1}$. Since for each t the time–t map Φ_t^X of the flow of X is a diffeomorphism, the set $\tilde{\mathcal{S}} = \bigcup_{t>0} \Phi_t^X(\mathcal{S})$ is an open neighbourhood of the orbit through m. Since the fibers of f are invariant under the flow of X, $f(\tilde{\mathcal{S}}) = f(\mathcal{S})$ and $y_1 \circ f, \ldots, y_{n-1} \circ f$ are functionally independent first integrals defined in $\tilde{\mathcal{S}}$.

iii. $\Longrightarrow$ *iv.* Any codimension–one submanifold on which the first integrals have rank $n-1$ is a slice.

iv. $\Longrightarrow$ *v.* Let $\mathcal{S}$ be a slice through a point $m \in M$. Since $\mathcal{S}$ is a Poincaré section and the flow is periodic, the first–return time coincides with the period. But the first–return time is smooth [31], hence continuous.

v. $\Longrightarrow$ *vi.* Take any point $m \in M$ and, around it, a smooth local Poincaré section $\mathcal{P}$ of the flow (that is, a codimension one submanifold which is everywhere transversal to the orbits of X; its existence is granted by the fact that, by hypothesis, $X(m) \neq 0$). Since the flow of X is periodic, by restricting it if necessary, we may assume that $\mathcal{P}$ intersects the orbit through m only in m. Hence, the first return time $F(m)$ of the point m and the period $T(m)$ of the orbit through m coincide, $F(m) = T(m)$. We now show that, after possibly restricting again $\mathcal{P}$, the same happens at all other points of $\mathcal{P}$. By contradiction, assume there is a sequence of points $m_k \to m$ such that $F(m_k) \neq T(m_k)$. Since the first return time is an integer multiple of the period, this implies that $F(m_k) \geq 2T(m_k)$. Since the first return time is smooth [31] and the period is continuous by hypothesis, in the limit we get $F(m) \geq 2T(m)$. Thus, $T(m') = F(m')$ for all $m' \in \mathcal{P}$ sufficiently close to m. The smoothness of the period now follows from that of the first–return time.

vi. $\Longrightarrow$ *vii.* Since the period T is smooth and nonzero, the rescaled vector field X/T defines a free smooth action of S^1 on M. Since S^1 is compact, such an action is also proper. Hence the quotient space is a differentiable manifold. Moreover, a free proper action of S^1 defines a principal bundle [17].

vii. $\Longrightarrow$ *i.* This is obvious.

i. $\Longrightarrow$ *viii.* $\Longrightarrow$ *iii.* The first implication follows from Lemma 1. The second implication is obvious. ■

Remarks: (i) The submersion $f : M \to N$ entering condition *ii.* needs not be the fibration $\pi : M \to B$ by the periodic orbits. Of course, the periodic orbits are the connected components of the fibers of f, which may however be union of (a possibly nonconstant) number of them. In

[2] A slice, or local cross section, of a foliation with leaves of dimension r is a submanifold of codimension r which intersects each leaf in at most one point, and the intersection is transversal.

practice, f could consist of $n-1$ (global) independent first integrals of X; this condition may be particularly simple to check (see e.g. Section 4).

(ii) The fact that the fibrating property is implied by the existence of a submersion $f: M \to N$, $\dim N = \dim M - 1$, is apriori not obvious because the connected components of a submersion with compact fibers need not be the fibers of a fibration. For this to be true, some further property should be satisfied, e.g. the submersion should be proper (Ehresmann theorem) or its compact fibers should all have the same number of connected components, see [30] for a recent general study of the problem. In our case, the connected components of the submersion are the orbits of a nowhere zero vector field.

(iii) The prototype of periodic flow which has non–continuous period, and which is not fibrating, is offered by two uncoupled harmonic oscillators with frequency ratio $2:1$. On each energy level all orbits are periodic with equal period T but one, which has period $T/2$. Moreover, no slice of the foliation by the periodic orbits exists in a neigbourhood of the points of these special orbits, so that the foliation is not a fibration.

3 Periodic flows and rank–two Poisson structures

A. Poisson manifolds. In this section we prove the three Theorems stated in the Introduction. First, we shortly review the basic definitions and facts about Poisson manifolds. For more details, see e.g [27, 29, 36].

A Poisson structure on a manifold is a twice contravariant antisymmetric tensor field Λ such that the Schouten bracket $[\![\Lambda, \Lambda]\!]$ vanishes. At each point $m \in M$, the restriction of $\Lambda_m : T^*_m M \times T^*_m M \to \mathbb{R}$ to the first factor defines a map $\Lambda^\sharp_m : T^*_m M \to T_m M$. The image of $\Lambda^\sharp_m$ is an even–dimensional subspace of $T_m M$, called the characteristic subspace of Λ. The dimension of such a subspace is called the *rank* of Λ. The vanishing of $[\![\Lambda, \Lambda]\!]$ is equivalent to the fact that the characteristic subspaces form a Frobenius integrable distribution of TM. Hence M is foliated into even–dimensional submanifolds. These submanifolds are called the *symplectic leaves* of M because they carry a symplectic structure, which is defined by the restriction of the Poisson structure to the symplectic leaves.

Given a function $f: M \to \mathbb{R}$, the vector field $X_f := \Lambda^\sharp(df)$ is tangent to the symplectic leaves and is called the Hamiltonian vector field of f. Functions with differentials in the kernel of $\Lambda^\sharp$ are called *Casimirs*; they are constant on the symplectic leaves and their Hamiltonian vector fields are identically zero. Since the symplectic leaves need not be embedded submanifolds, global Casimirs need not exist. Nevertheless, if the rank is constant, near each point of M there exist the maximal number of functionally independent *local* Casimirs.

B. Proof of Theorem 1. We use the fact, ensured by Theorem 4, that the fibration $\pi : M \to B$ by the periodic orbits is an S^1–principal bundle. Since S^1 is compact, there exists an S^1–invariant metric g on M (see e.g. [16], Section 8.3). Let ∇H be the gradient of H relative to this metric. Then:

i. ∇H is an S^1–invariant vector field.
ii. X and ∇H are everywhere linearly independent.
iii. The Lie derivative $[\nabla H, X]$ is parallel to X.

Property i. is proven with a straightforward computation, taking into account the fact that H is S^1–invariant, too. Property ii. follows from the fact that, since H has no critical points, ∇H is transversal to the level sets of H while X is tangent to them. Property iii. can be proven with a simple computation, using any semilocal system of trivializing coordinates u, φ as in Lemma 1: since the S^1–invariance of ∇H gives $[\nabla H, \frac{\partial}{\partial\varphi}] = 0$, one has $[\nabla H, X] = [\nabla H, \omega(u)\frac{\partial}{\partial\varphi}] = (L_{\nabla H}\omega)(u)\frac{\partial}{\partial\varphi}$.

The bivector field

$$\Lambda \;=\; g(\nabla H, \nabla H)^{-1}\, \nabla H \wedge X$$

satisfies $\Lambda^\sharp(dH) = X$. In fact, since H is a first integral of X and thus $L_X H = 0$, $\nabla H \wedge X(dH,\cdot) = (L_{\nabla H}H)X - (L_X H)\nabla H = g(\nabla H, \nabla H)X$. The characteristic distribution of Λ is spanned by ∇H and X and is therefore everywhere two–dimensional. It only remains to prove that the Schouten bracket $[\![\Lambda,\Lambda]\!]$ vanishes. Recall that, if X' and X'' are any two vector fields, then

$$[\![X' \wedge X'', X' \wedge X'']\!] \;=\; 2\, X' \wedge X'' \wedge [X', X''] \tag{3.1}$$

see [36]. Since $[\nabla H, X]$ is parallel to X, this shows that $[\![\nabla H \wedge X, \nabla H \wedge X]\!] = 0$. ■

Remark: The construction of Λ depends on the choice of an invariant metric on M and is therefore not unique.

C. Proof of Theorem 2. Let us first prove the uniqueness, given X, H and Σ, of the Poisson tensor Λ as in Theorem 2. To this end, we can work (semi)locally. Since H is a first integral of X with no critical points, we can use it as one of the u–coordinates in a semilocal system of trivializing coordinates near any point of M, see Section 2.A. Denote (H, y, φ) these coordinates and note that, within the coordinate domain, $X = \omega\frac{\partial}{\partial\varphi}$ and the leaves of Σ are the sets $y = $ const. The local representative in these coordinates of any bivector field Λ can be written as

$$\Lambda \;=\; A\frac{\partial}{\partial H} \wedge \frac{\partial}{\partial\varphi} \;+\; B_i\frac{\partial}{\partial H} \wedge \frac{\partial}{\partial y_i} \;+\; D_i\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial\varphi} \;+\; \frac{1}{2}\,E_{ij}\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$$

with real functions $A, B_i, D_i, E_{ij} = -E_{ji}$ $(i, j = 1, \ldots, n-2$ and we understand summation over repeated indexes). We now impose that Λ has characteristic distribution tangent to the leaves of Σ and is such that $\Lambda^\sharp(dH) = X$. Equating $\Lambda^\sharp(dH) = A\frac{\partial}{\partial\varphi} + B_i\frac{\partial}{\partial y_i}$ to $\omega\frac{\partial}{\partial\varphi}$ gives $A = \omega$ and all $B_i = 0$. Imposing $\Lambda^\sharp(d\varphi)$ to be tangent to Σ implies all $D_i = 0$. Imposing $\Lambda^\sharp(dy_i)$ to be tangent to Σ implies that all $E_{ij}\frac{\partial}{\partial y_j}$ are tangent to Σ and hence all $E_{ij} = 0$. Thus $\Lambda = \omega\frac{\partial}{\partial H} \wedge \frac{\partial}{\partial\varphi}$, proving the uniqueness.

Note that $\omega\frac{\partial}{\partial H} \wedge \frac{\partial}{\partial\varphi}$ is a Poisson tensor because its characteristic distribution is Frobenius integrable. Hence, the previous argument proves also the *semilocal* existence of the Poisson tensor with the required properties. As we have already mentioned, a very similar semilocal result is contained in [20]. In order to globalize this result we could cover M with an atlas consisting of semilocal systems of trivializing coordinates as above and, using the transition functions between these local coordinates (see Lemma 1), show that these semilocal tensors match in the right way so as to define a tensor on M. However, in order to facilitate the comparison with Theorem 1, we proceed in a way similar to that of the proof of Theorem 1, which uses a Riemannian metric on M. Of course, we need now a metric adapted to the foliation Σ.

Let Σ be a foliation of M with leaves Σ_x of constant dimension, $x \in M$. By a *leafwise metric on M adapted to* Σ we mean a smooth map $\tilde g$ which associates each point $x \in M$ with a symmetric and positive definite $(2,0)$–tensor $\tilde g(x)$ on $T_x\Sigma_x$. We note the following facts:

1. If Σ is any foliation of M with leaves of constant dimension then there is a leafwise metric on M adapted to Σ. Indeed, a leafwise metric on M adapted to Σ is a smooth section of the bundle of symmetric and positive definite $(2,0)$–tensors on the leaves of Σ. This bundle has contractible fiber and hence admits a smooth section (see e.g. [16] for this type of argument).
2. Let $\tilde g$ be a leafwise metric on M adapted to a foliation Σ and let H be a smooth function on M. Then, there is a unique smooth vector field $\tilde\nabla H$ on M such that $\tilde g(\tilde\nabla H,\cdot) = dH|_\Sigma$, where $dH|_\Sigma$ denotes, at each point $x \in M$, the restriction of $dH(x)$ to $T_x\Sigma_x$. The existence of such a vector field, which is tangent to the leaves of Σ, is obvious. Its smoothness follows from the smootheness of $\tilde g$. Note that, if the restriction of H to the leaves of Σ has no critical points, then $\tilde\nabla H$ is never zero.

Assume now that H and Σ are as in the Theorem and choose a leafwise metric $\tilde g$ on M adapted to Σ. The key fact is that the two vector fields X and $\tilde\nabla H$ span Σ. In fact, they are both tangent to Σ. Moreover, since dH is never zero, the condition that the level sets of H are transversal to the leaves of Σ implies that, for each $x \in M$, $T_x\Sigma_x \not\subset \ker dH(x)$ and hence the restriction of H to the leaf Σ_x has no critical points. As noticed above, this implies that $\tilde\nabla H$ is everywhere nonzero. Observe now that, since $\tilde\nabla H$ is tangent to the leaves of Σ, $dH(\tilde\nabla H) = dH|_\Sigma(\tilde\nabla H)$. Hence $L_{\tilde\nabla H} H = dH|_\Sigma(\tilde\nabla H) = \tilde g(\tilde\nabla H, \tilde\nabla H) \neq 0$. Since on the contrary $L_X H = 0$ and X is everywhere nonzero, this proves that X and $\tilde\nabla H$ are everywhere linearly independent and hence span Σ. It follows from this that the characteristic distribution of the bivector field

$$\Lambda \;:=\; \tilde g(\tilde\nabla H, \tilde\nabla H)^{-1}\, \tilde\nabla H \wedge X \tag{3.2}$$

coincides with the tangent spaces to the leaves of Σ and is thus Frobenius integrable, so that $[\![\Lambda, \Lambda]\!] = 0$. This proves that Λ is a rank–two Poisson tensor with symplectic foliation Σ. That $\Lambda^\#(dH) = X$ is proven as in the proof of Theorem 1. ∎

Remark: The Poisson tensor (3.2) constructed in the proof of Theorem 2 is exactly of the same type as that constructed in the proof of Theorem 1. The reason is that the leafwise metric $\tilde g$ can be extended to a Riemannian metric g on M such that $\tilde\nabla H$ is the gradient of H with respect to g. This fact is easily proven using a partition of unity argument. Let us consider a semilocal system of trivializing coordinates (H, φ, y), defined in an open set U. In these coordinates, $\tilde g$ has an expression of the type $a\, dH|_\Sigma^2 + b\, dH|_\Sigma d\varphi|_\Sigma + c\, d\varphi|_\Sigma^2$ with certain functions a, b, c. This local expression can be extended to a metric tensor $g_U = a\, dH^2 + b\, dH d\varphi + c\, d\varphi^2 + \sum_j dy_j^2$ on U. It is straightforward to check that g_U associates $\tilde\nabla H$ with dH. One can thus use a partition of unity to define a Riemannian metric g on M with the desired properties.

D. Proof of Theorem 3. Let us recall a few basic facts about Morse functions (see e.g. [10, 2]). A Morse–Bott function on a manifold M is a real valued function $f : M \to \mathbb{R}$ such that the connected components of C_f , the set of critical points of f, are manifolds and such that the Hessian of f at any critical point is non degenerate transversally to C_f. This means that at every point p in C_f the Hessian of f is a non-degenerate quadratic form in any algebraic complement N_p to T_pC_f in T_pM.

The Morse–Bott lemma states that, given a Morse–Bott function f, around any point $p \in C_f$ there exist coordinates (x, y) such that the critical manifold C_f is given by $\{x = 0\}$ and $f(x, y) = \text{const} \pm x_1^2 \pm \cdots \pm x_m^2$ (the signs depend on the signature of the Hessian).

We need to slightly extend this lemma, and show that there exist coordinates (x, y) around any point p of $\mathcal{E}$ such that the Hamiltonian H is of the form $H(x, y) = h(y) \pm x_1^2 \pm x_2^2$.

We know that the function H is a Morse function along the leaves of a foliation Σ with codimension equal to the dimension of $\mathcal{E}$ and such that, for each $p \in \mathcal{E}$, the leaf containing p is transverse to $\mathcal{E}$ in that point. Let us choose coordinates $(\tilde x, \tilde y)$ such that $\mathcal{E} = \{\tilde x = 0\}$ and that the leaves of Σ are of the form $\{\tilde y = \text{const}\}$.

The function $f(\tilde x, \tilde y) = H(\tilde x, \tilde y) - H(0, \tilde y)$ is a Morse–Bott function with critical manifold $\mathcal{E}$ which is zero on $\mathcal{E}$. Hence, with a change of coordinates $x = x(\tilde x, \tilde y)$, $y = \tilde y$ (see [3]), the local expression for f is $f(x, y) = \pm x_1^2 \pm x_2^2$. It follows that $H(x, y) = h(y) \pm x_1^2 \pm x_2^2$ for some function h. Observe that the leaves of Σ are, in the new coordinates, given by the equations $y = \text{const}$.

Let us now use the fact that H is a first integral for a vector field X tangent to the leaves of Σ. The vector field X must be of the form $X(x, y) = X_1(x, y)\frac{\partial}{\partial x_1} + X_2(x, y)\frac{\partial}{\partial x_2}$ for some functions X_1 and X_2. The condition $L_X H = 0$ becomes $\pm x_2 X_1 \pm x_1 X_2 = 0$ so that in these coordinates the vector field X has the expression $\omega(x, y)(x_1\frac{\partial}{\partial x_2} \pm x_2\frac{\partial}{\partial x_1})$ for some function ω. In turn, the eigenvalues condition iii. implies that ω is smooth and never vanishes and that the signs of the two terms are concordant. Hence, the smooth tensor $\tilde\Lambda_p$ defined in U_p by its local representative

$$\mathcal{C}_*\tilde\Lambda_p \;=\; \omega\, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$$

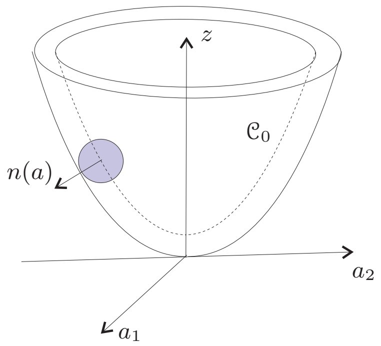

Figure 1: The sphere in the convex surface of revolution.

has everywhere rank two. This tensor satisfies $\tilde{\Lambda}_p^\sharp(dH) = X$ and is Poisson because, as one readily computes from (3.1), the Schouten bracket $[\![\tilde{\Lambda}_p, \tilde{\Lambda}_p]\!] = 0$. ∎

4 The heavy sphere rolling inside a convex surface of revolution

As an illustration of the previous theory, we consider now the system consisting of a sphere constrained to roll without sliding inside a convex surface of revolution with vertical axis, under the action of gravity.

This system is one of the prototypes of integrable nonholonomic systems and has a long history. The presence of a four–dimensional symmetry group allows the reduction to a four–dimensional reduced system. The integrability of the latter was known to Routh [34] who observed the existence of three independent integrals of motion: the energy E and two functions J_1 and J_2 which emerge as solutions of a certain system of nonautonomous linear differential equations.

More recently, Hermans [22, 23] and Zenkov [37] proved that the reduced system has periodic dynamics. From this, Hermans deduced the remarkable fact that the unreduced system has quasi–periodic dynamics. A similar conclusion, but restricted to the motion of the center of mass, was given by Zenkov.

It is for (a time–reparametrization of) the four–dimensional reduced system that Borisov, Mamaev and Kilin observed the existence of a rank–two Poisson structure [8]. That the rescaling is not necessary was pointed out in [33] and is once more a specificity of the assumption that the rank is two, since multiplying a rank–two Poisson tensor by a non zero function produces of course another rank–two Poisson tensor with the same symplectic foliation, see also [21, 5].

Our aim here is to show that the reduced system fits in our theory. To this end, we show that it satisfies one of the conditions of Theorem 4, which implies that its non–equilibrium periodic orbits are the fibers of a fibration and hence Theorems 1 and 2 apply. We then link this fact to the results of [8, 33] and discuss the non–uniqueness of the Poisson structure. Finally, we show that the conditions of Theorem 3 are satisfied, too, so as to include the equilibria.

To this purpose, we need to review in some detail the construction and properties of the reduced system. We rely as far as possible on existing results from [22, 37, 8], where greater details on some issues can also be found.

A. The system and its reduction. The starting point in the construction of the system is the holonomic system consisting of a (homogeneous) sphere of radius $r > 0$, the center of which is constrained to a single–valued convex surface of revolution $\mathcal{C}_0$ with vertical axis. More precisely, we assume that $\mathcal{C}_0$ is obtained by rotating around the z–axis the graph of an even, convex and smooth function $\phi : \mathbb{R} \to \mathbb{R}$. Thus, $\mathcal{C}_0$ is described by the equation

$$z = \phi\Big(\sqrt{x^2+y^2}\Big)\,,$$

see Figure 1. The system has five degrees of freedom and, after (right) trivialization of the tangent bundle of SO(3), lives on the ten–dimensional manifold

$$M_{10} \;=\; \mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^2 \times \mathbb{R}^3 \;\ni\; (a, \mathcal{R}, \dot a, \Omega)$$

where $a = (a_1, a_2)$ are the x and y coordinates of the center of mass, the matrix $\mathcal{R}$ fixes the attitude of the sphere, and Ω is the angular velocity.

Next, one introduces the nonholonomic constraint: the sphere rolls without sliding on a surface $\mathcal{C}$, the points of which have normal distance r from those of $\mathcal{C}_0$. The rolling constraint imposes to the point of the sphere in contact with $\mathcal{C}$ to have zero velocity. Denoting by $n(a)$ the exterior normal to $\mathcal{C}_0$ at the point of coordinates $\Big(a_1, a_2, \phi(\sqrt{a_1^2+a_2^2}\,)\Big)$, the velocity vector v of the center of the sphere satisfies $v = -r\,\Omega \times n(a)$. Therefore, the nonholonomically constrained system lives on the eight–dimensional submanifold M_8 of M_{10} given by $\dot a_1 = -r(\Omega \times n)_1$, $\dot a_2 = -r(\Omega \times n)_2$. This submanifold is diffeomorphic to $\mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^3$ and can be globally coordinatized with $(a, \mathcal{R}, \Omega)$. Instead of the three components of Ω one can use as well $\dot a_1, \dot a_2$ and $w := r\,\Omega \cdot n(a)$:

$$M_8 \;=\; \mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^2 \times \mathbb{R} \;\ni\; (a, \mathcal{R}, \dot a, w)\,.$$

The equations of motion of the system are the balance equations of linear momentum and angular momentum. Their precise form is not important for our purposes. What is important is that they are invariant under an action of $\mathrm{SO}(3) \times S^1$ obtained by rotating the sphere around its center and around the vertical axis of $\mathcal{C}_0$. The SO(3) and S^1 actions commute and can be reduced in stages.

Reduction of the SO(3)–action is straightforward and corresponds to cutting off the SO(3)–factor of M_8. Thus, the SO(3)–reduced system lives on the five–dimensional manifold

$$M_5 \;=\; \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \;\ni\; (a, \dot a, w)\,.$$

The SO(3)–reduced equations are equal to the unreduced ones because the matrix $\mathcal{R}$ does not appear at all in the latter. (Several treatments, including [37] and [8], consider in fact this system, not the eight–dimensional one. This amounts to study only the motion of the center of mass of the sphere.)

The remaining S^1–action on M_5 is given by $\theta.(a, \dot a, w) = (R_\theta a, R_\theta \dot a, w)$ where R_θ is the two–by–two matrix representing the rotation of angle θ. This action is free at all points of M_5 except those with $a = \dot a = 0$ (the sphere is standing on the bottom of the surface and spins around the vertical). The reduction procedure is well known, and elementary. In fact, the group S^1 does not act on the $\mathbb{R}$–factor of M^5 while its action on the factor $\mathbb{R}^2 \times \mathbb{R}^2$ is nothing but the familiar S^1–action of the $1:1$ oscillator. It is then a standard matter to verify that the (singularly) reduced space is a semialgebraic variety S_4 immersed in $\mathbb{R}^5$,

$$S_4 \;=\; \big\{ p \in \mathbb{R}^5 \,:\, p_0 \geq 0\,,\, p_1 \geq 0\,,\, 4p_0p_1 = p_2^2 + p_3^2 \big\}\,,$$

the quotient map $M_5 \to S_4$, $(a, \dot a, w) \mapsto (p_0, \ldots, p_4)$ being given by

$$p_0 \;=\; \frac{1}{2}\,|\dot a|^2\,, \quad p_1 \;=\; \frac{1}{2}\,|a|^2\,, \quad p_2 \;=\; a \cdot \dot a\,, \quad p_3 \;=\; a_1\dot a_2 - a_2\dot a_1\,, \quad p_4 \;=\; w$$

see [22]. (The functions $p_0, \ldots, p_4$ are in fact generators of the invariant polynomials of the S^1–action, see [15] for background on singular reduction). Note that S_4 consists of two strata:

- The 'singular' one–dimensional stratum

$$S_1 \;=\; \big\{ p \in \mathbb{R}^5 \,:\, p_0 = p_1 = p_2 = p_3 = 0 \big\} \,,$$

  which is obtained by projecting the one–dimensional subset of M_5 where $a = \dot{a} = 0$ (the sphere rotates around the vertical, standing at the bottom of the surface) where the action is not free.
- The four–dimensional 'regular' stratum

$$M_4 \;=\; \big\{ p \in \mathbb{R}^5 \,:\, p_0 \geq 0 \,,\; p_1 \geq 0 \,,\; 4p_0p_1 = p_2^2 + p_3^2 \,,\; p_0^2 + p_1^2 > 0 \big\} \tag{4.1}$$

  which is obtained by taking the quotient of the complementary set

$$M_5^{\mathrm{reg}} \;=\; \big(\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0,0\} \big) \times \mathbb{R} \;\ni\; (a, \dot{a}, w) \,.$$

The rank–two Poisson tensor introduced in [8] is defined in the regular stratum M_4 and because of this we shall restrict most of our analysis to the reduced system in M_4. To this regard, let us mention that, as remarked in [33], such a Poisson tensor can in fact be smoothly extended to the singular stratum S_1, but the extension identically vanishes on S_1.

As we have just noticed, M_4 is the quotient over the S^1 action of the submanifold M_5^{reg}. Such an action leaves the spheres $|a|^2 + |\dot{a}|^2 = $ const of the factor $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0,0\}$ of M_5^{reg} invariant and coincides on each of them with the Hopf fibration $S^3 \to S^2$. Hence, M_4 is diffeomorphic to $S^2 \times \mathbb{R} \times \mathbb{R}$ or, equivalently, to $\big(\mathbb{R}^3 \setminus \{0\}\big) \times \mathbb{R}$:

$$M_4 \;=\; \big(\mathbb{R}^3 \setminus \{0\}\big) \times \mathbb{R} \,.$$

One could thus introduce global coordinates on M_4 (e.g., $p_0 - p_1, p_2, p_3, p_4$) but in order to facilitate the comparison with [22] either we shall use the embedding of M_4 in $\mathbb{R}^5$ given by (4.1) or we shall cover M_4 with two charts, one with coordinates (p_1, p_2, p_3, p_4) where $p_1 > 0$ and the other with coordinates (p_0, p_2, p_3, p_4) where $p_0 > 0$.

B. The reduced equations of motion. In order to proceed towards the comparison with Theorem 2, we now remove the equilibria from the reduced phase space. To this end we need to write down the reduced equations of motion. The analysis is greatly simplified if we substitute the function $\phi : \mathbb{R} \to \mathbb{R}$ which describes the profile of the surface with a function $\Psi : \mathbb{R} \to \mathbb{R}$ such that

$$\Psi\Big(\frac{x^2}{2}\Big) = \phi(x) \,.$$

The existence and smoothness of such a function Ψ is proven as follows: Since $\phi : \mathbb{R} \to \mathbb{R}$ is even, convex and smooth it has a minimum in zero where all its odd derivatives vanish. Therefore, the right derivatives of the function $\Psi_+ : \mathbb{R}_+ \to \mathbb{R}$ given by $\Psi_+(x) = \phi(\sqrt{2x})$ for $x \geq 0$ satisfy $\Psi_+^{(k)}(0^+) = \frac{k!2^k}{(2k)!}\phi^{(2k)}(0)$ and hence are all finite. Consequently, Ψ_+ can be smoothly extended to a function Ψ which is defined on the whole real axis.

As it will be clear from the sequel, the use of Ψ rather than ϕ makes smoothness at $p_1 = 0$ evident. We also note that

$$\Psi'(p_1) > 0 \qquad \forall p_1 > 0$$

because $\phi'(\sqrt{2p_1}) = \sqrt{2p_1}\Psi'(p_1)$ is everywhere positive but at the origin given that ϕ is even and convex. However, $\phi''(\sqrt{2p_1}) = 2p_1\Psi''(p_1) + \Psi'(p_1)$ and Ψ need not be convex.

Remark: The extension Ψ of Ψ_+ is of course not unique, but this is irrelevant to our purposes. Moreover, we only need the existence of Ψ in an interval $\mathcal{I} = (-l, \infty)$ for some $l > 0$. (If $\phi : \mathbb{R} \to \mathbb{R}$ is real analytic then there is a unique analytic continuation of Ψ_+ to such an interval).

The equations of motion of the reduced system on S_4 are given in [22] and have the form

$$\dot p_0 = p_2\, F_0(p_0,p_1,p_2,p_3,p_4)\,, \qquad \dot p_2 = F_2(p_0,p_1,p_2,p_3,p_4) \tag{4.2}$$
$$\dot p_1 = p_2\,, \qquad \dot p_3 = p_2\, p_4\, F_3(p_1)\,, \qquad \dot p_4 = p_2\, p_3\, F_4(p_1) \tag{4.3}$$

where the functions F_0, F_2, F_3, F_4 are as follows:

$$\begin{aligned} F_0(p_0,p_1,p_2,p_3,p_4) &= \big[\big(Ap_3p_4 - p_2^2\Psi'(p_1)\big)\Psi''(p_1) - B\Psi'(p_1) - 2p_0\Psi'(p_1)^2\big]g(p_1)\\ F_2(p_0,p_1,p_2,p_3,p_4) &= \big[2p_0 - (Ap_3p_4 + 2Bp_1)\Psi'(p_1) + 2p_1p_2^2\Psi'(p_1)\Psi''(p_1)\big]\, g(p_1)\\ F_3(p_1) &= A\left[\Psi'(p_1) + 2p_1\Psi''(p_1)\right] g(p_1)\\ F_4(p_1) &= \left[\Psi'(p_1)^3 - \Psi''(p_1)\right] g(p_1) \end{aligned}$$

and

$$g(p_1) = \frac{1}{1+2p_1\Psi'(p_1)^2}\,, \qquad A = \frac{M}{M+mr^2}\,, \qquad B = \frac{mgr^2}{M+mr^2}\,,$$

M being the moment of inertia of the sphere relative to its center; thus $A=\frac{2}{5}$ and $B=\frac{5}{7}g$ for a homogeneous sphere. Equations (4.2), (4.3) define a vector field in $\mathbb{R}^5$ which, restricted to S_4, is tangent to its strata.

C. Excluding the equilibria. From (4.2) and (4.3) one sees that the equilibria of the (singularly) reduced system are the points $p\in S_4$ such that

$$p_2 = 0\,, \qquad 2p_0 - (Ap_3p_4 + 2Bp_1)\Psi'(p_1) = 0\,. \tag{4.4}$$

This includes

- All points of the one–dimensional 'singular' stratum S_1, which correspond to the sphere spinning around the vertical in the bottom of the surface.
- The points in the submanifold $\mathcal{E}$ of M_4 given by

  $$\mathcal{E} = \big\{p\in M_4 \,:\, p_2 = 0,\, 2p_0 = (Ap_3p_4 + 2Bp_1)\Psi'(p_1)\big\}$$

  which correspond to horizontal motions of the sphere (see [22, 23] for a thorough study of these motions).

That $\mathcal{E}$ is a submanifold can be checked routinely, e.g. verifying that the two functions at the left hand sides of equations (4.4) have rank two in each coordinate system of the considered atlas of M_4. Therefore, excluding the equilibria, we are eventually left with a nowhere zero vector field on the four–dimensional manifold

$$M = M_4 \setminus \mathcal{E}\,.$$

As we have already mentioned, Hermans proved that all motions of the reduced system are periodic [22]. His proof, which is based on a reversibility property of the reduced equations together with the compactness of the level sets of the energy, does not seem to be capable to prove the stronger condition that the periodic orbits with positive period are the fibers of a fibration. Zenkov's proof in [37] uses instead the fact that the reduced system has three functionally independent integrals of motion with compact common level sets. In view of Theorem 4, this property ensures that the flow is fibrating. However, Zenkov's treatment uses local coordinates on M_4 which are singular on the bottom of the surface, where $p_1 = 0$. Hence, his conclusions do not apply to the periodic motions with zero angular momentum, in which the sphere moves up and down in a vertical plane. Since we want our conclusions to be valid in all of M we need to supplement his analysis in this respect.

D. First integrals. It is immediate to verify that the reduced equations in S_4 have the energy integral

$$E(p_0,p_1,p_2,p_4) \;=\; p_0 + \frac{A}{2}p_4^2 + \frac{1}{2}\,p_2^2\Psi'(p_1)^2 + B\Psi(p_1) \tag{4.5}$$

(this differs by an inessential constant factor $M+mr^2$ from the 'true' energy integral). Moreover, as known since Routh [34], the reduced equations have two first integrals J_1 and J_2 which are constructed as follows. Consider the system of the two nonautonomous linear equations

$$p_3' \;=\; F_3(p_1)p_4\,, \qquad p_4' \;=\; F_4(p_1)p_3 \tag{4.6}$$

where the prime denotes derivative with respect to p_1. The two functions F_3 and F_4 are defined and smooth in an interval $\mathcal{I}=(-l,+\infty)$ for some $l>0$ (l is the first zero of $1+2p_1\Psi'(p_1)^2$ to the left of the origin). Therefore, these equations have two solutions

$$p_1 \mapsto \big(\pi_3(p_1),\pi_4(p_1)\big)\,, \qquad p_1 \mapsto \big(\sigma_3(p_1),\sigma_4(p_1)\big)$$

which are defined, smooth and independent in the whole interval $\mathcal{I}$. Hence, they satisfy

$$\pi_3(p_1)\sigma_4(p_1)-\sigma_3(p_1)\pi_4(p_1) \;\neq\; 0 \qquad \forall p_1\in\mathcal{I}\,. \tag{4.7}$$

It is immediate to verify that the two functions

$$J_1(p_1,p_3,p_4) \;:=\; p_3\pi_4(p_1)-p_4\pi_3(p_1)\,, \qquad J_2(p_1,p_3,p_4) \;:=\; p_3\sigma_4(p_1)-p_4\sigma_3(p_1)$$

are first integrals of the reduced system. These functions are defined and smooth for all $p_1\in\mathcal{I}$, $p_3,p_4\in\mathbb{R}$. Hence, thought of as functions in $\mathbb{R}^5\ni(p_0,\dots,p_4)$, they are defined and smooth in a half–space which contains entirely the singularly reduced space S_4. Therefore, they are globally defined in all of S_4. Moreover, because of (4.7), their restrictions to M_4 are everywhere functionally independent.

Remark: There is of course some arbitrariness in the choice of the solutions (π_3,π_4) and (σ_3,σ_4) of equations (4.6). Precisely, their initial values (at some point p_1) can be chosen arbitrarily, as long as (4.7) is satisfied. This arbitrariness reflects of course into some arbitrariness of the two integrals J_1 and J_2. Nevertheless, the submanifolds $(J_1,J_2)=\text{const}$ are uniquely determined. This is because different choices of (π_3,π_4) and (σ_3,σ_4) produce linear combinations of J_1 and J_2.

E. The fibration by the periodic orbits. We can now prove that the non–equilibrium periodic orbits of the reduced systems are the fibers of a fibration. Since we already know from Hermans' analysis that the orbits of the reduced system are periodic, in view of Theorem 4 it is sufficient to prove that

Proposition 1 *The map $(E,J_1,J_2): M\to\mathbb{R}^3$ is a submersion.*

Proof. (As we have already remarked, [37] proves that this is true in the subset of M where $p_1\neq 0$.) Let us consider the restrictions of the three functions E, J_1 and J_2 to the manifold M_4, which we still denote E, J_1 and J_2. The manifold M_4 is the subset of $\mathbb{R}^5$ given by

$$\mathcal{F}(p_0,p_1,p_2,p_3)=0\,, \quad p_0\geq 0\,, \quad p_1\geq 0\,, \quad p_0^2+p_1^2>0$$

for $\mathcal{F}(p_0,p_1,p_2,p_3)=\frac{1}{2}(p_2^2+p_3^2)-2p_0p_1$. In order to show that $(E,J_1,J_2):M\to\mathbb{R}^3$ is a submersion we introduce Lagrange multipliers μ, λ, λ_1 and λ_2 and show that, at each point of M, the equation

$$\mu\,d\mathcal{F}+\lambda\,dE+\lambda_1\,dJ_1+\lambda_2\,dJ_2 \;=\; 0$$

has only the trivial solution $\mu = \lambda = \lambda_1 = \lambda_2 = 0$. To simplify the notation, let us consider the function $\mathcal{G} = \mu\mathcal{F} + \lambda E + \lambda_1 J_1 + \lambda_2 J_2$, where the Lagrange multipliers have to be thought of as parameters. Clearly $\frac{\partial \mathcal{G}}{\partial p_0} = \lambda - 2p_1\mu$ and $\frac{\partial \mathcal{G}}{\partial p_2} = (\mu + \lambda\Psi'^2)p_2$ vanish simultaneously if and only if either $\mu = \lambda = 0$ or $p_2 = 0, \lambda = 2p_1\mu$. The first possibility would lead to $\lambda_1 dJ_1 + \lambda_2 dJ_2 = 0$ and hence, by the already noticed independence of J_1 and J_2 in all of M_4, to $\lambda_1 = \lambda_2 = 0$. So we assume $p_2 = 0$ and $\lambda = 2p_1\mu$ with nonzero λ and μ. To simplify the notation we may assume $\mu = 1$. The two conditions $\frac{\partial \mathcal{G}}{\partial p_3} = \frac{\partial \mathcal{G}}{\partial p_4} = 0$, evaluated at $p_2 = 0$ and $\lambda = 2p_1$, give

$$\lambda_1\pi_4(p_1) + \lambda_2\sigma_4(p_1) \; = \; -p_3\,, \qquad \lambda_1\pi_3(p_1) + \lambda_2\sigma_3(p_1) \; = \; 2Ap_1p_4\,.$$

In view of (4.7), these equations uniquely determine λ_1 and λ_2. It is immediate to check that the values of λ_1 and λ_2 determined in this way satisfy $\lambda_1\frac{\partial J_1}{\partial p_1} + \lambda_2\frac{\partial J_2}{\partial p_1} = Ap_3p_4\Psi'(p_1)$ and hence

$$\frac{\partial \mathcal{G}}{\partial p_1} \; = \; -2p_0 + 2p_1\big[p_2^2\Psi'(p_1)\Psi''(p_1) + B\Psi'(p_1)\big] + Ap_3p_4\Psi'(p_1)\,.$$

Thus, for $p_2 = 0$, $\frac{\partial \mathcal{G}}{\partial p_1} = 0$ is exactly the equilibrium condition (4.4). Hence, the restrictions to M_4 of the functions E, J_1, J_2 are functionally independent at all points but the equilibria, that is, in all of M. ■

Remarks: (i) In order to prove along these lines that the flow is periodic one needs to supplement Proposition 1 with the information that the level curves of (E, J_1, J_2) in M are compact. The proof of this fact, which is somehow technically involved, can be found in [35].

F. Poisson tensors. Applying Theorem 2 we can now conclude that *there exists a unique rank–two Poisson tensor Λ_{J_1,J_2} in M which has J_1, J_2 as Casimirs and makes X Hamiltonian with Hamiltonian E.* Being unique, this tensor coincides with that given in [8] (up to a factor corresponding to a time rescaling) and in [33], which in fact have J_1 and J_2 as Casimirs.

The expression of Λ_{J_1,J_2} in any system of coordinates on M can be easily worked out by observing that it is given by $\frac{\partial}{\partial E} \wedge X = \omega\frac{\partial}{\partial E} \wedge \frac{\partial}{\partial \varphi}$ in (semilocal) coordinates (E, J_1, J_2, φ). Let us use, for instance, the coordinates p_1, p_2, p_3, p_4. Since J_1 and J_2 are independent of p_2 we have $\frac{\partial}{\partial p_2} = \frac{\partial E}{\partial p_2}\frac{\partial}{\partial E} + \frac{\partial \varphi}{\partial p_2}\frac{\partial}{\partial \varphi}$. From here, observing that in these coordinate the energy E is given by (4.5) with p_0 replaced by $(p_2^2 + p_3^2)/(4p_1)$, one deduces

$$\frac{\partial}{\partial E} = \frac{2p_1}{p_2\big[1 + 2p_1\Psi'(p_1)^2\big]}\Big[\frac{\partial}{\partial p_2} - \frac{\partial \varphi}{\partial p_2}\,\frac{\partial}{\partial \varphi}\Big]\,.$$

The term containing $\frac{\partial}{\partial \varphi}$ clearly does not contribute to $\frac{\partial}{\partial E} \wedge X$. Using the expression of X given by (4.2) and (4.3) one eventually finds

$$\Lambda_{J_1,J_2} \; = \; \frac{2p_1}{1 + 2p_1\Psi'(p_1)^2}\,\frac{\partial}{\partial p_2} \wedge \Big[\frac{\partial}{\partial p_1} + p_4F_3(p_1)\frac{\partial}{\partial p_3} + p_3F_4(p_1)\frac{\partial}{\partial p_4}\Big]\,.$$

This expression coincides of course with that given in [33].

In view of Theorem 2 there are however infinitely many rank–two Poisson tensors in M which make X Hamiltonian with Hamiltonian E, in fact one for each two–dimensional foliation Σ of M which is transversal to E and invariant under the flow of X. Just as an example, let us note here that, written in local coordinates p_1, p_2, p_3, p_4, the Poisson tensor which has $J_1 + E$ and J_2 as Casimirs is given by

$$\begin{aligned}\Lambda_{J_1+E,J_2} \; = \; & \Big[p_2\frac{\partial}{\partial p_1} + F_3p_2p_4\frac{\partial}{\partial p_3} + F_4p_2p_3\frac{\partial}{\partial p_4} + F_2\frac{\partial}{\partial p_2}\Big] \\ \wedge \; & \Big[\sigma_3\frac{\partial}{\partial p_3} + \sigma_4\frac{\partial}{\partial p_4} + \Big(1 - \frac{\sigma_3p_3}{2p_1} - Ap_4\Big)\frac{2p_1}{p_2(1 + 2p_1\Psi'^2)}\frac{\partial}{\partial E}\Big]\,. \end{aligned} \tag{4.8}$$

The proof of this fact is a straightforward computation. Denoting $(E', J_1', J_2') = (E, J_1 + E, J_2)$ we have $\frac{\partial}{\partial J_1'} = \frac{\partial}{\partial J_1}$, $\frac{\partial}{\partial J_2'} = \frac{\partial}{\partial J_2}$, $\frac{\partial}{\partial E'} = \frac{\partial}{\partial E} - \frac{\partial}{\partial J_1}$ and hence $\Lambda_{J_1+E,J_2} := \omega\frac{\partial}{\partial\varphi}\wedge\frac{\partial}{\partial E'} \;=\; \omega\frac{\partial}{\partial\varphi}\wedge\Big(\frac{\partial}{\partial E} - \frac{\partial}{\partial J_1}\Big)$. The vector field $\frac{\partial}{\partial J_1}$ is readily computed from, e.g. $\frac{\partial}{\partial p_3} \;=\; \frac{p_3}{2p_1}\frac{\partial}{\partial E} + \pi_4(p_1)\frac{\partial}{\partial J_1} + \sigma_4(p_1)\frac{\partial}{\partial J_2}$, $\frac{\partial}{\partial p_4} \;=\; Ap_4\frac{\partial}{\partial E} - \pi_3(p_1)\frac{\partial}{\partial J_1} - \sigma_3(p_1)\frac{\partial}{\partial J_2}$, which give $\Delta\,\frac{\partial}{\partial J_1} = \Big(\frac{\sigma_3 p_3}{2p_1} + Ap_4\Big)\frac{\partial}{\partial E} - \sigma_3\frac{\partial}{\partial p_3} - \sigma_4\frac{\partial}{\partial p_4}$ where Δ is the wronskian $\pi_3\sigma_4 - \pi_4\sigma_3$. As already remarked, we can have $\Delta = 1$ with a suitable choice of (π_3, π_4), (σ_3, σ_4). After this, using the above expressions for the reduced vector field $\omega\frac{\partial}{\partial\varphi}$ and for $\frac{\partial}{\partial E}$ we eventually find (4.8).

G. Extension to equilibria. Finally, we show that the reduced system in M_4 fulfills all the hypotheses of Theorem 3 so that the rank–two Poisson structure Λ_{J_1,J_2} in M extends to all of M_4.

Hypothesis i. We have already observed that the equilibrium set $\mathcal{E}$ as given by (4.4) is a two–dimensional submanifold of M_4. We also know that J_1 and J_2 are everywhere independent in all of M_4. Therefore, their common level sets are the leaves of a two–dimensional foliation of M_4. It only remains to show that the level sets of (J_1, J_2) intersect transversally the equilibrium set $\mathcal{E} \subset M_4$. From the equilibrium conditions (4.4) one sees that if $\bar p = (\bar p_0, \bar p_1, \bar p_2, \bar p_3, \bar p_4)$ is an equilibrium in M_4 then $\bar p_1 > 0$, $\bar p_2 = 0$ and $\bar p_3 \neq 0$. (If it was $\bar p_3 = 0$ then the second equation (4.4) would imply $\bar p_1 = 0$, given that $\Psi'(p_1)$ never vanishes for positive p_1).

Since $\bar p_1 > 0$, in order to check transversality in $\bar p$ we can use the local coordinates p_1, p_2, p_3, p_4. The equilibrium conditions (4.4) then become $f_1 = 0$, $f_2 = 0$ for

$$f_1(p_1, p_2, p_3, p_4) \;=\; p_2\,, \qquad f_2(p_1, p_2, p_3, p_4) \;=\; p_2^2 + p_3^2 - 2(Ap_3p_4 + 2Bp_1)p_1\Psi'(p_1)\,.$$

Let us now observe that, in order to simplify the computations, we can choose the two solutions (π_3, π_4) and (σ_3, σ_4) of (4.6) in such a way that

$$\pi_3(\bar p_1) \;=\; \sigma_4(\bar p_1) \;=\; 1\,, \qquad \pi_4(\bar p_1) \;=\; \sigma_3(\bar p_1) \;=\; 0\,.$$

In fact, as noticed in a previous Remark, changing the solutions of (4.6) changes the two functions J_1 and J_2 but not their common level sets. Note that, correspondingly, $\pi_3'(\bar p_1) = \sigma_4'(\bar p_1) = 0$, $\pi_4'(\bar p_1) = F_4(\bar p_1)$, $\sigma_3'(\bar p_1) = F_3(\bar p_1)$.

We must now show that the Jacobian determinant of the map (f_1, f_2, J_1, J_2) at $\bar p$ is nonzero. A computation shows that this determinant is

$$32B\bar p_1^4\bar p_3^4\Psi' + 4\bar p_1^2\bar p_3^6\big[1 + 4A\Psi'^4\big]g + 64B^2p_1^6p_3^2\Psi'\big[\Psi' + 2p_1\Psi''\big]g + 16p_1^3p_3^6\Psi'\,\big[\Psi' + (1-A)p_1\Psi''\big]g$$

where it is understood that the functions g, Ψ' and Ψ'' are evaluated in $\bar p_1$. Here $g = g(p_1)$ is the function introduced in Section 4.B, which is everywhere positive. Since Ψ' is positive for positive arguments, the first two terms of the determinant are clearly positive. The third is non–negative because

$$\Psi' + 2p_1\Psi'' \;=\; \phi'' \geq 0$$

and the fourth is positive because

$$\Psi' + (1-A)p_1\Psi'' \;=\; \frac{1+A}{2}\Psi' + \frac{1-A}{2}\phi''$$

with $0 < A < 1$.

Hypothesis ii. That X is fibrating–periodic in $M = M_4 \setminus \mathcal{E}$ has just been proven. That E has no critical points in M is a fact that we have already mentioned several times and which can be proven with a direct computation.[3]

[3] In fact, E has no critical points in all of the (regularly) reduced space M_4, not even at the reduced equilibria. This might appear incompatible with the vanishing of the reduced vector field X at the equilibria because, as we have already mentioned, the reduced energy is the Hamiltonian of X with respect to an almost symplectic form. The fact is that such an almost symplectic form is singular at the reduced equilibria.

Hypothesis iii. That the restriction of E to the sets $(J_1, J_2)|_{M_4} = \mathrm{const}$ has a critical point with positive definite Hessian in $\mathcal{E}$ is proven by Zenkov in [37]. Specifically, Zenkov considers the more general case of a possibly non–convex surface of revolution and aims at proving that the reduced equilibria are Lyapunov–stable if and only if they are linearly stable. To this end, he considers the restriction of the energy E to the level sets of $(J_1, J_2)|_{M_4}$. This function has the reduced equilibria as its critical points. Zenkov shows that the Hessian of this function at a critical point is positive definite if and only if Routh's condition for linear stability is there satisfied. (See Proposition 1 in [37]). But Routh's condition (that is, equation (2.2) in [37]) is always satisfied if the surface of revolution is the graph of a convex function (since $\sin\alpha < 0$ in equation (2.2) of [37]). The fact that Zenkov uses a coordinate system singular for $p_1 = 0$ is irrelevant here, because $p_1 > 0$ at all reduced equilibria outside the singular stratum, see (4.4). Finally, that X is traversally elliptic at each point of $\mathcal{E}$, with nonzero eigenvalues, is proven in [23], Lemma 2.1.9.

5 Concluding remarks.

We make now some general remarks on the existence of Poisson structures.

First, it is natural to ask whether a vector field X with fibrating–periodic flow on an n–dimensional manifold is Hamiltonian also with respect to semilocal Poisson structures of ranks $4, 6, \ldots\ldots, [n/2]$, where [ ] denotes the integer part. Note that if n is even, then a Poisson structure of rank $n/2$ defines a symplectic structure. We limit ourselves to few considerations on this fact. When n is even, the existence of (infiniteley many) *semilocal* symplectic structures which make X Hamiltonian *with some Hamiltonian* is well known [26, 6, 18]. However, as already remarked in [28], the situation is different if the Hamiltonian is preassigned: since a Hamiltonian vector field with fibrating–periodic flow satisfies the period–energy relation, the frequency ω of its motions must be a function of the Hamiltonian alone, and this puts some conditions on the Hamiltonian.

Just to make an example, let us observe that in the case of the sphere rolling inside a convex surface of revolution one easily verifies (e.g. by numerical integration) that, at least for certain surfaces such as the paraboloid, the period of motions is not a function of the energy alone.

A similar problem is met with all Poisson tensors of rank greater than two: if the symplectic leaves have dimensions four or higher, then the restriction of the flow to each of them has three or more integrals of motion, one of which is the Hamiltonian. Hence, in order to fulfill the period–energy relation on each symplectic leaf, the Hamiltonian has to be chosen properly.

Another natural question is whether all this can be generalized to systems with *quasi–periodic* motions. Assume that X is a vector field with quasi–periodic dynamics: specifically, assume that there is an invariant fibration into k–dimensional tori on which the flow is linear. Assume moreover a nondegeneracy condition, that is, that a dense subset of these tori are closure of orbits. Assume, finally, that a first integral H is given, which has no critical points. Then, it is very easy to show that, *semilocally*, there are no obstructions whatsoever to the existence of a Poisson structure Λ of rank $2k$ which makes X Hamiltonian with Hamiltonian H. As we have already remarked, this has in fact been proved in [20]. We mention, however, that if $k > 1$ there are serious obstructions to the globality of such a structure: the attempt of extending any such semilocal structure to the whole subset of the phase space fibered by invariant tori fails, unless some very special property is met.

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References

[1] R. Abraham, J.E. Marsden and T.S. Ratiu, *Manifolds, Tensor Analysis, and Applications.* Second edition. (Springer, New York, 1988)

[2] M. Audin, *The Topology of Torus Actions on Symplectic Manifolds.* (Birkhäuser, Basel, 1991).

[3] A. Banyaga and D.E. Hurtubise, *A proof of the Morse–Bott lemma.* Expo. Math. **22**, 365–373 (2004).

[4] L. Bates and J. Śniatycki, *Nonholonomic reduction*, Rep. Math. Phys. **32** 99–115 (1993).

[5] M.Bertelson *Foliations associated to regular Poisson structures.* Commun. Contemp. Math. **3**, 441-456 (2001).

[6] O.I. Bogoyavlenskij, *Theory of tensor invariants of integrable Hamiltonian systems. I. Incompatible Poisson structures.* Comm. Math. Phys. **180**, 529-586 (1996)

[7] A.V. Borisov and I.S. Mamaev, *Rolling of a rigid body on a plane and sphere. Hierarchy of dynamics.* Regular & Chaotic Dynamics **7**, 177-200 (2002).

[8] A.V. Borisov, I.S. Mamaev and A.A. Kilin, *Rolling of a ball on a surface. New integrals and hierarchy of dynamics.* Regular & Chaotic Dynamics **7**, 201-219 (2002).

[9] A.V. Borisov, I.S. Mamaev and A.A. Kilin, *Dynamics of rolling disk.* Regular & Chaotic Dynamics **8**, 201-212 (2002).

[10] R. Bott, *Lectures on Morse theory, old and new.* Bull. Amer. Math. Soc. (N.S.) **7**, 331–358 (1982).

[11] R. Cushman, J. Hermans and D. Kemppainen, *The rolling disc*, Progr. Nonlinear Differential Equations Appl., 19, Birkhauser, Basel, 1996.

[12] R. Cushman, D. Kemppainen, J. Śniatycki and L. Bates, *Geometry of nonholonomic constraints*, Rep. Math. Phys. **36** 275–286 (1995).

[13] S.A. Chaplygin, *On the theory of motion of nonholonomic systems. Theorems on the reducing multiplier.* Mat. Sbornik **28**, 303-314 (1911).

[14] R.H. Cushman, *Routh's sphere*, Rep. Math. Phys. **42** 47–70 (1998).

[15] R.H. Cushman and L. Bates, *Global Aspects of Classical Integrable Systems.* (Birkhäuser, Basel, 1997)

[16] B.A. Dubrovin, A.T. Fomenko and S.P. Novikov, *Modern Geometry. Part II.* Graduate Texts in Mathematics **104**. (Springer, New York, 1985).

[17] J.J. Duistermaat and J.A.C. Kolk, *Lie groups* (Springer-Verlag, Berlin, 2000).

[18] F. Fassò, *Quasi periodicity of motions and complete integrability of Hamiltonian systems.* Ergodic Theory and Dynamical Systems **18**, 1349-1362 (1998).

[19] F. Fassò and N. Sansonetto, in preparation.

[20] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Bi-Hamiltonian partially integrable systems.* J. Math. Phys. **44** (2003).

[21] J. Grabowski, G. Marmo and A.M. Perelomov, *Poisson structures: towards a classification*, Modern Phys. Lett. A **8**, 1719–1733 (1993).

[22] J. Hermans, *A symmetric sphere rolling on a surface.* Nonlinearity **8**, 493-515 (1995).

[23] J. Hermans, *Rolling rigid bodies with and without symmetries.* PhD thesis, Univesity of Utrecht (1995).

[24] A.A. King and W.M. Schaffer, *The rainbow bridge: Hamiltonian limits and resonance in predator-prey dynamcis.* J. Math. Biol. **19**, 439-469 (1999).

[25] W.S. Koon and J.E. Marsden, *The Hamiltonian and Lagrangian approaches to the dynamics of nonholonomic systems*. Rep. Math. Phys. **40**, 21-62 (1997).

[26] V.V. Kozlov, *The Liouville property of invariant measures of completely integrable systems and the Monge–Ampère equation.* Math. Notes **53**, 389-393 (1993).

[27] P. Libermann and C.-M. Marle, *Symplectic geometry and analytical mechanics.* (D. Reidel, Dordrecht, 1987).

[28] N.K. Moschuk, *A qualitative analysis of the motion of a heavy solid of revolution on an absolutely rough plane.* PMM **52**, 159-165 (1988).

[29] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry.* Texts in Applied Mathematics **17** (Springer, New York, 1994).

[30] G. Meigniez, *Submersion, fibrations and bundles.* Trans. Am. Math. Soc. **354**, 3771-3787 (2002).

[31] K.R. Meyer and G.R. Hall, *Introduction to Hamiltonian Dynamical Systems and the N-body Problem.* (Springer-Verlag, New York, 1992).

[32] Ju.I. Neimark and N.A. Fufaev, *Dynamics of Nonholonomic Systems.* Translations of Mathematical Monographs **33** (AMS, Providence, 1972).

[33] A. Ramos, *Poisson structures for reduced non–holonomic systems.* J. Phys. A: Math. Gen. **37**, 4821-4842 (2004).

[34] E.J. Routh, *Treatise on the Dynamics of a System of Rigid Bodies (Advanced Part)* (Dover, New York, 1955).

[35] N. Sansonetto, *PhD Thesis*, in preparation.

[36] I. Vaisman, *Lectures on the geometry of Poisson manifolds.* (Birkhäuser, Basel, 1994)

[37] D.V. Zenkov, *The geometry of the Routh problem.* Journal of Nonlinear Science **5**, 503-519 (1995).