Quasi–periodicity of motions and complete integrability of Hamiltonian systems

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Abstract: *Consider a Hamiltonian system with d degrees of freedom whose motions are all linear on tori of some fixed dimension $n \le d$; is such a system necessarily completely (or else non–commutatively) integrable? We show that the answer is affermative under quite broad conditions, but not always, and we provide counterexamples.*

1. Introduction and sketch of results

A. Complete integrability, i.e., the existence of as many independent integrals of motion in involution as the number d of the degrees of freedom, plays a central role for the integrability of Hamiltonian systems. In particular, according to the well known Liouville–Arnol'd theorem, the compact regular common level sets of the integrals in involution are d–dimensional tori on which the flow is linear and, moreover, action–angle coordinates exist in a neighbourhood of each of these tori [1]. There also exist generalizations of this theorem for 'degenerate' systems, which possess more than d independent integrals of motion; specifically, if a system has $2d - n \ge d$ independent integrals with certain involutivity properties, then its motions are linear on tori of dimension $n \le d$ and 'generalized' action–angle coordinates do exist in a neighbourhood of each such torus; this is the so called 'noncommutative integrability' [1–5].

Even though it is clear that the existence of d integrals of motion which are not in involution is in general not sufficient to assure the integrability of a Hamiltonian system with d degrees of freedom, one may ask whether the requirement of involutivity (or its generalizations for the noncommutative setting, if there are more than d integrals) is really necessary. This problem seems to have a significant foundational interest for mechanics. To our knowledge, it has been considered by Rüssmann, Gallavotti and Ito, who found affirmative answers in special cases.

Specifically, Rüssmann [6] and Ito [7] considered a problem of this kind within the problem of the convergence of the Birkhoff series near an elliptic nonresonant equilibrium in $\mathbb{R}^{2d}$ (with the standard symplectic structure $\sum_i dp_i \wedge dq_i$). They showed that any set of d analytic integrals of motion defined in a neighbourhood of such an equilibrium are mutually in involution.

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In [8,9], Gallavotti raised the problem of whether a Hamiltonian system whose motions are all conjugate to linear motions on tori is necessarily completely integrable. Note that, in this approach, the existence of the integrals of motion is supplemented with the information that their common level sets are tori which carry linear motions. Gallavotti [10] showed that, under certain technical conditions which we will explain below, the answer to the question is affirmative for 'strictly anisochronous' systems, i.e., if the rank of the frequency map is everywhere d. One should note that this anisochronicity condition implies the nondegeneracy of the system. (Indeed, if the frequencies change from torus to torus, then a dense subset of the invariant d–dimensional tori support nonresonant motions and so are closures of trajectories; therefore, in no open invariant set there exist more than d independent integrals of motion.)

In this paper, we provide a general analysis of the problem along Gallavotti's lines. On the one hand, we extend his result by showing that the answer to the question he raised is affirmative under quite broad conditions, not only for strictly anysochronous systems; the consideration of degenerate systems is especially significant, given that various classical systems of mechanics are degnerate, like the rigid body with a fixed point ($d = 3$, $n = 2$) and the Kepler system ($d = 3$, $n = 1$). On the other hand, we prove that the answer to Gallavotti's question is not always affirmative, that is, that there exist Hamiltonian systems which are integrable, in the sense that all their motions are quasi periodic, but not completely (nor noncommutatively) integrable.

B. Specifically, as we explain in Section 2, we consider a Hamiltonian system on a symplectic manifold of dimension $2d$ which, in some open subset M of its phase space, to which we restrict our consideration, has all its motions conjugate to linear motions on tori of some fixed dimension n, $1 \leq n \leq d$. We shall make two assumptions. The first, as in [8–10], concerns the existence of suitable, possibly noncanonical, 'integrating coordinates'; this hypothesis amounts to the fact that the invariant tori are the fibers of a fibration, not only the leaves of a foliation. The second assumption is that in no open invariant subset of M there exist $2d - n + 1$ independent integrals of motion. We shall call *integrable with n frequencies* a system of this kind. The problem that we investigate, following [8,9], is whether the integrating coordinates can be chosen to be canonical, in which case they are (generalized) action–angle coordinates; when this happens, we say that the system is *canonically integrable.*

The notion of canonical integrability has the advantage of being easily defined but, for the interpretation of the results, it should be related to the standard characterizations of integrability. Thus, we show in the Appendix that canonical integrability is exactly equivalent to noncommutative integrability (and to complete integrability if $n = d$; if $n < d$, the equivalence with complete integrability is in a sense only local, the existence of sets of d independent integrals of motion being guaranted only in a neighbourhood of every invariant n–dimensional torus).

In Section 3 we derive a geometric characterization of the problem. Specifically, we show that a system which is integrable with n frequencies is canonically integrable if and only if the symplectic two–form is exact in a neighbourhood of every invariant torus, or else, if and only if the invariant tori are isotropic (Proposition 1). The latter property is obvious when $n = d$ (the Lagrangian character of the tori is indeed equivalent to the fact that the integrals of motion are in involution and so, by the Liouville–Arnol'd theorem, the system possesses action–angle coordinates and hence is canonically integrable) but, as we shall explain later, not if $n < d$.

As a simple application of this result, we derive in Section 4A a few sufficient conditions for integrability with n frequencies to imply canonical integrability (Proposition 2). One is, of course, the exacteness of the symplectic two–form of the phase space. This condition is particularly significant, being verified in all natural mechanical systems on cotangent bundles. However, since Hamiltonian systems on non–exact symplectic manifolds do exist and actually arise naturally (they appear, for instance, after reduction), we derive a few other sufficient conditions for canonical integrability: a mild nonisochronicity property (which generalizes Gallavotti's one), the fact that all motions are periodic ($n=1$), and the fact that the system has two frequencies ($n=2$). In a sense, these results constitute a converse to the Liouville–Arnol'd theorem (they deduce complete integrability from quasi–periodicity).

Finally, in Section 4B we provide some examples of Hamiltonian systems which have all motions linear on tori but are not canonically integrable. The prototype of these examples is the vector field

$$X = \sum_{i=1}^{3} \omega_i \frac{\partial}{\partial \varphi_i}\,, \qquad \omega = \begin{pmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \end{pmatrix},$$

on the manifold $\mathbb{R}^3 \times \mathbb{T}^3 \ni (I,\varphi)$, equipped with the (non–exact) symplectic two–form

$$\sigma = \sum_{i=1}^{3} dI_i \wedge d\varphi_i + \frac{1}{2}\sum_{i,j=1}^{3} G_{ij} d\varphi_i \wedge d\varphi_j\,, \qquad G = \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{2} \\ -\sqrt{3} & 0 & 1 \\ \sqrt{2} & -1 & 0 \end{pmatrix}.$$

The vector field X is Hamiltonian, with Hamiltonian $H(I) = \sum_i \omega_i I_i$, and its flow is linear on the tori $I = \mathrm{const.}$ However, $\{I_i, I_j\} = 3G_{ij}$ and the three integrals of motion I_1, I_2, I_3 are not involution (equivalently, $\sigma(\frac{\partial}{\partial\varphi_i}, \frac{\partial}{\partial\varphi_j}) = G_{ij}$ and the invariant tori are not Lagrangian). The existence of these systems, which are integrable but not completely integrable, raise some interesting questions that we shall shortly address at the end of the paper.

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2. Two integrability notions

In this section we make precise the class of systems with quasi–periodic motions that we consider. Let us recall, preliminarily, that a vector field X on a symplectic manifold (M,σ) is called *Hamiltonian* if the one–form $i_X\sigma$ is exact, and *locally Hamiltonian* if $i_X\sigma$ is closed; in the former case there exists a (global) Hamiltonian H for X, which satisfies $i_X\sigma = -dH$, while in the latter case the existence of Hamiltonians is assured only locally. Even though we are mostly interested to the Hamiltonian case, it is convenient to consider the more general case of a locally Hamiltonian vector field.

Throughout the paper, by a manifold (map, function) we mean a smooth manifold (map, function). Moreover, by saying that k functions $F_1, \ldots, F_k$ defined in a set M are *independent*, we always mean that their differentials are linearly independent at *every* point of M.

Definition 1: Let (M,σ) be a symplectic manifold of dimension $2d$, X a locally Hamiltonian vector field defined on M, and n an integer, $1 \le n \le d$.

(i) We say that X is *integrable with n frequencies* if

(H1) For every point $x \in M$ there exist a neighbourhood N of x and a diffeomorphism

$$\mathcal{C} = (b,\varphi) \ : \ N \to B \times \mathbb{T}^n \,, \tag{2.1}$$

where B is an open subset of $\mathbb{R}^{2d-n}$, which conjugates X to a vector field which is tangent to the tori $b =$ const and is constant on them, i.e.,

$$\mathcal{C}_* X \ = \ \omega \cdot \frac{\partial}{\partial \varphi}$$

with some map $\omega : B \to \mathbb{R}^n$, called the frequency map ($\mathcal{C}_*$ denotes the push–forward).

(H2) In no open invariant subset of M there exist $2d-n+1$ independent integrals of motion of X.

(ii) We say that X is *canonically integrable with n frequencies* if it is integrable with n frequencies and if, moreover, for every point $x \in M$ there exists a diffeomorphism $\mathcal{C} = (b,\varphi)$ as in (2.1) which is defined in a neighbourhood of x and is canonical, in the sense that, writing $b = (p_1,\ldots,p_{d-n},\, q_1,\ldots,q_{d-n},\, a_1,\ldots,a_n)$, one has

$$\mathcal{C}_*\sigma \ = \ \sum_{r=1}^{d-n} dp_r \wedge dq_r \ + \ \sum_{i=1}^{n} da_i \wedge d\varphi_i \,. \tag{2.2}$$

Note on terminology: The expressions 'integrable' and 'canonically integrable' are used by Gallavotti [8,9] in the case $n = d$ and with no analogue of hypothesis H2. The expression 'number of frequencies' is used by Nekhoroshev for analytic systems [3] (however, Nekhoroshev uses 'integrable' for our 'canonically integrable').

As explained in the Introduction, the purpose of this paper is to establish whether integrability with n frequencies implies canonical integrability. In the remaining of this Section, we add a few comments about these notions.

Hypothesis H1 implies that, in the set N, the flow of X is conjugate by $\mathcal{C}$ to the linear flow

$$(b,\varphi) \mapsto \big(b, \varphi + \omega(b)t \ \mathrm{mod}\, 2\pi\big) \,. \tag{2.3}$$

Therefore, M is foliated by n–dimensional invariant manifolds diffeomorphic to tori, on which the flow is linear; we shall refer to these submanifolds as the *invariant tori* of X. The diffeomorphisms $\mathcal{C} = (b,\varphi)$ will be called *integrating coordinates.*

We note that the existence of integrating coordinates is stronger than requiring that all motions are quasi–periodic. Specifically, the existence of the diffeomorphisms (2.1) amounts to the fact that the invariant tori are the fibers of a fibration. This follows from a special case of Reeb's stability theorem, according to which every foliation whose leaves are compact and have no holonomy is a fibration (see [11]); in the present case, the triviality of the holonomy is assured by the existence of the integrating coordinates. While this hypothesis is generally satisfied, there are some well known exceptions. For instance, the foliation by the periodic orbits of a system of two uncoupled harmonic oscillators with frequency ratio 2 : 1 has not the structure of a fibration in a neighbourhood of the special orbits of half period, in which only the slower oscillator moves; indeed, all the other orbits wind twice around these special orbits, and correspondingly, they form a so called Seifert's foliation [12], but not a fibration. (However, if one removes these special orbits, as well as the equilibrium, one still gets a system which is integrable with one frequency; see also the Appendix).

We also note that the integrating coordinates are not uniquely defined (for instance, any diffeomorphism $b' = b'(b)$ produces new integrating coordinates (b', φ)). Furthermore, as a consequence of the possible topological nontriviality of the fibration by the invariant tori, it may be impossible to cover the whole manifold M with a single set of integrating coordinates. However, one can always construct an atlas for M by patching together a suitable number of local systems of these coordinates.

Hypothesis H2 concerns the 'minimality' of the invariant tori of dimension n, in the sense that it implies that these tori cannot all be decomposed into tori of smaller dimension $n' < n$. Hence, once H2 is added to the definition, a system which is integrable with n frequencies is not integrable with $n' < n$ frequencies. Note that hypothesis H2 can be rephrased by saying that every integral of motion of the system which is defined in an invariant set is functionally dependent on the local coordinates $b_1, \ldots, b_{2d-n}$.

Finally, we observe that canonical integrability with n frequencies, with any $n \leq d$, obviously implies the existence of d integrals of motion in involution which are defined and are independent in a neighbourhood of any invariant torus (the coordinates a and p, for instance). Thus, a system which is canonically integrable with n frequencies is also completely integrable, in this local sense. The converse to this fact is discussed in the Appendix, where we also discuss the relation among canonical integrability and noncommutative integrability (which is relevant for the interpretation of Proposition 1 below).

3. Characterization of the problem

In this Section, we establish some necessary and sufficient conditions for integrability with n frequencies to imply canonical integrability. In the sequel, we make everywhere the convention that the indices $i, j, \ldots$ take the values $1, \ldots, n$, while the indices $r, s, \ldots$ take the values $1, \ldots, 2d-n$; moreover, we understand summation over repeated indices.

A. We begin by stating the following Lemma which describes the local structure, in a neighbourhood of an invariant torus, of the symplectic form and of the local Hamiltonians of a system which is integrable with n frequencies:

Lemma *Assume that the vector field X on the symplectic manifold (M, σ) is locally Hamiltonian and integrable with n frequencies. Then, for every local system of integrating coordinates $\mathcal{C} = (b, \varphi) : N \to B \times \mathbb{T}^n$, with N connected, there exist matrices $E(b)$, $F(b)$ and G (i.e. E and F depend on b but not on φ, while G is constant), with F and G antisymmetric, such that the vector $g := G\omega(b)$ is constant in all of B and one has*

$$\mathcal{C}_* \sigma \; = \; E_{rj} db_r \wedge d\varphi_j \; + \; \frac{1}{2} F_{rs} db_r \wedge db_s \; + \; \frac{1}{2} G_{ij} d\varphi_i \wedge d\varphi_j \, . \tag{3.1}$$

Moreover, every local Hamiltonian H of $\mathcal{C}_*X = \omega \cdot \frac{\partial}{\partial\varphi}$, defined in any set $B' \times V \subset B \times \mathbb{T}^n$, has the form

$$H(b,\varphi) \;=\; h(b) \;+\; g\cdot\varphi\,, \tag{3.2}$$

where $h : B' \to \mathbb{R}$ is a function such that $\frac{\partial h}{\partial b} = E\omega$.

Note that, by its very construction, the vector field $\mathcal{C}_*X = \omega \cdot \frac{\partial}{\partial\varphi}$ in $B \times \mathbb{T}^n$ is locally Hamiltonian with respect to $\mathcal{C}_*\sigma$; however, it is Hamiltonian if and only if $\omega(b) \in \ker(G)$ at every point $b \in B$, i.e., if and only if $g = 0$ (otherwise, H is not single valued).

Proof. Expression (3.1) holds with some matrices E, $F = -F^T$, and $G = -G^T$ dependent on the coordinates (b,φ) and 2π–periodic in the variables $\varphi_1,\dots,\varphi_n$. In order to show that these matrices have the specified properties we begin by using the fact that $\omega\cdot\frac{\partial}{\partial\varphi}$ is locally Hamiltonian with respect to $\mathcal{C}_*\sigma$. This implies that, for every $t \in \mathbb{R}$, the map at time t of its flow, here denoted Φ_t, is symplectic, that is, $\Phi_t^*\mathcal{C}_*\sigma = \mathcal{C}_*\sigma$. Now, since $\Phi_t(b,\varphi) = \big(b, \varphi + \omega(b)t \bmod 2\pi\big)$, for sufficiently small t (e.g., $|t| < \pi \min_i |\omega_i(b)|^{-1}$) one has

$$\begin{aligned}\Phi_t^*\mathcal{C}_*\sigma \;&= (E_{rj}\circ\Phi_t)db_r \wedge d(\varphi_j + t\omega_j) \;+\; \frac{1}{2}\,(F_{rs}\circ\Phi_t)db_r\wedge db_s\\ &\quad+\;\frac{1}{2}(G_{ij}\circ\Phi_t)d(\varphi_i + t\omega_i)\wedge d(\varphi_j+t\omega_j)\,.\end{aligned}$$

Therefore, for small t, the condition $\Phi_t^*\mathcal{C}_*\sigma = \mathcal{C}_*\sigma$ leads to the equations

$$G_{ij}\circ\Phi_{-t} \;=\; G_{ij}\,, \tag{3.3a}$$

$$E_{rj}\circ\Phi_{-t} \;=\; E_{rj} \;+\; t\,G_{ij}\frac{\partial\omega_i}{\partial b_r}\,, \tag{3.3b}$$

$$F_{rs}\circ\Phi_{-t} \;=\; F_{rs} \;+\; 2tE_{rj}\frac{\partial\omega_j}{\partial b_s} \;+\; t^2\,G_{ij}\frac{\partial\omega_i}{\partial b_r}\frac{\partial\omega_j}{\partial b_s}\,, \tag{3.3c}$$

for all $i,j = 1,\dots,n$ and all $r,s = 1,\dots,2d-n$. Using the group property of the flow Φ_t, one verifies that equations (3.3) hold for all times $t \in \mathbb{R}$.

Observe now that, on account of hypothesis H2, equations (3.3a) imply $G = G(b)$. For the same reason, and by virtue of the periodicity of E, equations (3.3b) imply $E = E(b)$ and $G_{ij}\frac{\partial\omega_i}{\partial b_r} = 0$ for all j and r, that is,

$$G_{ij}d\omega_i \;=\; 0\,, \qquad\quad j = 1,\dots,n\,. \tag{3.4}$$

Similarly, equations (3.3c) give $F = F(b)$. Furthermore, closedness of $\mathcal{C}_*\sigma$ and the fact that E does not depend on φ imply

$$0 \;=\; d\mathcal{C}_*\sigma\Big(\frac{\partial}{\partial b_r},\frac{\partial}{\partial\varphi_i},\frac{\partial}{\partial\varphi_j}\Big) \;=\; \frac{1}{2}\left(\frac{\partial G_{ij}}{\partial b_r} - \frac{\partial G_{ji}}{\partial b_r}\right) \;=\; \frac{\partial G_{ij}}{\partial b_r}$$

for all $i,j = 1,\dots,n$ and all $r = 1,\dots,2d-n$. This shows that the matrix G is constant (since B is connected). Therefore, by (3.4), $G\omega$ is constant, too. Finally, one has

$$i_{\omega\cdot\frac{\partial}{\partial\varphi}}\mathcal{C}_*\sigma \;=\; -E_{ri}\omega_i db_r \;+\; G_{ij}\omega_i d\varphi_j\,;$$

equating this expression to $-dH$, and using the constancy of $G\omega$, one proves (3.2). ∎

B. We now provide a geometric characterization of the equivalence of integrability and canonical integrability with n frequencies. Let us recall that a submanifold of a symplectic manifold (M,σ)

is said to be *isotropic* if σ vanishes on its tangent spaces. If M has dimension $2d$, its isotropic submanifolds may have any dimension $n \le d$. An isotropic submanifold of the maximal dimension $n = d$ is called *Lagrangian.* In the case at hand, within every system of integrating coordinates, the condition which expresses the isotropy of the invariant tori of X is

$$\mathcal{C}_*\sigma\left(\frac{\partial}{\partial\varphi_i},\frac{\partial}{\partial\varphi_j}\right) \;=\; 0 \qquad \text{for all} \;\; i,j = 1,\ldots,n\,,$$

that is, with the notations entering equation (3.1), $G = 0$.

Proposition 1 *Let X be a locally Hamiltonian vector field on a symplectic manifold (M,σ) which is integrable with n frequencies. Then the following statements are equivalent:*

(S1) X is canonically integrable with n frequencies.

(S2) The invariant tori of X are isotropic.

(S3) Every invariant torus of X has a neighbourhood in which σ is exact.

Proof. The fact that S1 implies S3 follows from the very existence of canonical integrating coordinates: $\sigma|_N = d(\sum_r p_r dq_r + \sum_i a_i d\alpha_i)$.

In order to prove that S3 implies S2, let us assume $\sigma|_U = d\theta$, U being a neighbourhood of an invariant torus which is contained within the domain of a system of integrating coordinates $\mathcal{C} = (b,\varphi)$. Integrating $\mathcal{C}_*\sigma = d\mathcal{C}_*\theta$ as given in (3.1) over the two–dimensional tori

$$\gamma_{ij} \;=\; \big\{(b,\varphi) \in B \times \mathbb{T}^n : \, b = \text{const}, \varphi_l = \text{const if } l \neq i,j\big\}\,, \qquad i,j = 1,\ldots,n\,,$$

using the constancy of G_{ij} and Stokes theorem, we get

$$4\pi^2\, G_{ij} \;=\; \int_{\gamma_{ij}} \mathcal{C}_*\sigma \;=\; \int_{\partial\gamma_{ij}} \mathcal{C}_*\theta \;=\; 0\,.$$

This proves the claim since, as already observed, the tori $b = \text{const}$ are isotropic if and only if $G = 0$.

Finally, we prove that S2 implies S1. It clearly suffices to prove this fact within a local system of integrating coordinates. Let us refer to the expression (3.1) of $\mathcal{C}_*\sigma$. The closedness of $\mathcal{C}_*\sigma$, the fact that F is independent of the angles φ, and the constancy of G imply that the two–form $E_{ri}db_r \wedge d\varphi_i$ is closed. In turn, since $E = E(b)$, this implies that the n one–forms $E_{ri}db_r$ $(i = 1,\ldots,n)$ are closed, too. Hence, every point $b \in B$ has a neighbourhood $B' \subset B$ in which are defined n real functions $a_1,\ldots,a_n$ such that $E_{ri}db_r = da_i$. Since $G = 0$ by the isotropy hypothesis, one has $i_{\partial/\partial\varphi_i}\mathcal{C}_*\sigma = -da_i$ for every $i = 1,\ldots,n$, showing that $\frac{\partial}{\partial\varphi_i}$ is the Hamiltonian vector field of the function a_i; hence, the linear independence of these vector fields and the nondegeneracy of the symplectic form imply that the differentials $da_1,\ldots,da_n$ are everywhere linearly independent in B'. Moreover, since $\{a_i,a_j\} = \sigma(\frac{\partial}{\partial\varphi_i},\frac{\partial}{\partial\varphi_j}) = 0$, the functions $a_1,\ldots,a_n$ are pairwise in involution.

(At this point, the proof could be concluded just by observing that the existence of the functions $a_1,\ldots,a_n$ implies the noncommutative integrability of the system. Nevertheless, it is easy to directly complete the proof.)

On account of a theorem of Lie and Caratheodory (see for instance [3,1,13]), the existence of functions $a_1,\ldots,a_n$ with the stated properties implies that every point $(b_0',\varphi_0) \in B' \times \mathbb{T}^n$ has a neighbourhood $B'' \times W$, $W \subset \mathbb{T}^n$, with coordinates $(a_i,p_s,q_s,\tilde{\alpha}_i)$, $i = 1,\ldots,n$, $s = 1,\ldots,d-n$, which are canonical, in the sense that $\mathcal{C}_*\sigma = da_i \wedge d\tilde{\alpha}_i + dp_s \wedge dq_s$. This expression of $\mathcal{C}_*\sigma$ implies that, for every $i = 1, \ldots, n$, the local coordinate $\tilde{\alpha}_i$ is the time along the flow of the Hamiltonian vector field of a_i, which as already observed is $\frac{\partial}{\partial \varphi_i}$. Therefore, there exist functions $\psi_i(b)$ defined in B'' such that $\tilde{\alpha}_i(b, \varphi) = \varphi_i + \psi_i(b)$ if $(b, \varphi) \in B'' \times W$. But then, one can extend the canonical coordinates $(a, p, q, \tilde{\alpha})$ to canonical coordinates (a, p, q, α) defined in all of $B'' \times \mathbb{T}^n$ by letting $\alpha_i = \varphi_i + \psi_i \mod 2\pi$. These coordinates are local generalized action–angle coordinates for X, which is therefore canonically integrable. ∎

Remarks: (i) The heart of the the proof of Proposition 1 is showing that S2 implies S1. As we have already observed in the Introduction, this fact is obvious in the nondegenerate case $n = d$. The situation is different in degenerate cases ($n < d$), since not every fibration with compact isotropic fibers admits canonical integrating coordinates: according to the noncommutative generalizations of the Liouville–Arnol'd theorem, this happens indeed if and only if such a fibration has a special property, i.e., it possesses a 'polar' foliation (see the Appendix). Therefore, the above proof amounts to showing that, in the case under consideration, the existence of the polar foliation is implied *de facto* by the assumption that the invariant tori carry a locally Hamiltonian linear flow which satisfies hypothesis H2.

(ii) The same arguments used in the proofs of the Lemma and of Proposition 1 could be used to prove that, if a locally Hamiltonian system has an embedded invariant torus on which the flow is linear and nonresonant, and if the symplectic two–form is exact (in a tubular neighbourhood of such a torus), then such a torus is isotropic. This generalizes a known result for the Lagrangian case (see [14]).

4. Integrability with n frequencies and canonical integrability

A. As an application of the results of the previous section, we give now a few sufficient conditions for a system which is integrable with n frequencies to be canonically integrable:

Proposition 2 *Let X be a locally Hamiltonian vector field on a symplectic manifold (M, σ) which is integrable with n frequencies. Then, X is canonically integrable with n frequencies whenever one of the following conditions is satisfied:*

(C1) σ is exact.

(C2) M is connected and, in some system of integrating coordinates (2.1), the derivative of the frequency map $\omega : B \to \mathbb{R}^n$ has rank either n or $n-1$ at one point of B.

(C3) $n = 1$.

(C4) $n = 2$ and X is Hamiltonian.

Proof. The sufficiency of conditions C1 and C3 follows from, respectively, conditions S3 and S2 of Proposition 1 (recall that every one–dimensional submanifold is isotropic).

In order to prove the sufficiency of condition C4, we can clearly restrict ourselves to a system of integrating coordinates (2.1). By equation (3.2), the existence of a global Hamiltonian for

X implies $G\omega(b) = 0$ at every point $b \in B$. If $n = 2$ this implies $G = 0$, since every non–zero 2×2 antisymmetric matrix is non–singular. Hence the tori are isotropic and the system is noncommutatively integrable.

We now prove the sufficiency of C2. By hypothesis, there is a system of integrating coordinates $(b, \varphi) : N \to B \times \mathbb{T}^n$ such that the derivative of the map $b \mapsto \omega(b)$ has rank $r \geq n-1$ at a point $b_0 \in B$. Let us restrict initially to this coordinate system. According to the Lemma, both G and $G\omega$ are constant, so that

$$G\left(\omega(b') - \omega(b'')\right) \;=\; 0 \qquad \text{for all } \; b', b'' \in B\,. \tag{4.1}$$

But ω maps a neighbourhood of b_0 onto an r–dimensional submanifold of $\mathbb{R}^n$, so that equation (4.1) implies $\dim(\ker G) \geq r$. Since $r \geq n-1$ and G is antisymmetric, this in turn implies that the range of G has dimension either zero or one. Hence $G = 0$, since the range of any antisymmetric non–zero matrix is at least two–dimensional. This proves that the invariant tori of X are isotropic, and thus that X is canonically integrable, within the considered domain N. We now extend this result to the whole manifold M. Let us consider any other system of integrating coordinates $\mathcal{C}' = (b', \varphi') : N' \to B' \times \mathbb{T}^n$, with $N' \cap N \neq \emptyset$, and write

$$\mathcal{C}'_* \sigma \;=\; E'_{rj} db'_r \wedge d\varphi'_j \;+\; \frac{1}{2} F'_{rs} db'_r \wedge db'_s \;+\; \frac{1}{2} G'_{ij} d\varphi'_i \wedge d\varphi'_j\,.$$

Hypothesis H2 implies that, in $N \cap N'$, one has $b' = b'(b)$; consequently, denoting $J = \partial\varphi'/\partial\varphi$, one has $G = J^T G' J$. Thus, if M is connected, the condition $G = 0$ is transported to all local integrating coordinate systems forming an atlas for M, proving that all the invariant tori of X are isotropic. ∎

The conditions given in Proposition 2 are just indicative. In particular, conditions C2–C4 are sufficient conditions for the more general condition range$(d\omega) \subset \ker(G)$. In the nondegenerate case $n = d$, the sufficiency of conditions similar to C2 and S2 was already known to Gallavotti [10].

B. According to Propositions 1 and 2, integrability with n frequencies and canonical integrability with n frequencies are equivalent notions in most situations. Nevertheless, there is some room left for Hamiltonian systems on non–exact symplectic manifolds which are integrable with $n \geq 3$ frequencies but not canonically integrable. As is clear from the Lemma and from Proposition 1, the possibility of constructing systems of this kind depends on the possibility of constructing a frequency map ω which is nonresonant at a dense set of points, satisfies $\partial h/\partial b = E\omega$ with some function $h(b)$ and some matrix $E(b)$, and is such that $G\omega = 0$ with some non–zero antisymmetric constant matrix G.

It is easy to construct examples of this situation: given any integer $n \geq 3$ and any nonresonant constant vector $\omega \in \mathbb{R}^n$, consider a non–zero antisymmetric real matrix G such that $\omega \in \ker(G)$ and define the two–form

$$\sigma \;=\; dI_i \wedge d\varphi_i \;+\; \frac{1}{2} G_{ij} d\varphi_i \wedge d\varphi_j \tag{4.2}$$

on $\mathbb{R}^n \times \mathbb{T}^n$; then, the vector field $X = \omega \cdot \frac{\partial}{\partial\varphi}$ on $(\mathbb{R}^n \times \mathbb{T}^n, \sigma)$ is Hamiltonian, with Hamiltonian $H(I) = \omega \cdot I$, and is integrable with n frequencies but not completely integrable.

An example of this type, with $n = 3$, was given in the Introduction. It can be of interest to observe that, if $n = 2$, one can construct locally Hamiltonian (although not Hamiltonian) vector fields which are integrable with two frequencies but not canonically integrable. An example is $X = \frac{\partial}{\partial\varphi_1} + \sqrt{2}\,\frac{\partial}{\partial\varphi_2}$ on $\mathbb{R}^2 \times \mathbb{T}^2$ with symplectic form $\sigma = dI_1 \wedge d\varphi_1 + dI_2 \wedge d\varphi_2 + d\varphi_1 \wedge d\varphi_2$.

These examples of systems which are integrable but not completely integrable are certainly very special and artificial. We do not know whether any system of this kind arises from situations of physical interest – even though the results of this paper seem to pose quite severe limitations to this possibility.

However, it seems to us that the very existence of these examples raises the question of whether complete (or noncommutative) integrability provides the ultimate characterization of integrability for Hamiltonian systems. In particular, it would be interesting to establish whether there exists an integrability criterion, which makes reference to properties of the integrals of motion and of the symplectic manifold alone (not to properties of the motions, as we did in Definition 1), and which includes both complete integrability and integrability with n frequencies.

Remarks: (i) In the above example, the vector field $\omega \cdot \frac{\partial}{\partial\varphi}$ is Hamiltonian (incidentally, with the same Hamiltonian $\omega \cdot I$) also with respect to the standard symplectic structure $\sum dI_i \wedge d\varphi_i$ on $\mathbb{R}^n \times \mathbb{T}^n$, and with respect to such a structure it is obviously completely integrable. This fact could be interpretatively relevant, but we do not know whether it is true in general. For instance, it is not difficult to see that a second symplectic structure for which the invariant tori are isotropic always exists locally, within the domain of a system of integrating coordinates, if the matrix E entering (3.1) is the identity (or more generally, if E has everywhere rank n since, on account of the nondegeneracy of σ, one can thus make a local change $b' = b'(b)$, $\varphi' = \varphi$ of integrating coordinates so as to get $E = \mathrm{id}$.) However, even assuming that the matrices E can be taken to be the identity within every coordinate system, it is plausible that, if M is a manifold, there are obstructions to the global existence of a symplectic structure with this property.

(ii) Since the two symplectic forms (4.2) and $\sum dI_i \wedge d\varphi_i$ form a pencil, the vector fields $\omega \cdot \frac{\partial}{\partial\varphi}$ considered in the above example are biHamiltonians. Hence, they are examples of biHamiltonian systems which are completely integrable with respect to one structure but not to the other.

Appendix

In this Appendix we give some examples of systems which are integrable with n frequencies and we discuss the equivalence between canonical and complete or noncommutative integrability.

A. First of all, we observe that hypothesis H2 entering Definition 1 is satisfied whenever a dense subset of the invariant n–dimensional tori are closure of trajectories. (The difference between this condition and hypothesis H2 is very slight, and they are in fact equivalent in the analytic case, see [3]). This in turn amounts to the fact that, in every local system of integrating coordinates, the frequency vector $\omega(b)$ is nonresonant (i.e., $\omega(b) \cdot \nu \neq 0$ for all $\nu \in \mathbb{Z}^n$, $\nu \neq 0$) at a dense set of points $b \in B$. The latter condition is satisfied, in particular, in the two opposite cases in which, within every system of integrating coordinates, either $\omega \in \mathbb{R}^n$ is a constant nonresonant vector or the derivative of the frequency map $\omega \,:\, B \,\to \mathbb{R}^n$ has everywhere rank n.

Using this fact, it is easy to verify that various systems of mechanics, restricted to the subset M of their phase space fibered by n–dimensional invariant tori, are integrable with n frequencies, for instance: the free rigid body with a fixed point ($d = 3$, $n = 1$), any system of $d \geq 1$ uncoupled harmonic oscillators ($n = d - k$, if k is the number of independent resonance relations), Kepler ($d = 3$, $n = 1$) and, more generally, every system with all motions periodic whose period does not identically vanish in any open set ($n = 1$).

B. As an immediate consequence of Proposition 1, we have the following

Proposition A1 *For a system which is integrable with n frequencies, with any $1 \leq n \leq d$, canonical integrability is equivalent to the fact that every invariant torus has a neighbourhoud in which are defined d independent integrals of motion in involution.*

Proof. One implication is obvious, as already observed in Section 2. For the other observe that, if any set of d independent integrals of motion do exist in an open invariant set, then, in that set, every n–dimensional invariant torus is contained within a level set of these functions. If these functions are in involution, then their level sets are Lagrangian, and so the invariant tori are isotropic. By Proposition 1, this implies that the system is canonically integrable. ■

Note that Proposition A1 does not follow from the Liouville–Arnol'd theorem, since it is not assumed that the d integrals in involution have compact level sets or that their Hamiltonian vector fields are complete.

According to Proposition A1, for systems which are integrable with n frequencies, canonical integrability is equivalent to complete integrability in a 'local' sense. Globally, however, the situation is different.

One reason is that, even if the system is nondegenerate ($n = d$), it may happen that the fibration by the invariant tori cannot be described *globally* by a single set of independent functions in involution. As is obvious, this fact can be overcome by focusing on the fibration, rather than on the functions used to describe it, and defining complete integrability as the existence of a fibration with compact, Lagrangian, invariant fibers (see [15]; one could obviously relax the requirement of compactness, but this is the case of interest here). Once this has been done, canonical integrability with the maximal number d of frequencies turns out to be equivalent to complete integrability. (For a proof, see Proposition A2 below).

On the other hand, in the case of a degenerate system ($n < d$), it may happen that there is no invariant Lagrangian fibration which is defined in the whole subset M of the phase space fibered by the n–dimensional invariant tori. Examples of this situations are met for instance with the rigid body, Kepler, the $1:1$ resonance (see [16]). In these cases, in order to get a global picture of the system, one should focus on the fibration by the n–dimensional tori, and this is exactly what noncommutative integrability does.

Noncommutative integrability is often defined, in a local way (analog of the existence of d integrals in involution for complete integrability), by the existence of $2d-n$ independent integrals of motion $F_1, \ldots, F_{2d-n}$ such that the Poisson brackets $\{F_i, F_j\}$ are functionally dependent on $F_1, \ldots, F_{2d-n}$ and are such that the matrix with entries $\{F_i, F_j\}$ has everywhere rank $2d - 2n$ [1,2]. The following is a possible globalization of this definition [4,5]:

Definition A1. A locally Hamiltonian system on (M, σ) is said to be *noncommutatively integrable* if there exists a fibration of M which has compact isotropic invariant fibers of a certain dimension $n \leq d$ and which possesses a polar foliation.

The polar foliation of a fibration, if it exists, is the foliation whose leaves have for tangent spaces the symplectic orthogonals to the tangent spaces of the fibers of the fibration. Since every Lagrangian fibration coincides with its own polar, when $n = d$ noncommutative integrability reduces to complete integrability (in the 'global' sense).

According to some generalizations of the Liouville–Arnol'd theorem [2–5], the connected components of the fibers of an isotropic fibration which possesses a polar foliation are n–dimensional tori, and every one of them has a neighbourhood N equipped with 'generalized action–angle coordinates', that is, a diffeomorphism $\mathcal{C} = (p, q, a, \alpha) : N \to B \times \mathbb{T}^n$ which satisfies (2.2) and is such that the fibers of the fibration are described by $(a, p, q) = \text{const}$.

We may now prove the following

Proposition A2. *Let X be a locally Hamiltonian vector field defined on a symplectic manifold (M, σ) which is integrable with n frequencies. Then, X is canonically integrable if and only if it is noncommutatively integrable.*

Proof. Assume that X is integrable with n frequencies and noncommutatively integrable. On account of hypothesis H2, in the intersection of the domain of a system of integrating coordinates (b, φ) and of a system of generalized action–angle coordinates (p, q, a, α), one has $b = b(a, p, q)$, and viceversa. Hence, the generalized action–angle coordinates are local integrating coordinates; since they are canonical, X is canonically integrable.

Conversely, if X is canonically integrable with n frequencies, then its invariant tori are isotropic. Thus, the only thing we must prove is that the fibration by these tori has a polar foliation. Now, within the domain of any system of canonical integrating coordinates, the equation $a = \text{const}$ defines a foliation which is polar to the invariant tori. So, it suffices to prove that the leaves of these local foliations coincide in the intersection of their domains. This amounts to the fact that, if (p, q, a, α) and (p', q', a', α') are two local systems of canonical integrating coordinates, then in the intersection of their domains one has $a' = a'(a)$. This follows indeed from the fact that the transition functions between any two such coordinate systems have the form

$$a' = Za + z\,, \qquad \alpha' = (Z^T)^{-1}\alpha + \mathcal{F}(a, p, q)\,, \qquad p' = \mathcal{P}(a, p, q)\,, \qquad q' = \mathcal{Q}(a, p, q)\,,$$

where Z and z are, respectively, a constant matrix and a constant vector, and $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{Q}$ are maps. The proof of this fact can be found in [3] (section 2; one should take into account that, because of hypothesis H2, α and α' are coordinates on the same tori). ∎

Notes added in proofs: (i) During the refereeing process of this paper, there appeared two papers by O.I. Bogoyavlenskij (*Commun. Math. Phys.* **180**, 529, 1996 and **184**, 301, 1997) which have a significant overlapping with the present one. Indeed, they contain formula (3.1) for the local symplectic structures, an analogue of (Gallavotti's) condition C2 for the nondegenerate case $n = d$, and, what is very interesting, a generalization of complete and noncommutative integrability which embraces the examples of Section 4.B (as well as those mentioned in point ii below) and answers the question raised at the end of Section 4.

(ii) A different class of systems which are integrable but not completely integrable had already been discovered by I.O. Parasjuk (*Ukr. Math. J.* **36**, 380, 1984; **46**, 572 and 994, 1994); these are systems with motions linear on coisotropic tori.

(iii) A minor modification of the proof of the Lemma shows that, if the symplectic two–form is exact, then an invariant torus of a Hamiltonian system on which the flow is linear and dense is necessarily isotropic.

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