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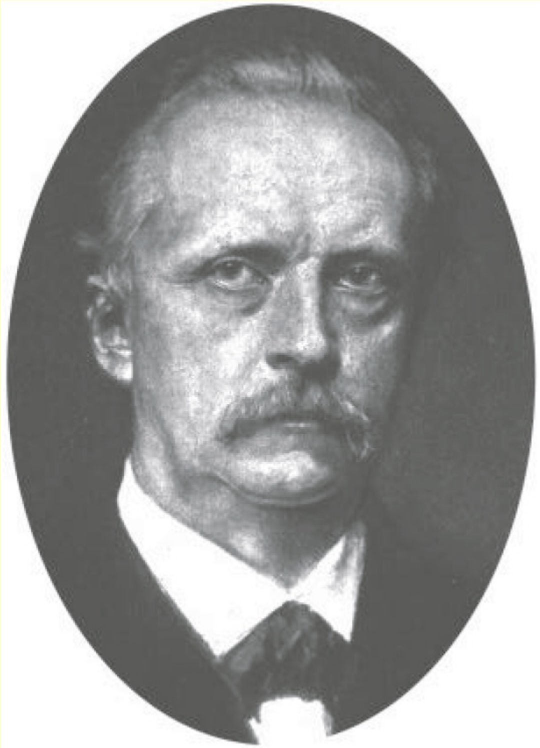
Vita(, Morte) e Miracoli dello Spostamento Termico

“A Thermodynamics Day”
Padova, 5 giugno 2009

VITA

Thermal displacement, what the hell is that?

In principio erat HELMHOLTZ (1821-1894)



- **H von H, Prinzipien der Statik monocyklischer Systeme (1884)**
- **H von H, Studien zur Statik monocyklischer Systeme (1884)**

In fine (?) erat GAM

- **C. Dascalu & GAM, The thermoelastic material-momentum equation. J. Elasticity 39, 201-212 (1995)**
- …

Given that *in principio* erat Helmholtz and

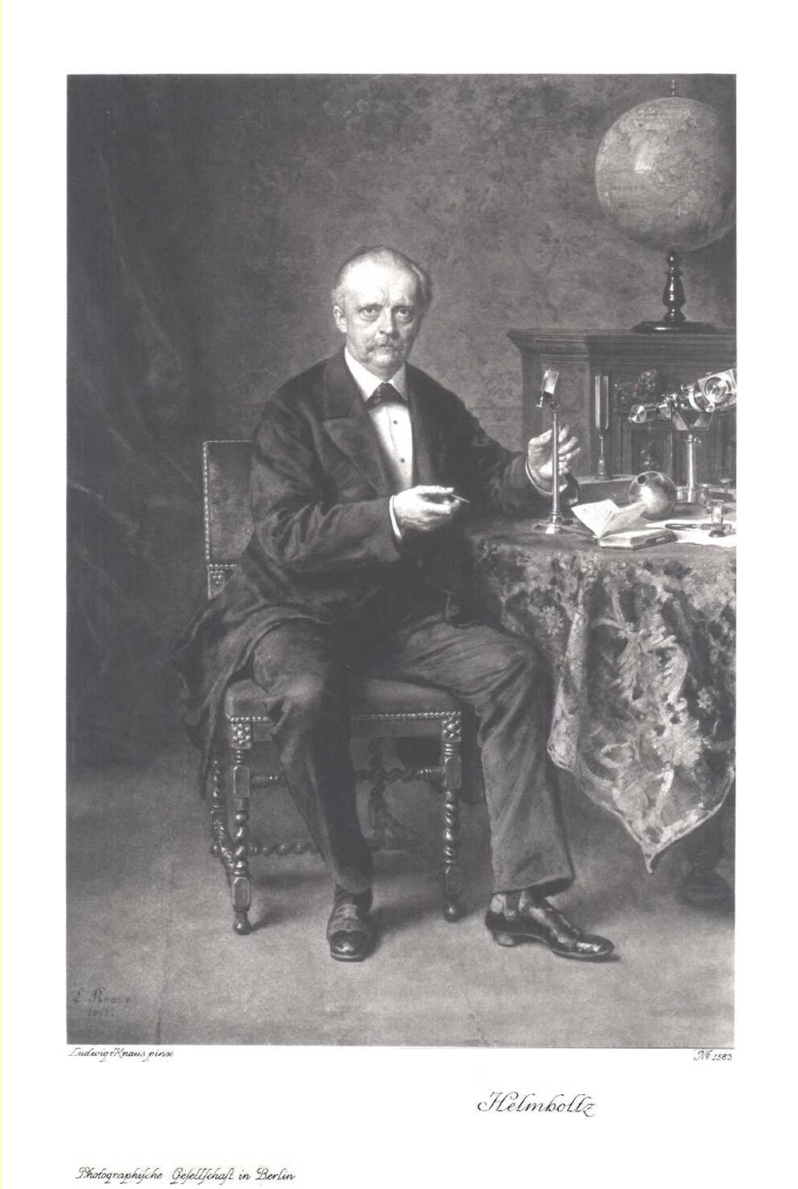
***in fine* erat GAM,**

Q1. Was there anything *in medio*?

Q2. What comes next?

Before we try and answer, . . .

…back to H von H



with the help of Cornelius Lanczos* and of . . .

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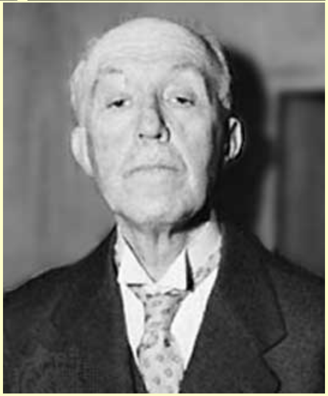
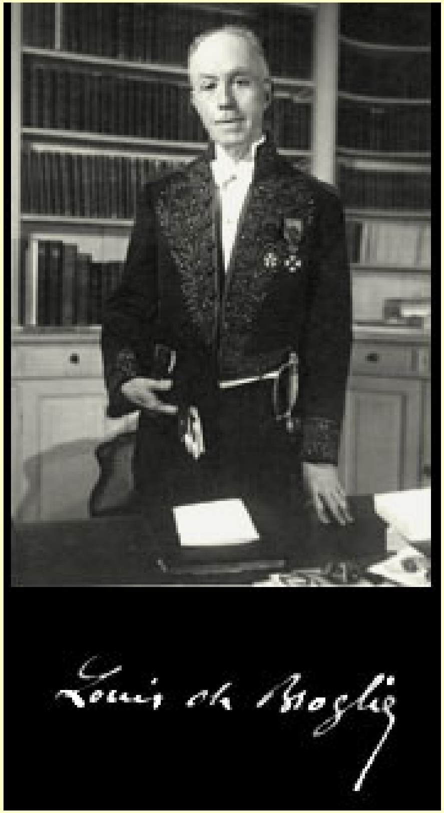
* **C. L., The Variational Principles of Mechanics, Dover, 1986**

Louis de Broglie (1892-1987)



L de B, ***La Thermodynamique de la particule isolée (ou Thermodynamique cachée des particules)*****, Gauthier-Villars, Paris (1964)**

Helmholtz's *Monocyclic* Lagrangian Systems. 1

Consider a lagrangian where the $n-$th coordinate is missing:

$$L = L(q_1, \ldots, q_{n-1}, \not{q}_n; \dot{q}_1, \ldots, \dot{q}_n; t).$$

Then,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = 0 \quad \Leftrightarrow \quad p_n = \frac{\partial L}{\partial \dot{q}_n} = c_n, \text{ a motion constant} \quad \Rightarrow$$

$$\dot{q}_n = f(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_{n-1}; c_n; t). \qquad (*)$$

Strategy

- **With the use of ($*$), obtain $t \mapsto (q_1(t), \ldots, q_{n-1}(t))$, the time evolution of the 'slow' coordinates;**
- **having this information, revert to ($*$), and obtain the time evolution of the 'fast' coordinate q_n by quadrature.**

Note **A fast coordinate is not observable, its time rate is.**

Helmholtz's *Monocyclic* Lagrangian Systems. 2

Require the action $\mathcal{A} = \int_{t_1}^{t_2} L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n; t)\, dt$ **be stationary under the constraint**

$$\dot{q}_n = f(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_{n-1}; c_n; t). \qquad (*)$$

Formally,

$$\delta \left(\int_{t_1}^{t_2} \left(L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n; t) + \lambda \dot{q}_n \right) dt \right) = 0$$

implies

$$\lambda = -\frac{\partial L}{\partial \dot{q}_n} = -p_n = -c_n \, .$$

Accordingly, one can introduce the *modified lagrangian*

$$\overline{L}(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_{n-1}; c_n; t) =: L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n; t) - c_n \dot{q}_n,$$

where $(*)$ **specifies** $\dot{q}_n$**.**

Helmholtz's *Monocyclic* Lagrangian Systems. 3

Split $L = K - U$, with $U = U(q_1, q_2, \ldots, q_{n-1}; t)$ the *potential energy* and

$$K = \frac{1}{2}\sum_{i=1}^{n-1} a_{ik}\dot{q}_i\dot{q}_k + \left(\sum_{i=1}^{n-1} a_{in}\dot{q}_i\right)\dot{q}_n + \frac{1}{2}a_{nn}\dot{q}_n^2$$

the *kinetic energy*. This implies that $\overline{L} = \overline{K} - \overline{U}$, with

$$\overline{K} := \frac{1}{2}\sum_{i=1}^{n-1} a_{ik}\dot{q}_i\dot{q}_k + \frac{1}{2}\Big(\sum_{i=1}^{n-1} a_{in}\dot{q}_i\Big)^2 + a_{nn}^{-1}c_n\sum_{i=1}^{n-1} a_{in}\dot{q}_i, \quad \overline{U} := U + \frac{1}{2}a_{nn}^{-1}c_n^2\,.$$

- **If $a_{in} \neq 0$, then (i) *kinetic coupling* between fast variable q_n and slow variable q_i; (ii) *gyroscopic term* $a_{in}\dot{q}_i$ in the kinetic energy.**
- ***potential energy augmented by a positive contribution of kinetic origin*, a manifestation of the 'ghost' variable q_n, the only manifestation in case of no kinetic coupling.**

Helmholtz's Heat Theorem. 1

Stripped-to-the-bones motion equations are:

$$\frac{\partial L}{\partial q_s} + \cancel{g}_s = 0, \quad -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_f}\right) + g_f = 0 \;\Leftrightarrow\; -\dot{p}_f + g_f = 0$$

(one fast coordinate, one slow; $g_f \equiv$ only external force).
The incremental working performed on the system is:

$$dQ = g_f dq_f = \dot{p}_f \dot{q}_f dt = \dot{q}_f dp_f \quad \Rightarrow \quad \frac{dQ}{2\,K} = d(\log p_f).$$

Formally, on setting

$$2\,K =: temperature\;\; T \quad \textbf{and} \quad \log p_f := entropy\;\; S,$$

we have:

$$\frac{dQ}{T} = dS,$$

with '*coldness*' T^{-1} as the integrating factor. Note that this conclusion does not depend on the form of the potential energy.

Helmlholtz's Heat Theorem. 2

When generalized so as to apply to a *multidimensional ergodic system*, this result takes the form of the *heat theorem*:

$$\frac{dE + PdV}{T} = dS_\Phi, \tag{1}$$

where S_Φ is the *volume entropy*:

$$S_\Phi(E) := \log \Phi(E), \quad \Phi(E) := \int_{P_h S_p} H(E - \mathcal{H}(\mathbb{Q}, \mathbb{P}))\, d\mathbb{Q} d\mathbb{P}\,,$$

and $\quad T := 2 < K >, \quad E = K + U, \quad P := \left\langle -\frac{\partial U}{\partial q} \right\rangle.$

Recall that

$$\Phi'(E) = \Omega(E) = \int_{P_h S_p} \delta(E - \mathcal{H}(\mathbb{Q}, \mathbb{P}))\, d\mathbb{Q} d\mathbb{P},$$

$$< F > (E) := \frac{1}{\Omega(E)} \int_{P_h S_p} F(\mathbb{Q}, \mathbb{P})\, \delta(E - \mathcal{H}(\mathbb{Q}, \mathbb{P}))\, d\mathbb{Q} d\mathbb{P}\,.$$

LES GRANDS PROBLÈMES DES SCIENCES

ouvrages réunis par Mme P. Février

N° 17

La Thermodynamique de la particule isolée

(ou Thermodynamique cachée des particules)

par Louis de Broglie

de l'Académie française,

Secrétaire perpétuel de l'Académie des Sciences

1964

GAUTHIER-VILLARS ÉDITEUR

PARIS

[from p. 63] "Le schéma canonique de la Thermo-dynamique de Helmholtz part essentiellement de l'introduction d'*une vari-able α dont la température est la dérivée par rapport au temps*, mais *la significa-tion de la variable α reste mysterieuse*."

Thermal displacement, additional references

(just to hint at what erat *in medio*)

References from the physical literature

- **M. von Laue, *Relativitätstheorie*, Vol. 1, Braunschweig (1921).**
- **D. van Dantzig, On the phenomenological thermodynamics of moving matter. Physica 6, 673-704 (1939).**
- **…**
- **V.E. Kuzmichev and V.V. Kuzmichev, Accelerating Quantum Universe. Acta Phys. Pol. B 39 (2008)**

NOTE! For von L., thermacy = *minus* thermal displacement; van D. calls the thermal displ. thermasy, just as the authors to follow, who are in search of a hamiltonian structure for relativistic perfect fluids:

- **D. Bao, J. Marsden, and R. Walton, The Hamiltonian Structure of General Relativistic Perfect Fluids. Comm. Math. Phys. (1985)**
- **J. Kijowski, A. Smólski, and A. Górnicka, Hamiltonian theory of self-gravitating perfect fluid …Phys. Rev. D 41 (1990)**
- **…**

CM references, where authors are in search of

a Hamiltonian structure
for nonrelativistic dissipative materials

- **G.A. Maugin**, Towards an analytical mechanics of dissipative materials. Rend. Sem. Mat. Univ. Pol. Torino 58 (2), 171-180 (2000).
- **G.A. Maugin** and **V.K. Kalpakides**, The slow march towards an analytical mechanics of dissipative materials. Tech. Mech. 22 (2), 98-103 (2002).
- **G.A. Maugin** and **V.K. Kalpakides**, A Hamiltonian formulation for elasticity and thermoelasticity. J. Phys. A: Math. Gen. 35, 10775-10788 (2002).

CM references where authors are in search of an alternative formulation for thermomechanics

- **A.E. Green** and **P.M. Naghdi**, A re-examination of the basic postulates of thermomechanics. Proc. Royal Soc. London A432, 171-194 (1991).
- **A.E. Green** and **P.M. Naghdi**, A demonstration of consistency of an entropy balance with balance of energy. ZAMP 42, 159-168 (1991).
- **A.E. Green** and **P.M. Naghdi**, Thermoelasticity without energy dissipation. J. Elasticity 31, 189-208 (1993).
- **V.K. Kalpakides** and **C. Dascalu**, On the configurational force balance in thermomechanics. Proc. R. Soc. London. A458, 3023-3039 (2002).
- **S. Bargmann** and **P. Steinmann**, Theoretical and computational aspects of non-classical thermoelasticity. Comp. Meth. Appl. Mech. Eng. 196, 516-527 (2006).

MIRACOLI

A Virtual-Working Format for Thermomechanics*

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* **P. P-G**, Continuum Mechanics and Thermodynamics, Volume 20 (8), pp. 479-487 (2009)

building upon an idea sketched in

P. P-G and **A. Tiero**, Un formato tipo lavori virtuali per la termodinamica dei processi omogenei. Proc. XIV Congr. Naz. Meccanica Teor. Appl. (Como, Italy - October 1999)

The Standard Mechanical & Thermal Structures

- ***mechanical* body structure** **based on**
  - **Virtual Working Principle**

$$\int_{\mathcal{P}} \boldsymbol{S} \cdot \nabla \boldsymbol{v} = \int_{\mathcal{P}} \boldsymbol{d} \cdot \boldsymbol{v} + \int_{\partial \mathcal{P}} \boldsymbol{c} \cdot \boldsymbol{v} , \quad \boldsymbol{v} \in \mathcal{V}, \ \mathcal{P} \subset \mathcal{B} ,$$

  implying force balance ;

- **thermal body structure based on**
  - **energy balance**

$$\dot{\varepsilon} = -\mathbf{div}\, \boldsymbol{q} + r ,$$

  - **entropy imbalance**

$$\dot{\eta} \geq -\mathbf{div}\, \boldsymbol{h} + s, \quad \textbf{with} \ \ \boldsymbol{h} = \vartheta^{-1} \boldsymbol{q}, \ \ s = \vartheta^{-1} r .$$

Body as a Composition of Mechanical & Thermal Structures. i. Kinetics

- **kinetic variables are**
  - **mechanical displacement $\boldsymbol{u}$, with**

$$\boldsymbol{v} = \dot{\boldsymbol{u}} \;\equiv\; \textbf{velocity},$$

  - **thermal displacement α, with**

$$\vartheta = \dot{\alpha} \;\equiv\; \textbf{temperature},$$

  defined over *space-time cylinder* $\mathcal{B} \times (0, T)$

- **process $(x,t) \mapsto \big(\boldsymbol{u}(x,t), \alpha(x,t)\big)$, with**

  $\big(\boldsymbol{v}(x,t), \vartheta(x,t)\big) \;\equiv\;$ **realizable velocity pair**

- $\big(\delta\boldsymbol{u}(x,t), \delta\alpha(x,t)\big) \;\equiv\;$ **virtual velocity pair**

Body as a Composition of Mechanical & Thermal Structures. ii. Dynamics

For $\mathcal{P}$ a *subbody* of $\mathcal{B}$, and $I = (t_i, t_f)$ a *subinterval* of $(0, T)$, dynamics specified by

(i) *internal virtual working*

$$\delta\mathcal{W}^{(i)} = \int_{\mathcal{P}\times I} (\boldsymbol{s}\cdot\delta\boldsymbol{u} + \boldsymbol{S}\cdot\nabla\delta\boldsymbol{u} + h\,\delta\alpha + \boldsymbol{h}\cdot\nabla\delta\alpha)\,,$$

where

- $\boldsymbol{s}$ **and** $\boldsymbol{S}$ $\equiv$ $0-$**th and** $1-$**st order mechanical interactions**
- h **and** $\boldsymbol{h}$ $\equiv$ $0-$**th and** $1-$**st order thermal interactions**

Note

Treatments of mechanical and thermal entities should be kept *as parallel as possible*. *Parallelism broken by*

- ***invariance requirements***

  in a *galilean observer change*,

$$\boldsymbol{v} \mapsto \boldsymbol{v}^+ = \boldsymbol{v} + \boldsymbol{t}, \quad \dot{\alpha} \mapsto \dot{\alpha}^+ = \dot{\alpha}.$$

  Thus, translational invariance of $\delta\mathcal{W}^{(i)}$ implies a symmetry-breaking conclusion:

  ***the 0th order stress* $\boldsymbol{s}$ *is null*.**

- ***habit***

***habitual entropy flux* = *minus the 1-st order thermal interaction* h.**

(ii) ***external virtual working***

$$\delta\mathcal{W}^{(e)} = \int_{\mathcal{P}\times I} (\boldsymbol{d}\cdot\delta\boldsymbol{u} + \boldsymbol{p}\cdot\dot{\overline{\delta\boldsymbol{u}}} + s\,\delta\alpha + \eta\,\dot{\overline{\delta\alpha}})$$
$$+\int_{\partial\mathcal{P}\times I}\big(\boldsymbol{c}\cdot\delta\boldsymbol{u} + c\,\delta\alpha\big) + \int_{\mathcal{P}\times\partial I}[\![\boldsymbol{p}\cdot\delta\boldsymbol{u} + \eta\,\delta\alpha]\!]$$

- $(\boldsymbol{d}, \boldsymbol{c})$ **&** $(s, c) \equiv$ ***mech.*** **&** ***therm.*** **(*distance*, *contact*)** ***interactions*;**
- $\boldsymbol{p}$ $\equiv$ **momentum**, $\big(\boldsymbol{S}\boldsymbol{n}$ $\equiv$ **momentum flux**$\big)$

  $\boldsymbol{d}$ $\equiv$ **momentum source** $\equiv$ **(*noninertial* distance force);**
- η $\equiv$ **entropy**, **(**$\boldsymbol{h}\cdot\boldsymbol{n}$ $\equiv$ **entropy flux)** s $\equiv$ **entropy source;**
- $\boldsymbol{c}$ $\equiv$ **contact force**, c $\equiv$ **contact heating;**
- $\boldsymbol{p}_f, \ldots, \eta_i$ $\equiv$ **mechanical** **&** **thermal** ***external actions*, *at time boundaries* of** $\mathcal{P}\times I$ **:**

$$\int_{\partial I}[\![\boldsymbol{p}\cdot\delta\boldsymbol{u} + \eta\,\delta\alpha]\!] := \boldsymbol{p}_f(x)\cdot\delta\boldsymbol{u}(x,t_f) + \eta_f(x)\,\delta\alpha(x,t_f)$$
$$+\boldsymbol{p}_i(x)\cdot\delta\boldsymbol{u}(x,t_i) + \eta_i(x)\,\delta\alpha(x,t_i)\,.$$

Notes

- **Just as** α **called** ***thermal displacement*** **to allude to role analogy with mechanical displacement** $\boldsymbol{u}$**,**

  why not to regard η **as the** ***thermal momentum*****, by analogy with the mechanical momentum** $\boldsymbol{p}$**?**

- **Just as** $\boldsymbol{p}$ **thought of as measuring** ***reluctance to quiet*****,**

  why not to think of η **as measuring** ***reluctance to order*****?**

The Virtual Working Axiom

(VW) ***The internal and the external working should be equal:***

$$\delta\mathcal{W}^{(i)} = \delta\mathcal{W}^{(e)},$$

for each virtual velocity pair defined over the closure of any subcylinder $\mathcal{P} \times I$ ***of*** $\mathcal{B} \times (0, T)$ ***and such as to vanish at the end of*** I ***itself.***

Implications of VW Axiom

- ***momentum*** **and** ***entropy balances*****:**

$$\dot{\boldsymbol{p}} = \textbf{Div}\,\boldsymbol{S} - \boldsymbol{s} + \boldsymbol{d}, \quad \dot{\eta} = \textbf{Div}\,\boldsymbol{h} - h + s \quad \textbf{in} \;\; \mathcal{P} \times I$$

- ***initial conditions*****:**

$$\boldsymbol{p}(x, t_i) = \boldsymbol{p}_i(x), \quad \eta(x, t_i) = \eta_i(x) \quad \textbf{for} \;\; x \in \mathcal{P}$$

- ***boundary conditions*** **on** $\partial\mathcal{P} \times I$**:**

$\boldsymbol{S}\boldsymbol{n} = \boldsymbol{c} \quad \equiv$ **balance of contact forces:** $\boldsymbol{c} + \boldsymbol{S}(-\boldsymbol{n}) = \boldsymbol{0}$;

$\boldsymbol{h} \cdot \boldsymbol{n} = c \quad \equiv$ **continuity cnd. on contact heating:** $c = (-\boldsymbol{h}) \cdot (-\boldsymbol{n})$

establishing $-\boldsymbol{h}$ **as a measure of specific** ***entropy influx*** **at a point of an oriented surface of normal** $\boldsymbol{n}$**.**

Conservation of Internal Action. Preliminaries

An integral consequence of momentum and entropy balances is:

$$W(\mathcal{P})+H(\mathcal{P})=\frac{d}{dt}\Big(\int_{\mathcal{P}}(\boldsymbol{p}\cdot\boldsymbol{v}+\eta\,\vartheta)\Big)+\int_{\mathcal{P}}\textit{\textbf{stuff}},$$

where

- ***noninertial working*** $\qquad W(\mathcal{P}):=\int_{\mathcal{P}}\boldsymbol{d}\cdot\boldsymbol{v}+\int_{\partial\mathcal{P}}\boldsymbol{c}\cdot\boldsymbol{v},$
- ***heating*** $\qquad H(\mathcal{P}):=\int_{\mathcal{P}}s\,\vartheta+\int_{\partial\mathcal{P}}c\,\vartheta\,,$
- $$\int_{\mathcal{P}}\textit{\textbf{stuff}}=\frac{d}{dt}\Phi(\mathcal{P}),$$ **where** ***internal action*** $\quad \Phi(\mathcal{P}):=\int_{\mathcal{P}}\varphi,$

whence

$$\textit{\textbf{stuff}}=\boldsymbol{s}\cdot\boldsymbol{v}+\boldsymbol{S}\cdot\nabla\boldsymbol{v}-\boldsymbol{p}\cdot\dot{\boldsymbol{v}}+h\,\vartheta+\boldsymbol{h}\cdot\nabla\vartheta-\eta\,\dot{\vartheta}:=\dot{\varphi}.$$

The Axiom of Conservation of Internal Action

(CIA) ***In a cycle, the noninertial working plus the heating supplied to or extracted from $\mathcal{P}$ sum to null*:**

$$\oint \big(W(\mathcal{P}) + H(\mathcal{P})\big) = 0\,.$$

Equivalently,

(CIA)′ ***In a cycle, the internal action is conserved*:**

$$\oint \Phi(\mathcal{P}) = 0\,.$$

Implications of CIA Axiom. The 1st Law

For $\tau \;\equiv\;$ **specific *total energy*:**

$$\tau := \varphi + \boldsymbol{p}\cdot\boldsymbol{v} + \eta\,\vartheta, \quad T(\mathcal{P}) := \int_{\mathcal{P}} \tau,$$

we have:

$$\dot{T}(\mathcal{P}) = W(\mathcal{P}) + H(\mathcal{P}),$$ **the First Law of TD.**

To see this,

- **set** ***entropy inflow*** $(-\boldsymbol{h}, s)$ **proportional to** ***energy inflow*** $(-\boldsymbol{q}, r)$ **through** ***coldness*****:**

$$\boldsymbol{h} = \vartheta^{-1}\boldsymbol{q}, \;\; s = \vartheta^{-1} r\,;$$

- **accept standard notion of specific** ***kinetic energy*** κ**:**

$$\kappa := \frac{1}{2}\boldsymbol{p}\cdot\boldsymbol{v}, \quad \textbf{with}\;\; \dot{\boldsymbol{p}}\cdot\boldsymbol{v} = \boldsymbol{p}\cdot\dot{\boldsymbol{v}},\; (\textbf{so that}\;\; \dot{\kappa} + (-\dot{\boldsymbol{p}})\cdot\boldsymbol{v} = 0)\,.$$

- **set**

$$\varepsilon := \tau - \kappa, \quad \varphi := \psi - \kappa,$$

  and interpret

  - $\varepsilon \equiv$ **specific** ***internal energy***
  - $\psi \equiv$ **specific** ***Helmholtz free energy*** $= \varepsilon - \eta\,\vartheta$**,**

  whence the interpretations for both the ***total energy*** τ **and the** ***internal action*** φ**.**

The Dissipation Axiom

(D) ***Whatever the process*** $(x,t) \mapsto \big(\boldsymbol{u}(x,t), \alpha(x,t)\big),$

$$h\,\dot{\alpha} \leq 0$$

over the ***space-time cylinder*** $\mathcal{B} \times (0,T)$**.**

Implications of D Axiom. The 2nd Law

- **Main implication is the generalized *dissipation inequality*:**

$$\dot{\psi} \leq -\eta\,\dot{\vartheta} + \boldsymbol{h}\cdot\nabla\vartheta + \boldsymbol{s}\cdot\boldsymbol{v} + \boldsymbol{S}\cdot\nabla\boldsymbol{v}\,.$$

- **If** $\vartheta \geq 0$ **(an unnecessary assumption so far),**

  then *entropy balance & D axiom imply*:

$$\dot{\eta} \geq \textbf{div}\,\boldsymbol{h} + s \quad \left(\textit{entropy inbalance} \equiv \textbf{ the Second Law of TD}\right).$$

- ***Standard dissipation inequality & entropy imbalance follow for***

$$\boldsymbol{h} = -\vartheta^{-1}\boldsymbol{q}\,,\quad s = \vartheta^{-1}\,r\,;\qquad \boldsymbol{s} = \boldsymbol{0}.$$

Thermal displacement, what the hell is that?

Is there a statistical notion of thermal displacement?

Listening to Green & Naghdi (1991):

"The temperature T (on the macroscopic scale) is generally regarded as representing (on the molecular scale) some 'mean' velocity magnitude or 'mean' (kinetic energy)$^{1/2}$. With this in mind, we introduce a scalar $\alpha = \alpha(X, \tau)$ through an integral of the form

$$\alpha = \int_{t_0}^{t} T(X, \tau) d\tau + \alpha_0, \qquad (7.3)$$

where t_0 denotes some reference time and the constant α_0 is the initial value of α at time t_0. In view of the above interpretation associated with T and the physical dimension of the quantity defined by (7.3), the variable α may justifiably be called thermal displacement magnitude or simply thermal displacement. Alternatively, *we may regard the scalar α (on the macroscopic scale) as representing a 'mean' displacement magnitude on the molecular scale* [italics are mine] and then $T = \dot{\alpha}$."

Listening to Kalpakides & Dascalu (2002):

"Taking the view of Green & Naghdi, we consider that $\alpha(X,t)$ *represents a second motion of the continuum, taking place at a different scale from the macroscopic motion $y(X,t)$*. This could be *a continuous representation of the lattice vibration*, a phenomenon of quantum-mechanical origin. [my italics] This justifies us to postulate complete independence in the corresponding observers of the two motions, hence to introduce two kind[s] of observer changes . . ."

Summing up:

- **(Green & Naghdi)**
  - **primary variable: either temperature or thermal displacement;**
  - **both variables should be given a statistical interpretation as some ensemble average.**
- **(Kalpakides & Dascalu)**
  - **thermal displacement as primary variable;**
  - **reference to lattice vibration appropriate only when thinking of a lattice is;**
  - **use of *microscopic observer changes*:**

$$\boldsymbol{v} \mapsto \boldsymbol{v}^+ = \boldsymbol{v}, \quad \dot{\alpha} \mapsto \dot{\alpha}^+ = \dot{\alpha} + \beta, \quad \beta = \textbf{an arbitrary scalar},$$

has equivalent consequences to my use of virtual velocities $\delta\alpha$.
(Recall: *galilean o. c.*, $\boldsymbol{v} \mapsto \boldsymbol{v}^+ = \boldsymbol{v} + \boldsymbol{t}, \quad \dot{\alpha} \mapsto \dot{\alpha}^+ = \dot{\alpha}.$)