The Impact of Disorder in the Critical Dynamics of Mean Field Models

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joint work with Paolo Dai Pra

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Outline

1. Introduction
2. The Curie-Weiss Model
3. The Kuramoto Model

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Step I: Limiting Dynamics

- Mean Field Interacting N-Particle System evolving as a CTMC on its state space for $t \in [0, T]$, T fixed

- Dynamics of the system $\leftrightsquigarrow$ Dynamics of its order parameter

- Order parameter: it is a function of the empirical measure

$$\rho_N := \frac{1}{N} \sum_{j=1}^{N} \delta_{j\text{-th state variable}} ,$$

  whose evolution is Markovian. A given system may admit or not a finite dimensional order parameter.

As $N \longrightarrow +\infty$, the limiting dynamics of the order parameter are deterministic (driven by a system of ordinary differential equations) and exhibit a phase transition (multiple equilibria).

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Step II: Fluctuations

We consider the fluctuations of the order parameter around its limiting dynamics.

Theorem

For $t \in [0, T]$, with T fixed, a Central Limit Theorem holds true for the order parameter; in other words, the fluctuations of the order parameter converge to a Gaussian process.

Whenever we deal with long time-interval, typically with $T = T(N)$, we can observe different behaviors:

1. Supercritical Regime: Metastability
2. Subcritical Regime: Central Limit Theorem continues to hold
3. Critical Regime: Critical Dynamics

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The Critical Dynamics describe the behavior of the fluctuations at the critical point in the infinite volume limit.

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Description of the Model

- We consider a system composed by N sites: $j = 1, \dots, N$
- At each site j we associate a spin value, which is a random variable σ_j taking values in $\{-1, +1\}$
- A configuration of the system is $\underline{\sigma} = (\sigma_j)_{j=1}^N \in \{-1, +1\}^N$

Example (N=18)

$$\text{site } j \rightsquigarrow \begin{cases} \bullet & \text{if} \quad \sigma_j = +1 \\ \circ & \text{if} \quad \sigma_j = -1 \end{cases}$$

$$\underline{\sigma} = (+1, -1, +1, +1, -1, +1, +1, +1, +1, -1, -1, +1, -1, +1, +1, -1, -1, +1)$$

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Dynamics of the System (I)

- The interaction is of the **mean field** type: each site interacts with all the others in the same way
  - $\Rightarrow$ there is no spatial geometry
  - $\Rightarrow$ it depends on the magnetization

$$m_N^\sigma(t) := \frac{1}{N}\sum_{j=1}^N \sigma_j(t)$$

- The rates of transitions are of the form

$$\sigma_j \longrightarrow -\sigma_j \quad \text{at rate} \quad e^{-\beta\sigma_j m_N^\sigma}$$

with $\beta > 0$ the inverse of the temperature

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Remark

No simultaneous jumps can happen!

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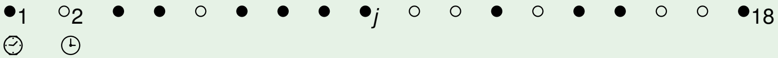

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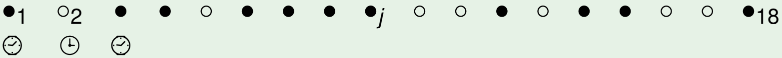

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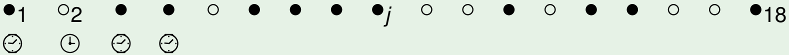

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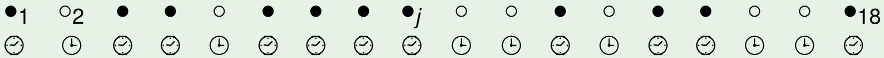

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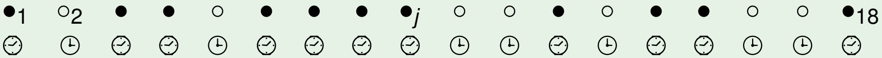

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● $\rightsquigarrow$ ⏲🔔 $\rightsquigarrow$ $e^{\frac{2}{9}\beta}$

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Dynamics of the System (III)

For $t \in [0, T]$, with T fixed, the process $\underline{\sigma}(t) = (\sigma_j(t))_{j=1}^N$ evolves as a Continuous Time Markov Chain with Infinitesimal Generator L_N, acting on functions $f : \{-1, +1\}^N \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{\sigma}) = \sum_{j=1}^{N} e^{-\beta \sigma_j m_N^{\sigma}} \nabla_j^{\sigma} f(\underline{\sigma})$$

- $\nabla_j^{\sigma} f(\underline{\sigma}) = f(\underline{\sigma}^j) - f(\underline{\sigma})$

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$\underline{\sigma}^j$: $\bullet_1$ $\circ_2$ $\bullet$ $\bullet$ $\circ$ $\bullet$ $\bullet$ $\bullet$ $\circ_j$ $\circ$ $\circ$ $\bullet$ $\circ$ $\bullet$ $\bullet$ $\circ$ $\circ$ $\bullet_{18}$

Addition of the Magnetic Field

- A configuration: $\underline{\sigma} = (\sigma_j)_{j=1}^N \in \{-1, +1\}^N$
- A realization of the magnetic field: $\underline{\eta} = (\eta_j)_{j=1}^N \in \{-1, +1\}^N$ a sequence of i.i.d. random variables, with law $\mu \sim \mathrm{Be}(1/2)$

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$$\text{site } j \rightsquigarrow \begin{cases} \uparrow & \text{if} \quad \eta_j = +1 \\ \downarrow & \text{if} \quad \eta_j = -1 \end{cases}$$

$$\underline{\eta} = (+1, -1, +1, -1, -1, +1, -1, +1, +1, +1, +1, -1, +1, -1, -1, -1, +1, +1)$$

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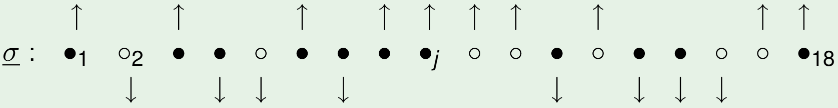

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Dynamics of the System (Final)

For given $\underline{\eta}$ and $t \in [0, T]$, T fixed, the process $\underline{\sigma}(t) = (\sigma_j(t))_{j=1}^N$ evolves as a Continuous Time Markov Chain with Infinitesimal Generator L_N, acting on functions $f : \{-1, +1\}^N \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{\sigma}) = \sum_{j=1}^{N} e^{-\beta \sigma_j (m_N^{\sigma} + h \eta_j)} \nabla_j^{\sigma} f(\underline{\sigma})$$

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Limiting Dynamics ($N \longrightarrow +\infty$)

Theorem

For $t \in [0, T]$, the order parameter converges to the solution of the following system of ordinary differential equations

$$\begin{aligned} \dot{m}_t^\sigma &= -2\, m_t^\sigma \cosh(\beta h) \cosh(\beta m_t^\sigma) - 2\, m_t^{\sigma\eta} \sinh(\beta h) \sinh(\beta m_t^\sigma) \\ &\quad + 2 \cosh(\beta h) \sinh(\beta m_t^\sigma) \\ \dot{m}_t^{\sigma\eta} &= -2\, m_t^\sigma \sinh(\beta h) \sinh(\beta m_t^\sigma) - 2\, m_t^{\sigma\eta} \cosh(\beta h) \cosh(\beta m_t^\sigma) \\ &\quad + 2 \sinh(\beta h) \cosh(\beta m_t^\sigma) \end{aligned} \tag{LD}$$

Remark

The limiting dynamics $(m_t^\sigma, m_t^{\sigma\eta})$ are deterministic.

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Stationary Solution(s) and Phase Diagram

Any equilibrium solution of (LD) is of the form

$$m_*^\sigma = \frac{1}{2}\left[\tanh(\beta(m_*^\sigma + h)) + \tanh(\beta(m_*^\sigma - h))\right]$$

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$(0, \tanh(\beta h))$ is always a stationary solution.

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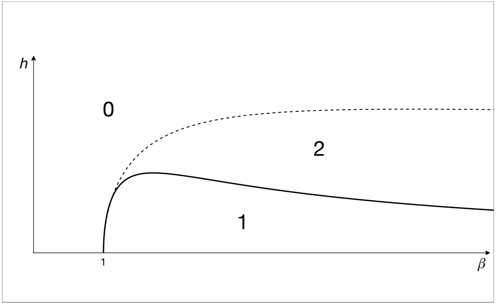

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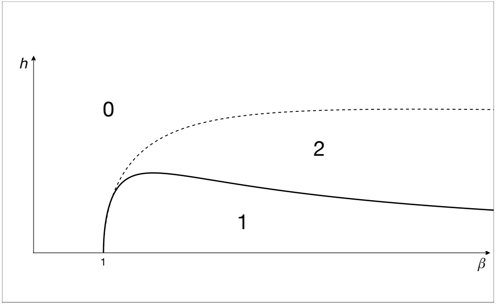

There are three phases corresponding to 0, 1 and 2 ferromagnetic solutions. The lower curve is $\beta \mapsto h(\beta) = \frac{1}{\beta}\text{arccosh}(\sqrt{\beta})$ for $\beta \geq 1$.

Central Limit Theorem

Theorem

For $t \in [0, T]$, with T fixed, the fluctuations of the magnetization

$$\begin{aligned} x_N^{(1)}(t) &:= \sqrt{N}\left(m_N^{\sigma}(t) - m_t^{\sigma}\right) \\ x_N^{(2)}(t) &:= \sqrt{N}\left(m_N^{\sigma\eta}(t) - m_t^{\sigma\eta}\right) \end{aligned}$$

converge to a limiting Gaussian Process $(x^{(1)}(t), x^{(2)}(t))$, which is the unique solution of

$$\begin{pmatrix} dx^{(1)}(t) \\ dx^{(2)}(t) \end{pmatrix} = 2\mathscr{H} A_1(t)\, dt + 2A_2(t) \begin{pmatrix} x^{(1)}(t) \\ x^{(2)}(t) \end{pmatrix} dt + D(t) \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$$

where $B_1(t)$, $B_2(t)$ are independent standard Brownian motions, $\mathscr{H}$ is a Gaussain random variable and $A_1(t)$, $A_2(t)$, $D(t)$ are suitable matrices.

Critical Dynamics ($\beta = \cosh^2(\beta h)$)

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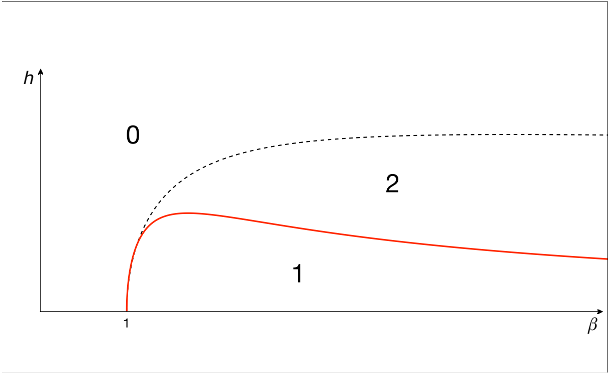

- Critical Direction: $m_N^\sigma(t) = \frac{1}{N}\sum_{j=1}^N \sigma_j(t)$

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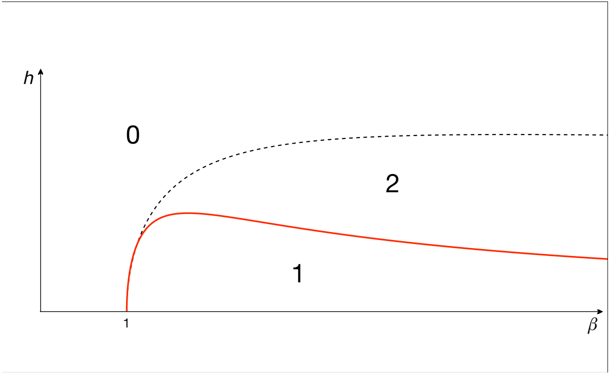

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Critical Dynamics ($\beta = \cosh^2(\beta h)$)

Theorem

For $t \in [0, T]$, if we consider the two-dimensional critical fluctuation process

$$\begin{aligned} y_N^{(1)}(t) &:= N^{1/4} m_N^{\sigma}(N^{1/4} t) \\ y_N^{(2)}(t) &:= N^{1/4} \left(m_N^{\sigma\eta}(N^{1/4} t) - \tanh(\beta h) \right) \end{aligned}$$

then, as $N \longrightarrow +\infty$, $y_N^{(2)}(t)$ collapses and $y_N^{(1)}(t)$ converges to the limiting Gaussian process

$$y^{(1)}(t) = 2\mathscr{H} \sinh(\beta h) t \,,$$

where $\mathscr{H}$ is a Gaussian random variable.

Critical Dynamics of the Homogeneous Model ($\mu \sim \delta_0$)

- Order parameter = critical direction: $m_N^\sigma(t) = \frac{1}{N}\sum_{j=1}^N \sigma_j(t)$

Theorem ($\beta = 1$)

For $t \in [0, T]$, if we consider the critical fluctuation process

$$y_N(t) := N^{1/4} m_N^\sigma(\sqrt{N}t)$$

then, as $N \longrightarrow +\infty$, $y_N(t)$ converges to a limiting Non-Gaussian process $y(t)$, which is the unique solution of

$$\begin{cases} dy(t) & = & -\frac{2}{3}y^3(t)dt + 2dB(t) \\ y(0) & = & 0 \end{cases}$$

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Outline

1. Introduction

2. The Curie-Weiss Model

3. The Kuramoto Model

Motivation

The transition **from inchoerence to collective synchronization** is a phenomenon occurring in biological, chemical, physical and social systems. It consists in a family of individuals spontaneously locking to a cooperative behavior, despite their intrinsic differences.

Examples: fireflies, applause, hearth cells, arrays of lasers, superconducting Josephson junctions,. . .

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- Kuramoto, 1975: analyzed a model of oscillators running at arbitrary frequencies and coupled through the sine of their phase differences

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`http://oldweb.ct.infn.it/~cactus/laboratorio/5-Metronome-Synchronisation.divx`

Dynamics of the System

Let $\underline{\eta} = (\eta_j)_{j=1}^N \in \{-1, +1\}^N$ be a sequence of i.i.d. random variables, with law $\mu \sim \mathrm{Be}(1/2)$.

For given $\underline{\eta}$, let $\underline{x}(t) = (x_j(t))_{j=1}^N$, with $t \in [0, T]$, be the N-diffusion system evolving in accord with the stochastic differential equations

$$dx_j(t) = \left[\omega \eta_j + \frac{\theta}{N} \sum_{k=1}^{N} \sin(x_k(t) - x_j(t)) \right] dt + dB_j(t)$$

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Limiting Dynamics ($N \longrightarrow +\infty$)

Theorem

For $t \in [0, T]$, the order parameter converges to solution of the following partial differential equation

$$\frac{\partial q_t}{\partial t}(x,\eta) = \frac{1}{2}\frac{\partial^2 q_t}{\partial x^2}(x,\eta) - \frac{\partial}{\partial x}\{[\omega\eta - \theta r_t \sin x] q_t(x,\eta)\} \tag{LD}$$

where

$$r_t = \int e^{ix} q_t(x,\eta)\, \mu(d\eta)\, dx\,.$$

Remark

The limiting dynamics q_t are deterministic.

We are going to determine the stationary solution(s).

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Any equilibrium solution of (LD) is of the form

$$q_*(x,\eta) = (Z_*)^{-1} \cdot e^{2(\omega\eta x+\theta r_* \cos x)} \left[e^{4\pi\omega\eta} \int_0^{2\pi} e^{-2(\omega\eta x+\theta r_* \cos x)} dx + (1-e^{4\pi\omega\eta}) \int_0^{x} e^{-2(\omega\eta y+\theta r_* \cos y)} dy \right],$$

where Z_* is a normalizing factor and r_* satisfies the self-consistency relation

$$\begin{array}{rcl} r_* & = & \Phi(r_*) \\ \Phi(r_*) & = & \int e^{ix}\, q_*(x,\eta)\, \mu(d\eta)\, dx \end{array} \qquad \text{(SC)}$$

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$r_* = 0$ is always a solution of (SC) and the corresponding stationary distribution is $q_*(x,\eta) = \frac{1}{2\pi}$.

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Solutions with $r_* = 0$ are called **incoherent**, while those with $r_* > 0$ are called **synchronized**.

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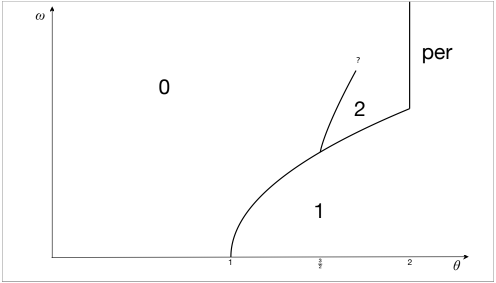

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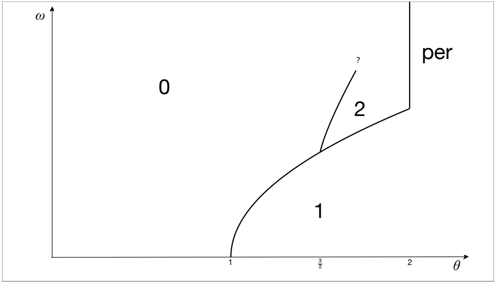

There are three phases corresponding to 0, 1 and 2 synchronized solutions. The lower curve is $\theta \mapsto \omega(\theta) = \frac{1}{2}\sqrt{\theta - 1}$ for $1 \leq \theta \leq 2$. The value $\theta = 2$ turns out to be a value above which non-stationary periodic solutions occur.

Central Limit Theorem

For $h \geq 1$, we are interested in the evolution of integrals of the type

$$X_h^{(1,N)}(t) := \int \cos(hx) d\hat{\rho}_N(t) \qquad X_h^{(2,N)}(t) := \int \sin(hx) d\hat{\rho}_N(t)$$

$$X_h^{(3,N)}(t) := \int \eta \cos(hx) d\hat{\rho}_N(t) \qquad X_h^{(4,N)}(t) := \int \eta \sin(hx) d\hat{\rho}_N(t)$$

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Theorem

The process $\left(X_h^{(1,N)}, X_h^{(2,N)}, X_h^{(3,N)}, X_h^{(4,N)} \right)_{h\geq 1}$ *converges to the Gaussian process* $\left(X_h^{(1)}, X_h^{(2)}, X_h^{(3)}, X_h^{(4)} \right)_{h\geq 1}$ *which is the unique solution of an infinite dimensional linear diffusion equation.*

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Theorem

The process $\left(X_h^{(1,N)}, X_h^{(2,N)}, X_h^{(3,N)}, X_h^{(4,N)}\right)_{h\geq 1}$ *converges to the Gaussian process* $\left(X_h^{(1)}, X_h^{(2)}, X_h^{(3)}, X_h^{(4)}\right)_{h\geq 1}$ *which is the unique solution of an infinite dimensional linear diffusion equation.*

Critical Dynamics ($\theta = 1 + 4\omega^2$)

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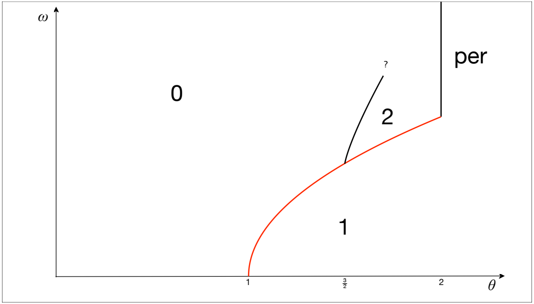

- Critical direction:

$$\text{span}\left\{ \int (\sin x + 2\omega\eta \cos x)\, d\rho_N, \int (\cos x - 2\omega\eta \sin x)\, d\rho_N \right\}$$

- We consider more "moderate" fluctuations

$$\widetilde{\rho}_N := N^{-1/4}\hat{\rho}_N = N^{1/4}\left[\frac{1}{N}\sum_{j=1}^{N}\delta_{(x_j,\eta_j)} - \frac{1}{2\pi}\right]$$

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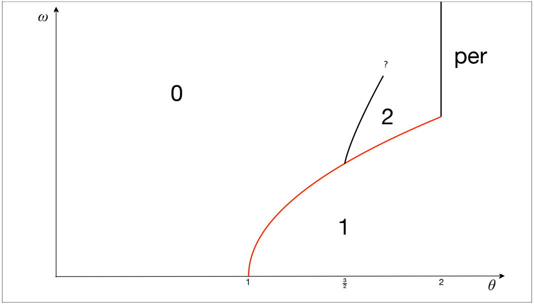

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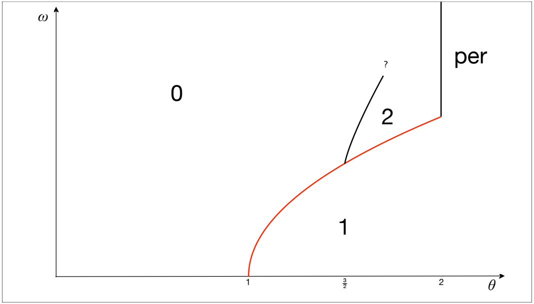

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Critical Dynamics ($\theta = 1 + 4\omega^2$)

Theorem

Assume $\omega < \frac{1}{2}$. *For* $t \in [0, T]$, *the process*

$$V^{(1,N)}(t) := \int (\sin x + 2\omega\eta \cos x)\, d\widetilde{\rho}_N(\sqrt{N}t)$$

$$V^{(2,N)}(t) := \int (\cos x - 2\omega\eta \sin x)\, d\widetilde{\rho}_N(\sqrt{N}t)$$

converges, as $N \longrightarrow +\infty$, *to a limiting Non-Gaussian process* $(V^{(1)}(t), V^{(2)}(t))$, *which is the unique solution of*

$$\begin{cases} dV^{(1)}(t) = -\frac{1}{4}\frac{(1+4\omega^2)^2}{(1-4\omega^2)^3} V^{(1)}(t)\left[\left(V^{(1)}(t)\right)^2 + \left(V^{(2)}(t)\right)^2\right] dt + \sqrt{\frac{1+4\omega^2}{2}}\, dB^{(1)}(t) \\ dV^{(2)}(t) = -\frac{1}{4}\frac{(1+4\omega^2)^2}{(1-4\omega^2)^3} V^{(2)}(t)\left[\left(V^{(1)}(t)\right)^2 + \left(V^{(2)}(t)\right)^2\right] dt + \sqrt{\frac{1+4\omega^2}{2}}\, dB^{(2)}(t) \\ V^{(1)}(0) = V^{(2)}(0) = 0 \end{cases}$$

where $B^{(1)}(t)$ *e* $B^{(2)}(t)$ *are independent standard Brownian motions.*

Critical Dynamics of the Homogeneous Model ($\mu \sim \delta_0$)

- Order parameter: $\rho_N := \frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}$
- Critical direction: span$\{\int \cos x \, d\rho_N, \int \sin x \, d\rho_N\}$

Theorem ($\theta = 1$)

For $t \in [0, T]$, the process

$$Y^{(1,N)}(t) := \int \cos x \, d\widetilde{\rho}_N(\sqrt{N}t) \qquad Y^{(2,N)}(t) := \int \sin x \, d\widetilde{\rho}_N(\sqrt{N}t)$$

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Summary

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Thanks for your attention!