Secondo compitino di Fisica Matematica 12/3/2009

A

Si consideri il sistema dinamico nel piano $\mathbb{R}^2$:

$$\begin{cases} \dot{x} &= x(3-x-2y) \\ \dot{y} &= y(2-x-y) \end{cases}$$

Determinarne gli equilibri e studiarne la stabilità.

Soluzione:

2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling (on the grass, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many sheep, the odds of a rabbit encountering a sheep would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

A specific model that incorporates these assumptions is

$$\dot{x} = x(3-x-2y)$$
$$\dot{y} = y(2-x-y)$$

where

$x(t)$ = population of rabbits,
$y(t)$ = population of sheep

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

To find the fixed points for the system, we solve $\dot{x}=0$ and $\dot{y}=0$ simultaneously. Four fixed points are obtained: (0,0), (0,2), (3,0), and (1,1). To classify them, we compute the Jacobian:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}.$$

Now consider the four fixed points in turn:

(0,0): Then $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

The eigenvalues are $\lambda = 3, 2$ so (0,0) is an *unstable node.* Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $\mathbf{v} = (0,1)$, which spans the y-axis. (Recall the general rule: at a node, trajectories are tangential to the slow eigendirection, which is the eigendirection with the smallest $|\lambda|$.) Thus, the phase portrait near (0,0) looks like Figure 6.4.1.

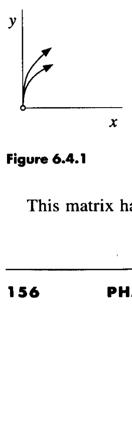

Figure 6.4.1

(0,2): Then $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$.

This matrix has eigenvalues $\lambda = -1, -2$, as can be seen from inspection, since

the matrix is triangular. Hence the fixed point is a *stable node.* Trajectories approach along the eigendirection associated with $\lambda = -1$; you can check that this direction is spanned by $\mathbf{v} = (1,-2)$. Figure 6.4.2 shows the phase portrait near the fixed point (0,2).

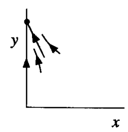

Figure 6.4.2

(3,0): Then $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ and $\lambda = -3, -1$.

This is also a *stable node.* The trajectories approach along the slow eigendirection spanned by $\mathbf{v} = (3,-1)$, as shown in Figure 6.4.3.

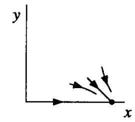

Figure 6.4.3

(1,1): Then $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, which has $\tau = -2$, $\Delta = -1$, and $\lambda = -1 \pm \sqrt{2}$.

Hence this is a *saddle point.* As you can check, the phase portrait near (1,1) is as shown in Figure 6.4.4.

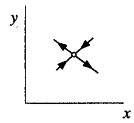

Figure 6.4.4

Combining Figures 6.4.1–6.4.4, we get Figure 6.4.5, which already conveys a good sense of the entire phase portrait. Furthermore, notice that the x and y axes contain straight-line trajectories, since $\dot{x}=0$ when $x=0$, and $\dot{y}=0$ when $y=0$.