Algebra 2 - February 12, 2013

First name and family name:

Matricola:

Es 1	Es 2	Es 3	Es 4	Es 5	Es 6	Es 7	Tot

1. Let H be a subgroup, and N a normal subgroup of the group G. A function $\phi : H \to G/N$ is defined by $\phi(h) = hN$ for each $h \in H$. Prove that
   (a) ϕ is a group homomorphism;
   (b) $\phi(H) = HN/N$;
   (c) $H/H \cap N \cong HN/N$.
2. (a) Partition the elements of the symmetric group S_4 in conjugacy classes.
   (b) Partition the elements of the alternating group A_4 in conjugacy classes.
3. Let p be a prime number, and G a finite group.
   (a) Write the definition: A p-Sylow subgroup of G is...
   (b) Let N be a normal subgroup of G. Show that, if the order of N is a power of p, then N is contained in every p-Sylow subgroup of G.
4. Put $u = \sqrt{5} + i\sqrt{7}$.
   (a) Prove that u is algebraic over $\mathbb{Q}$.
   (b) Is $\mathbb{Q}(u) = \mathbb{Q}(\sqrt{5}, i\sqrt{7})$?
   (c) Find the minimum polynomial $f(x)$ of u over $\mathbb{Q}$.
   (d) Check whether $\mathbb{Q}(u)$ is a splitting field of $f(x)$ over $\mathbb{Q}$.
5. Let $z \in \mathbb{C}$ be a primitive p-th root of 1 (p is a prime). Show that $1 + x + \ldots + x^{p-1}$ is the minimum polynomial of z over $\mathbb{Q}$.
6. Put $F = \mathbb{Z}/3\mathbb{Z}$, and $f(x) = x^3 + 2x + 2 \in F[x]$.
   (a) Show that $f(x)$ is irreducible in $F[x]$.
   (b) Let α be a root of $f(x)$ in a suitable field extension of F, and let $K = F(\alpha)$. How many are the elements of K ?
   (c) How many the subfields of K ?
   (d) Write $(\alpha^2 + 2)^{-1}$ as a polynomial expression in α with coefficients in F.

7. (a) Show that the ring $\mathbb{Z}/(17 \cdot 19)\mathbb{Z}$ is isomorphic to $\mathbb{Z}/17\mathbb{Z} \times \mathbb{Z}/19\mathbb{Z}$ (direct product of rings).

   (b) Consider the equation $x^2 = x$ in $\mathbb{Z}/(17 \cdot 19)\mathbb{Z}$. How many solutions does it have?

   (c) Write explicitly the solutions of that equation.