Spectral stability and boundary homogenization for polyharmonic operators

Francesco Ferraresso
Joint work with Pier Domenico Lamberti

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Principal references

J. Arrieta, P.D.Lamberti,
Higher order elliptic operators on variable domains. Stability results and boundary oscillations for intermediate problems, preprint, online at arXiv:1502.04373v2 [math.AP]

F.F., P.D.Lamberti,
Spectral convergence of higher order operators on varying domains and polyharmonic boundary homogenization, in preparation.

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Are they continuous?

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$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ \nu \Delta u + (1-\nu)\frac{\partial^2 u}{\partial n^2} = 0, & \text{on } \partial\Omega, \\ (1-\nu)\mathrm{div}_{\partial\Omega}(Hu \cdot n) + \frac{\partial \Delta u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

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ν is the Poisson coefficient of the material ($0 < \nu < 1/2$).

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compact convergence $\Rightarrow$ spectral convergence

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Theorem

Assume that $||g_\epsilon - g||_{C^1(\bar{W})} \to 0$ as $\epsilon \to 0$ and $||g_\epsilon||_{C^2(\bar{W})} < M$ for all $\epsilon > 0$, then the compact convergence holds.

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Then $H^{-1}_{V(\Omega_\epsilon)} \to H^{-1}_{V(\Omega)}$ with respect to the compact convergence.

Lemma

Suppose that $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$ *for some* $1 \leq k < m$. *If for all* $\epsilon > 0$ *there exists* $\kappa_\epsilon > 0$ *such that*

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where $\alpha > 0$, and $g : \mathbb{R}^{N-1} \to \mathbb{R}$ is a periodic smooth positive function (with period Y, say the unit cell in $\mathbb{R}^{N-1}$)

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Let $m \geq 2$, *and let* $H_{\Omega_\epsilon, I}$ *be the operator* $(-1)^m \Delta^m + \mathbb{I}$ *on* Ω_ϵ

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Let also $H_{\Omega,D}$ be the same operator with Dirichlet boundary conditions on $W \times \{0\}$ and intermediate boundary conditions on the rest of $\partial\Omega$.

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- [Spectral stability] *If $\alpha > 3/2$, then $H^{-1}_{\Omega_\epsilon,I} \xrightarrow{\mathcal{C}} H^{-1}_{\Omega,I}$.*

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- [Spectral stability] *If $\alpha > 3/2$, then $H^{-1}_{\Omega_\epsilon,I} \xrightarrow{\mathcal{C}} H^{-1}_{\Omega,I}$.*
- [Instability] *If $\alpha < 3/2$, then $H^{-1}_{\Omega_\epsilon,I} \xrightarrow{\mathcal{C}} H^{-1}_{\Omega,D}$.*

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- characterize (*) in terms of a microscopic problem for an auxiliary function.

Limit problem for the biharmonic operator

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \\ \Delta u - K(U)\frac{\partial u}{\partial n} = 0, & \text{on } \Gamma \end{cases}$$

What about the critical case $\alpha = 3/2$?

and for the triharmonic operator:

$$\begin{cases} \Delta^3 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \\ \nabla u = 0, & \text{on } \Gamma, \\ \Delta(\partial_{x_N} u) - K(V)\Delta u = 0, & \text{on } \Gamma \end{cases}$$

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where $K(U)$, $K(V)$ are respectively

$$K(U) = \int_{Y\times(-\infty,0)} |D^2 U|^2 dy, \quad K(V) = \int_{Y\times(-\infty,0)} |D^3 V|^2 dy;$$

in particular they are not zero!

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The functions U and V are the solutions of a suitable PDE, which catches the microscopic behaviour of the system.

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The functions U and V are the solutions of a suitable PDE, which catches the microscopic behaviour of the system.

For example, the function V solves

$$\begin{cases} \Delta^3 V = 0, & \text{in } Y \times (-\infty, 0), \\ V(\bar{y}, 0) = 0, & \text{on } Y, \\ \frac{\partial V}{\partial y_N} = g(\bar{y}), & \text{on } Y, \\ \frac{\partial^3 V}{\partial y_N^3} = 0, & \text{on } Y. \end{cases}$$

and is periodic in the first $N - 1$ coordinates.

Thank you
for your attention

Conjecture

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If $\alpha = 3/2$, the limit problem for $\Delta^m + \mathbb{I}$ with strong intermediate b.c. satisfies the following b.c. on W:

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$$\begin{cases} u = 0, \\ \frac{\partial^k u}{\partial x_N^k} = 0, & \text{for any } k \leq m-2 \\ \frac{\partial^m u}{\partial x_N^m} - K\frac{\partial^{m-1} u}{\partial x_N^{m-1}} = 0. \end{cases}$$

where the factor K is given by

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$$K = -\int_Y B_{m-2}(V) \mathrm{d}\bar{y} = \int_{Y \times (-\infty, 0)} |D^m V|^2 \, \mathrm{d}y,$$

and the function V satisfies the following

$$\begin{cases} \Delta^m V = 0, & \text{in } Y \times (-\infty, 0), \\ V(\bar{y}, 0) = 0, \quad \frac{\partial^k V}{\partial y_N^k} = 0, & \text{on } Y, \text{ for all } 1 \leq k \leq m-3, \\ \frac{\partial^{m-2} V}{\partial y_N^{m-2}} = g(\bar{y}), & \text{on } Y, \\ \frac{\partial^m V}{\partial y_N^m} = 0, & \text{on } Y, \end{cases}$$