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A Unifying Framework for Submodular Mean Field Games

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Abstract. We provide an abstract framework for submodular mean field games and identify verifiable sufficient conditions that allow us to prove the existence and approximation of strong mean field equilibria in models where data may not be continuous with respect to the measure parameter and common noise is allowed. The setting is general enough to encompass qualitatively different problems, such as mean field games for discrete time finite space Markov chains, singularly controlled and reflected diffusions, and mean field games of optimal timing. Our analysis hinges on Tarski's fixed point theorem, along with technical results on lattices of flows of probability and subprobability measures.

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Keywords: mean field games • submodularity • complete lattice of measures • Tarski's fixed point theorem • Markov chain • singular stochastic control • reflected diffusion • optimal stopping

1. Introduction

Mean field games (MFGs in short) are limit models for noncooperative symmetric N-player games with interaction of mean field type as the number of players N tends to infinity. They have been proposed independently by Huang et al. [39] and Lasry and Lions [42], and since their introduction, they have attracted increasing interest in various fields of mathematics ranging from the theory of partial differential equations (PDEs) to stochastic analysis and game theory as well as in applications in economics, finance, biology, and engineering among others; we refer, for instance, to the recent book by Carmona and Delarue [19] for an extensive presentation of theoretical results and applications.

The interest in identifying a key property that allows us to prove the existence and approximation of equilibria for a general class of MFGs has motivated our study. Inspired by the early contribution of Topkis [57] on submodular N-player games (i.e., games in which players minimize cost functions, which are submodular in the vector of strategies chosen by all the players) in a static setting, we identify submodularity as a relevant structural condition and explore the flexibility of lattice-theoretical techniques in MFGs enjoying a submodular structure. Submodular MFGs have already been considered in the literature; see Adlakha et al. [1] for a class of stationary discrete time games, Carmona et al. [20] for optimal timing MFGs, Dianetti et al. [25] for MFGs involving a regularly controlled one-dimensional Itô diffusion, and Więcek [60] for a class of finite state MFGs with exit. It is crucial to underline that (see Dianetti et al. [25] or Remark 2) when it comes to MFGs, the submodularity property represents an antithetic version of the well-known Lasry–Lions monotonicity condition, which is typically related to the uniqueness of equilibria (see Carmona and Delarue [19]).

In this work, we push the analysis of ours (Dianetti et al. [25]) much forward, and we provide an abstract framework for submodular MFGs, which embeds qualitatively different problems and allows us to show the existence and approximation of their mean field equilibria. The results of our work can be informally presented as follows.

1. The submodular structure of the game yields an alternative way of establishing the existence of MFG solutions by using the lattice-theoretical Tarski's fixed point theorem rather than topological fixed point results. This allows us to treat systems with coefficients that are possibly discontinuous in the measure variable as well as to prove the existence of strong solutions in settings involving a common noise.

2. The set of MFG solutions enjoys a lattice structure so that there exist a minimal solution and a maximal solution with respect to a suitable order relation.

3. A learning procedure, which consists of iterating the best-response map (thus, computing a new flow of measures as the best response to the previous measure flow), converges to the minimal (or the maximal) MFG solution for appropriately chosen initial measure flows.

These claims are made precise in Theorem 1 under suitable assumptions that are formulated at a general abstract level. Those requirements do not involve nondegeneracy of the underlying noise and are satisfied in a variety of formulations of the mean field game problem, including deterministic frameworks. Clearly, the setting of our previous work (Dianetti et al. [25]) is included. Furthermore, in this paper we highlight the flexibility of the approach by considering four qualitatively different problems, in which the representative agent minimization problem involves as a state variable (i) a finite state discrete time Markov chain (see Section 4); (ii) a singularly controlled Itô diffusion, possibly affected by a common noise (see Section 5); (iii) an Itô diffusion facing a reflecting boundary condition (see Section 6); and (iv) a general progressive stochastic process whose evolution can be stopped by the representative player (see Section 7). Here, a common source of noise is also allowed. For each of these examples, existence and approximation results are derived through a suitable application of Theorem 1. It is worth noting that fine properties of lattices of probability and subprobability measures are needed in order to apply Theorem 1 in the different examples. As we were unable to find a precise reference for those properties, we present them in brevity in Section 3. Given the generality of the setting in which they are obtained, we believe that those findings are of interest on their own and might be a useful technical tool in other works as well.

The approach that we follow in this paper focuses exclusively on the representative agent minimization problem, without reformulating the problem in terms of a related forward-backward system or of the master equation. Whether those reformulations of the mean field game problem allow one to obtain results of a similar fashion as ours is, to the best of our knowledge, an open question that we leave for future research.

1.1. Existence and Approximation Results in MFGs

Questions of existence and approximation of mean field equilibria have been addressed in the literature at various degrees of generality and through different mathematical techniques.

General existence results for solutions to the MFG problem can be obtained through Banach's fixed point theorem if the time horizon is small (see Huang et al. [39]). For an arbitrary time horizon, a version of the Brouwer–Schauder fixed point theorem, including generalizations to multivalued maps, can be used (see Cardaliaguet [17] and Lacker [41]; see also Fu and Horst [31] in the context of MFGs with singular controls). In the presence of a common noise (i.e., an aggregate source of randomness), the existence of a weak MFG solution (i.e., not adapted to the common noise) can be established for a general class of MFGs. On the other hand, the existence of a strong MFG solution (i.e., adapted to the common noise) has been addressed mainly under conditions that imply uniqueness of equilibria. For example, in Carmona et al. [20], an analogue of the famous result by Yamada and Watanabe is derived, and it is used to prove the existence and uniqueness of a strong solution under the Lasry–Lions monotonicity conditions (see Lasry and Lions [42]). Under a lack of uniqueness, the existence of strong solutions remains mainly an open question.

Because uniqueness of equilibria in game theory is the exception rather than the rule, it is not surprising that multiple solutions also often arise in MFGs. This phenomenon has been investigated mainly on a case by case basis, and specific examples with multiple solutions have been presented in the recent literature: Bardi and Fischer [3], Cecchin et al. [22], Delarue and Tchuendom [23], and Tchuendom [56] among others. Interestingly, the submodularity assumption appears implicitly in a number of classical linear-quadratic models (see, e.g., Bensoussan et al. [9]) and in Bardi and Fischer [3], Campi et al. [14], Cecchin et al. [22], and Delarue and Tchuendom [23], although this property is not exploited therein. The increasing interest in the nonuniqueness of solutions together with the perspective of characterizing many models through a unique key structural property has been one of the main motivations for our study of submodular MFGs.

Once existence is established, it is natural to investigate how and whether solutions to MFGs can be approximated in a constructive way. This problem has been addressed by Cardaliaguet and Hadikhanloo [18]. They analyze a learning procedure—similar to what it is known as "fictitious play" (see Hofbauer and Sandholm [38] and the references therein)—where the representative agent, starting from an arbitrary flow of measures, computes a new flow of measures by updating the average over past measure flows according to the best response to that average. For potential mean field games, the authors establish convergence of this kind of fictitious play via PDE methods. Similar approaches have been further developed in some more recent works (see Elie et al. [28], Perrin et al. [51], and Xie et al. [62] among others) with the help of machine learning techniques, providing a rich set of tools able to address computational aspects in MFGs. As already discussed, our result also contributes to the approximation question because the submodularity condition provides convergence of a simple learning procedure à la Topkis [58], consisting of iterating the best-response map (see Dianetti et al. [25] and also, see Dianetti and Ferrari [24] in the context of N-player games). In particular, this type of algorithm seems to be quite promising when combined with reinforcement learning methods, as shown in the recent work by Lee et al. [43] for stationary discrete time finite state MFGs with complementarities.

1.2. Examples

We now discuss in more detail the applications of Theorem 1 that we present in this work by also reviewing the related literature.

1.2.1. Submodular Mean Field Games with Finite State Discrete Time Markov Chains. We start with a simple class of finite state discrete time MFGs where expected costs are to be minimized over a finite time horizon. Control acts on the Kolmogorov equation: that is, on the transition matrix that determines the evolution of the state probability vector. Mean field interaction only occurs through the measure variable appearing in the cost coefficients. We relate our model to the general setup of Section 2 and provide sufficient conditions so that part (a) of Theorem 1 applies, yielding the existence of solutions. We also give a simple example of a class of two-state models satisfying those conditions. They are related to the continuous time two-state MFGs studied in Cecchin et al. [22] and Gomes et al. [33]; also see Bayraktar and Zhang [4], which exhibits multiple solutions.

The study of finite state discrete time MFGs goes back to Gomes et al. [32], where existence and convergence to equilibrium for a class of finite horizon problems were established. For discrete MFGs of this type satisfying an analogue of the Lasry–Lions monotonicity condition, convergence of a fictitious play learning procedure is proved in Hadikhanloo and Silva [36]. There, discrete models are also shown to approximate corresponding continuous time and space MFGs. Existence of solutions for a general class of finite and infinite horizon discrete MFGs is established in Doncel et al. [26], and their connection with the underlying N-player games is investigated. Discrete time MFGs with more general state space have been studied recently under various optimality criteria; see Saldi et al. [53, 54] for infinite horizon discounted cost and risk-sensitive problems, respectively; Więcek [61] for ergodic MFGs; and Bonnans et al. [12] for risk-averse problems. Existence of solutions in those works is established through a topological fixed point theorem; in particular, cost coefficients are assumed to depend continuously on the measure variable. Although our simple discrete models fall under the framework of, for instance, Doncel et al. [26], the continuity assumptions there are not needed here because here, as in the aforementioned Adlakha et al. [1] and Więcek [60], we rely on an order-theoretic fixed point result. Lastly, we mention that our finite state MFGs do not involve common noise. Choosing a common noise for finite state problems is in fact less straightforward than in the usual continuous space setting; see the recent works by Bayraktar et al. [7, 8] for continuous time finite state problems.

1.2.2. Submodular Mean Field Games with Singular Controls. The number of papers considering MFGs of singular stochastic controls is still relatively limited. Fu and Horst [31] employs a relaxed approach in order to establish the existence for a general class of MFGs involving singular controls, whereas the more recent work of Fu [30] extends the analysis to MFGs, in which interaction takes place both through states and through controls. In Campi et al. [14] and Guo and Xu [35], MFGs for finite fuel follower problems are considered. By employing the connection to problems of optimal stopping and PDE methods, respectively, the structure of the mean field equilibrium as well as its connection to Nash equilibria for the corresponding N-player stochastic differential games is derived. Finally, Cao and Guo [15] and Cao et al. [16] study stationary MFGs (i.e., games in which the interaction comes through the stationary distribution of the population of players). Cao et al. [16] considers ergodic and discounted performance criteria and studies the relation across the corresponding equilibria; in Cao and Guo [15], the representative player can employ two-sided controls in order to adjust geometric dynamics and optimize a certain discounted payoff. It is worth noting that none of the previous contributions allow for the presence of common noise, which we can instead treat in our analysis. We can indeed show that the class of submodular MFGs with geometric dynamics that we consider in Section 5.1 admits strong equilibria (i.e., adapted to the common noise), which can in fact also be approximated through the previously discussed learning algorithm à la Topkis. In the case of a general nonconvex setting, a weak formulation of the singular control MFG is employed, and existence of mean field equilibria is proved by means of an approximation result through Lipschitz-continuous controls. Furthermore, convergence of the learning procedure is also established (see Section 5.2).

1.2.3. Submodular Mean Field Games with Reflecting Boundary Conditions. Theorem 1 also yields the existence and approximation of equilibria for submodular MFGs in which the representative player can employ regular controls in order to adjust the drift of a one-dimensional Itô diffusion, which is constrained, via a Skorokhod reflection, to live in a bounded interval (see Section 6). These models have received recent interest because they naturally arise as suitable limits of interacting queuing systems; see Bayraktar et al. [5, 6]. As in Bayraktar et al. [6], we employ a weak (distributional) approach, and by enforcing additional mild technical requirements on the data of the problem, an application of Tanaka's formula for continuous semimartingales allows us to embed the considered MFG into the class of abstract submodular MFGs for which Theorem 1 holds. Then, the existence and approximation of mean field equilibria follow.

1.2.4. Supermodular Mean Field Games with Optimal Stopping.

In Section 7, we consider a class of MFGs where the representative agent can choose a stopping time in order to stop the evolution of a general multidimensional progressive process while maximizing a certain reward functional. The model is formulated by including the presence of a common noise. By assuming that the running profit function is increasing with respect to the stochastic order put on the lattice of subprobability measures, the game enjoys a supermodular (rather than submodular because here, we are dealing with a maximization problem) structure that allows us to invoke Theorem 1 and show the existence of equilibria. Furthermore, under suitable continuity requirements, convergence of a learning procedure is obtained.

Models involving MFGs of optimal stopping have been considered in the economic literature mostly in stationary settings (see Luttmer [45] and Miao [48] in the context of industry equilibria) and more recently, under greater generality in the mathematical literature; see Aïd et al. [2], Bertucci [10], and Bouveret et al. [13]. Using a relaxed solution approach, in Aïd et al. [2] and Bouveret et al. [13], an Itô-diffusive setting not allowing for a common noise is considered (see also Example 3 in Section 7). In Bertucci [10], an analytical approach to MFGs of optimal stopping is developed through the study of the associated variational inequality. Explicit use of the supermodular property and of Tarski's fixed point theorem in an MFG of stopping with common noise is made in Carmona et al. [21] (see also Example 4 in Section 7).

1.3. Outline of the Paper

The rest of the paper is organized as follows. Section 2 presents the general approach to submodular MFGs. There, we state and prove Theorem 1. Section 3 derives the properties of lattices of probability and subprobability measures needed in the paper. The remaining sections deal with applications of the abstract setup. Section 4 deals with MFGs having discrete time finite space Markov chains as state variables. Section 5 considers MFGs with singular controls. Section 6 treats MFGs with reflecting boundary conditions, whereas MFGs of optimal stopping are addressed in the Section 7. For the reader's convenience, we collect some lattice-theoretical preliminaries in the appendix.

1.3.1. General Notation.

For a fixed finite time horizon $T \in (0, \infty)$, we introduce the following canonical spaces.

1. $\mathcal{C}$ denotes the space of $\mathbb{R}$-valued continuous functions defined on $[0, T]$, endowed with the supremum norm and the Borel σ-algebra $\mathcal{B}(\mathcal{C})$ generated by the supremum norm.
2. For a set $A \subset \mathbb{R}$, let Λ denote the set of *deterministic relaxed controls* on $[0, T] \times A$: that is, the set of positive measures λ on $[0, T] \times A$ such that $\lambda([s, t] \times A) = t - s$ for all $s, t \in [0, T]$ with $s < t$. The set Λ is endowed with the topology of weak convergence of probability measures, and $\mathcal{B}(\Lambda)$ denotes the related Borel σ-algebra.
3. $\mathcal{D}$ denotes the Skorokhod space of $\mathbb{R}$-valued càdlàg functions, defined on $[0, T]$, endowed with the Borel σ-algebra $\mathcal{B}(\mathcal{D})$ generated by the Skorokhod topology. On the space $\mathcal{D}$, consider the *pseudopath topology* τ_{pp}^T: that is, the topology on $\mathcal{D}$ induced by the convergence in the measure $dt + \delta_T$ on the interval $[0, T]$, where dt denotes the Lebesgue measure and δ_T denotes the Dirac measure at the terminal time T. For the topological space $(\mathcal{D}, \tau_{pp}^T)$, the Borel σ-algebra induced by the topology τ_{pp}^T coincides with the σ-algebra induced by the Skorokhod topology (see the appendix in Li and Žitković [44]).
4. $\mathcal{D}_\uparrow$ denotes the set of elements of $\mathcal{D}$, which are nonnegative and nondecreasing, endowed with the Borel σ-algebra $\mathcal{B}(\mathcal{D}_\uparrow)$ induced by the Skorokhod topology. Note that $\mathcal{D}_\uparrow$ is a closed subset of the topological space $(\mathcal{D}, \tau_{pp}^T)$.
5. $\mathcal{V}$ denotes the set of elements of $\mathcal{D}$ with bounded total variation endowed with the Borel σ-algebra $\mathcal{B}(\mathcal{V})$ induced by the Skorokhod topology. Furthermore, the space $\mathcal{V}$ is a closed subset of the topological space $(\mathcal{D}, \tau_{pp}^T)$.

2. A General Approach to Submodular MFGs

In this section, we consider an abstract version of a mean field game. The aim of this section is to collect fundamental structural conditions and arguments, which provide a common basis for the examples treated in the next sections. For the lattice-theoretical notions and preliminaries that are used throughout this section and the rest of this paper, we refer to the appendix.

2.1. Formulation of the Abstract Model

Let $(L, \leqslant^L)$ be a complete and Dedekind super complete lattice, which represents the set of possible distributions of players; see Definition A.1. Let E be the set of strategies of the representative player. The set E is endowed with a topology and a map $p: E \to L$, which can be interpreted as a projection, that maps each strategy to a related distribution. The representative player wants to minimize a cost functional $J: E \times L \to \mathbb{R}$, depending also on the distribution of her opponents.

We make the following assumption (see also Remark 1 for a generalization).

Assumption 1. *For every $\mu \in L$, we assume the following.*

1. *The set* $\arg\min_E J(\cdot, \mu)$ *is nonempty, and* $J(\cdot, \mu)$ *is lower semicontinuous.*
2. *For any sequence* $(\nu^n)_n \subset \arg\min_E J(\cdot, \mu)$ *such that* $p(\nu^n)$ *is nondecreasing or nonincreasing in* L*, there exists a subsequence* $(n_j)_{j\in\mathbb{N}}$ *and* $\nu \in \arg\min_E J(\cdot, \mu)$ *such that* ν^{n_j} *converges to* ν *as* $j \to \infty$ *and* $p\nu = \sup_j p(\nu^{n_j})$ *or* $p\nu = \inf_j p(\nu^{n_j})$*, respectively.*

For $\mu \in L$, we define the set of best responses $R(\mu) \subset L$ by

$$R(\mu) := p\Big(\arg\min_{\nu \in E} J(\nu, \mu)\Big).$$

Definition 1. We say that $\mu \in L$ is a *mean field game equilibrium* if $\mu \in R(\mu)$ (i.e., μ is a fixed point of the best-response map).

2.2. Submodularity Conditions and Properties of the Best-Response Map

Existence of MFG solutions is subject to the following abstract structural condition.

Assumption 2 (Submodularity Conditions). *There exist operations* $\wedge^E, \vee^E : E \times E \to E$.

1. *The projection p behaves like a homeomorphism of lattices: that is,*

$$p(\nu \wedge^E \bar{\nu}) \leqslant^L p\nu \wedge^L p\bar{\nu} \leqslant^L p\nu \vee^L p\bar{\nu} \leqslant^L p(\nu \vee^E \bar{\nu}), \quad \text{for each } \nu, \bar{\nu} \in \mathrm{E}.$$

2. *The cost functional satisfies the following submodularity properties*

$$J(\nu \vee^E \bar{\nu}, \bar{\mu}) - J(\bar{\nu}, \bar{\mu}) \leqslant J(\nu \vee^E \bar{\nu}, \mu) - J(\bar{\nu}, \mu) \leqslant J(\nu, \mu) - J(\nu \wedge^E \bar{\nu}, \mu),$$

for each $\nu, \bar{\nu} \in E$ *and* $\mu, \bar{\mu} \in L$ *with* $\mu \leqslant^L \bar{\mu}$.

We underline that condition 2 in Assumption 2 coincides with the conditions in Topkis [57] only in the case in which $(E, \wedge^E, \vee^E)$ is a lattice.

We start our analysis with the following result on the structure of the sets of best responses.

Lemma 1. *Under Assumptions 1 and 2, we have the following.*

a. *The set* $R(\mu)$ *is directed (i.e., for every* $\eta^1, \eta^2 \in R(\mu)$*, there exist* $\eta^\wedge, \eta^\vee \in R(\mu)$ *such that* $\eta^\wedge \leqslant^L \eta^1 \wedge^L \eta^2$ *and* $\eta^\vee \geqslant^L \eta^1 \vee^L \eta^2$*).*

b. *For all* $\mu, \bar{\mu} \in L$ *with* $\mu \leqslant^L \bar{\mu}$, $\inf R(\mu) \leqslant^L \inf R(\bar{\mu})$*, and* $\sup R(\mu) \leqslant^L \sup R(\bar{\mu})$.

c. *For every* $\mu \in L$, $\inf R(\mu) \in R(\mu)$ *and* $\sup R(\mu) \in R(\mu)$.

Proof. We prove each of the claims separately. To this end, let $\mu, \bar{\mu} \in L$ with $\mu \leqslant^L \bar{\mu}$. Moreover, let $\eta^1 \in R(\mu)$ and $\eta^2 \in R(\bar{\mu})$.

a. By definition of $R(\mu)$ and $R(\bar{\mu})$, there exist

$$\nu^1 \in \arg\min_{\nu\in E} J(\nu, \mu) \quad \text{and} \quad \nu^2 \in \arg\min_{\nu\in E} J(\nu, \bar{\mu}) \quad \text{with} \quad p\nu^1 = \eta^1 \quad \text{and} \quad p\nu^2 = \eta^2.$$

By condition 1 in Assumption 2, we can define $\nu^\wedge, \nu^\vee \in E$ by $\nu^\wedge := \nu^1 \wedge^E \nu^2$ and $\nu^\vee := \nu^1 \vee^E \nu^2$, leading to $p\nu^\wedge \leqslant^L p\nu^1 \wedge^L p\nu^2$ and $p\nu^\vee \geqslant^L p\nu^1 \vee^L p\nu^2$. The optimality of ν^1 and ν^2 for η^1 and η^2, respectively, together with condition 2 in Assumption 2 implies that

$$0 \leqslant J(\nu^\vee, \bar{\mu}) - J(\nu^2, \bar{\mu}) \leqslant J(\nu^\vee, \mu) - J(\nu^2, \mu) \leqslant J(\nu^1, \mu) - J(\nu^\wedge, \mu) \leqslant 0.$$

This shows that $\eta^\wedge := p\nu^\wedge \in R(\mu)$ and $\eta^\vee := p\nu^\vee \in R(\bar{\mu})$. Now, the statement in (a) directly follows by choosing $\mu = \bar{\mu}$.

b. By part (a), $\eta^\wedge \in R(\mu)$ and $\eta^\vee \in R(\bar{\mu})$ imply that

$$\inf R(\mu) \leqslant^L \eta^\wedge = p\nu^\wedge \leqslant^L p\nu^2 = \eta^2 \quad \text{and} \quad \eta^1 = p\nu^1 \leqslant^L p\nu^\vee = \eta^\vee \leqslant^L \sup R(\bar{\mu}).$$

Taking the infimum over all $\eta^2 \in R(\bar{\mu})$ and the supremum over all $\eta^1 \in R(\mu)$ yields the assertion in (b).

c. We now prove the claim only for the infimum; the statement for the supremum follows in an analogous manner. Because L is, by assumption, Dedekind super complete, there exists a sequence $(\mu^n)_n \subset R(\mu)$ such that $\inf R(\mu) = \inf_n \mu^n$. Therefore, we can find a sequence $(\nu^n)_n \subset \arg\min J(\cdot, \mu)$ with $p\nu^n = \mu^n$. We can inductively define a new sequence $(\nu^{\wedge,n})_n$ by setting

$$\nu^{\wedge,1} := \nu^1 \quad \text{and} \quad \nu^{\wedge,n+1} := \nu^{\wedge,n} \wedge^E \nu^{n+1}, \quad n \geqslant 1.$$

As shown in the proof of part (a), we have that $\nu^{\wedge,1} \in \arg\min J(\cdot, \mu)$, and by induction, we deduce that $\nu^{\wedge,n} \in \arg\min J(\cdot, \mu)$ for each $n \in \mathbb{N}$. Define now the sequence $(\mu^{\wedge,n})_n$ setting, $\mu^{\wedge,n} := p\nu^{\wedge,n}$ for each $n \in \mathbb{N}$, and note that $\mu^{\wedge,n} \in R(\mu)$. Moreover, condition 1 in Assumption 2 implies that

$$\mu^{\wedge,n+1} = p(\nu^{\wedge,n+1}) = p(\nu^{\wedge,n} \wedge^E \nu^{n+1}) \leqslant^L p\nu^{\wedge,n} \wedge^L p\nu^{n+1} = \mu^{\wedge,n} \wedge^L \mu^{n+1},$$

which at the same time, implies that $(\mu^{\wedge,n})_n$ is nonincreasing in L and that $\mu^{\wedge,n} \leqslant^L \mu^n$ for each $n \in \mathbb{N}$. Hence, we have

$$\inf R(\mu) = \inf_n \mu^n = \inf_n \mu^{\wedge,n}.$$

Moreover, by Assumption 1, there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ and a limit point $\nu \in \arg\min_E J(\cdot, \mu)$ such that

$$p\nu = \inf\nolimits_j p(\nu^{\wedge,n_j}) = \inf_j \mu^{\wedge,n_j} = \inf R(\mu),$$

so that $\inf R(\mu) \in R(\mu)$. □

2.3. Existence and Approximation of MFG Solutions

For the approximation of MFG solutions, we will enforce the following additional continuity requirements (see again Remark 1 for a generalization).

Assumption 3. *For any sequence $(\nu^n)_n \subset \{\arg\min_E J(\cdot, \mu) \mid \mu \in L\}$ such that $p(\nu^n)$ is increasing or decreasing in L, there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ and $\nu \in E$ such that ν^{n_j} converges to ν as $j \to \infty$ and $p\nu = \sup_j p(\nu^{n_j})$ or $p\nu = \inf_j p(\nu^{n_j})$, respectively.*

Moreover, respectively, for any nondecreasing or nonincreasing sequence $(\mu^n)_n \subset L$, we assume that

1. *for any $\nu \in E$, $J(\nu, \sup_n \mu) = \lim_n J(\nu, \mu^n)$ or $J(\nu, \inf_n \mu) = \lim_n J(\nu, \mu^n)$ and*
2. *for any sequence $(\nu^n)_n \subset E$ converging to $\nu \in E$, we have that $J(\nu, \sup_n \mu) \leqslant \liminf_n J(\nu^n, \mu^n)$ or $J(\nu, \inf_n \mu) \leqslant \liminf_n J(\nu^n, \mu^n)$.*

We can then state the main result of this section.

Theorem 1. *Under Assumptions 1 and 2, we have that*

a. *the set of mean field game equilibria M is nonempty with $\inf M \in M$ and $\sup M \in M$. If $R(\mu)$ is a singleton for all $\mu \in L$, then M is a nonempty complete lattice.*

Moreover, if Assumption 3 is satisfied, then

b. *the learning procedure $\underline{\mu}^0 := \inf L$ and $\underline{\mu}^n := \inf R(\underline{\mu}^{n-1})$, for $n \in \mathbb{N}$, is monotone increasing and converges to $\inf M$, and*

c. *the learning procedure $\bar{\mu}^0 := \sup L$ and $\bar{\mu}^n := \sup R(\bar{\mu}^{n-1})$, for $n \in \mathbb{N}$, is monotone decreasing and converges to $\sup M$.*

Proof.

a. Follows directly from Lemma 1 together with Tarski's fixed point theorem applied to the maps $\mu \longmapsto \inf R(\mu)$ and $\mu \longmapsto \sup R(\mu)$.

b. By Lemma 1, it follows that the sequence $(\underline{\mu}^n)_{n\in\mathbb{N}_0}$ is increasing. By completeness of the lattice L, we can set $\mu_* := \sup_n \underline{\mu}^n$. We next want to prove that $\mu_* = \inf M$.

For any $n \in \mathbb{N}$, by Lemma 1 and the definition of $\underline{\mu}^n$, we can find

$$\nu^n \in \arg\min_E J(\cdot, \underline{\mu}^{n-1})$$

with $p\nu^n = \underline{\mu}^n$. By Assumption 3, we can take a subsequence $(\nu^{n_j})_j$ and a limit point ν_* such that ν^{n_j} converges to ν_* and $p\nu^{n_j}$ converges to $p\nu_*$ as $j \to \infty$. This implies that $p\nu_* = \underline{\mu}_*$. Moreover, we have

$$J(\nu^{n_j}, \underline{\mu}^{n_j-1}) \leqslant J(\nu, \underline{\mu}^{n_j-1}), \quad \text{for any } \nu \in E \text{ and } j \in \mathbb{N}.$$

Exploiting the continuity properties of J in Assumption 3, we may pass to the limit as $j \to \infty$ in the previous inequality and obtain that

$$J(\nu_*, \underline{\mu}_*) \leqslant J(\nu, \underline{\mu}_*), \quad \text{for any } \nu \in E \text{ and } j \in \mathbb{N}.$$

This, in turn, implies that $\nu_* \in \arg\min_E J(\cdot, \underline{\mu}_*)$, so that $\underline{\mu}_* = p\nu_* \in R(\underline{\mu}_*)$. Therefore, $\underline{\mu}_*$ is an MFG solution.

We next want to prove that $\underline{\mu}_*$ is the minimal MFG solution. Let $\mu \in M$ be another mean field game equilibrium. Then, $\underline{\mu}^0 \leqslant^L \mu$, which by Lemma 1, implies that $\underline{\mu}^1 = R(\underline{\mu}^0) \leqslant^L R(\mu)$. Inductively, one obtains that $\underline{\mu}^n \leqslant^L \mu$ for all $n \in \mathbb{N}_0$, which implies that $\underline{\mu}_* \leqslant^L \mu$. Because $\underline{\mu}_* \in M$, it follows that $\underline{\mu}_* = \inf M$.

c. Follows by arguments analogous to the one used in the proof of part (b). □

The following remark proposes a set of purely order-theoretical conditions, alternative to those in Assumptions 1 and 3. These will be employed in the proof of Proposition 1.

Remark 1. The proofs of Lemma 1 and Theorem 1 show that all stated properties remain valid if Assumption 1 is replaced by the following purely order-theoretic assumptions.

- For every $\mu \in L$, the set $\arg\min_E J(\cdot, \mu)$ is nonempty, and the set $R(\mu)$ is closed under monotone sequences; that is, for any nondecreasing or nonincreasing sequence $(\mu^n)_n \subset R(\mu)$, there exists $\nu \in \arg\min_E J(\cdot, \mu)$ such that $p\nu = \sup_n \mu^n$ or $p\nu = \inf_n \mu^n$, respectively.

If Assumption 3 is replaced by the following two order-theoretic conditions,

- for any $\nu \in E$, $J(\nu, \cdot)$ is continuous over monotone sequences in L, and
- for any sequence $(\nu^n, \mu^n)_n \in E \times L$ such that $p\nu^n$ and μ^n are nondecreasing or nonincreasing, there exist $\nu \in E$ such that $p\nu = \sup_n p\nu^n$ or $p\nu = \inf_n p\nu^n$ and $J(\nu, \sup_n \mu^n) \leqslant \liminf_n J(\nu^n, \mu^n)$ or $J(\nu, \inf_n \mu^n) \leqslant \liminf_n J(\nu^n, \mu^n)$, respectively.

3. Lattices of Measures Related to Submodular MFG

In this section, we discuss lattices of measures arising in the context of submodular MFGs. Again, we refer to the appendix for the lattice-theoretic preliminaries. Throughout this section, let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$ and $\mathcal{M}_{\leqslant 1}$ denote the set of all subprobability measures (i.e., the set of all (nonnegative) measures on $\mathcal{B}(\mathbb{R})$ with $\mu(\mathbb{R}) \leqslant 1$). We identify a distribution $\mu \in \mathcal{M}_{\leqslant 1}$ by its survival function μ_0, given by

$$\mu_0(s) := \mu((s, \infty)) \quad \text{for all } s \in \mathbb{R}.$$

On $\mathcal{M}_{\leqslant 1}$, we consider the partial order $\leqslant^{\text{st}}$ arising from *first-order stochastic dominance*, given by

$$\mu \leqslant^{\text{st}} \nu \quad \text{if and only if} \quad \mu_0(s) \leqslant \nu_0(s) \quad \text{for all} \quad s \in \mathbb{R}.$$

Recall that, for $\mu, \nu \in \mathcal{M}_{\leqslant 1}$, $\mu \leqslant^{\text{st}} \nu$ if and only if

$$\int_{\mathbb{R}} h(x)\, d\mu(x) \leqslant \int_{\mathbb{R}} h(x)\, d\nu(x) \tag{1}$$

for all nondecreasing functions $h: \mathbb{R} \to [0, \infty)$. In particular, $\mu(\mathbb{R}) \leqslant \nu(\mathbb{R})$. Note that (1) holds for all nondecreasing functions $h: \mathbb{R} \to \mathbb{R}$ if and only if $\mu \leqslant^{\text{st}} \nu$ and $\mu(\mathbb{R}) = \nu(\mathbb{R})$. For a detailed discussion on the properties of the partial order $\leqslant^{\text{st}}$ for probability measures, we refer to section 1.A in the book by Shaked and Shanthikumar [55].

By identifying a subprobability measure μ with its survival function μ_0, the set $\mathcal{M}_{\leqslant 1}$ coincides with the set of all nonincreasing right-continuous functions $F: \mathbb{R} \to [0, \infty)$ with $\lim_{s \to -\infty} F(s) \leqslant 1$ and $\lim_{s \to \infty} F(s) = 0$. In particular, the partial order $\leqslant^{\text{st}}$ induces a lattice structure on $\mathcal{M}_{\leqslant 1}$ via

$$(\mu \vee^{\text{st}} \nu)(s) := \mu_0(s) \vee \nu_0(s) \quad \text{and} \quad (\mu \wedge^{\text{st}} \nu)(s) := \mu_0(s) \wedge \nu_0(s) \quad \text{for all} \;\; s \in \mathbb{R}.$$

We point out that although the firs- order stochastic dominance induces a partial order on the set of probability measures $\mathcal{P}(\mathbb{R}^d)$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, for all $d \in \mathbb{N}$, it does *not* induce a lattice order on $\mathcal{P}(\mathbb{R}^d)$ for $d > 1$; see Müller and Scarsini [49].

Recall that the weak convergence coincides with the pointwise convergence of survival functions at every continuity point (i.e., $\mu^n \to \mu$ weakly as $n \to \infty$ if and only if

$$\mu_0^n(s) \to \mu_0(s) \quad \text{as} \quad n \to \infty \quad \text{for every continuity point } s \in \mathbb{R} \text{ of } \mu_0)$$

and that the weak topology on $\mathcal{M}_{\leqslant 1}$ (i.e., the topology induced by the weak convergence of subprobability measures) is metrizable. As a consequence, the lattice operations $(\mu, \nu) \longmapsto \mu \vee^{\text{st}} \nu$ and $(\mu, \nu) \longmapsto \mu \wedge^{\text{st}} \nu$ are continuous maps $\mathcal{M}_{\leqslant 1} \times \mathcal{M}_{\leqslant 1} \to \mathcal{M}_{\leqslant 1}$, and the weak topology is finer than the interval topology (see Definition A.3 in the appendix) because every closed interval is weakly closed.

Lemma 2. *Every bounded and nondecreasing or nonincreasing sequence $(\mu^n)_{n\in\mathbb{N}} \subset \mathcal{M}_{\leqslant 1}$ converges weakly to its supremum or infimum with respect to (w.r.t.) $\leqslant^{\text{st}}$, respectively.*

Proof. First, observe that a nonincreasing function $\mathbb{R} \to \mathbb{R}$ is right continuous if and only if it is lower semicontinuous. Hence, for every sequence $(\mu^n)_{n\in\mathbb{N}} \in \mathcal{M}_{\leqslant 1}$, which is bounded above, the supremum $\sup_{n\in\mathbb{N}}\mu^n$ w.r.t. $\leqslant^{\text{st}}$ exists, and it is exactly the pointwise supremum of the survival functions $(\mu_0^n)_{n\in\mathbb{N}}$.

For a nonincreasing function $F : \mathbb{R} \to \mathbb{R}$, we define its lower semi-continuous envelope $F_* : \mathbb{R} \to \mathbb{R}$ by

$$F_*(s) := \sup_{\delta>0} F(s+\delta) \quad \text{for } s \in \mathbb{R}.$$

Then, $F(s) \geqslant F_*(s) \geqslant F(s+\varepsilon)$ for all $s \in \mathbb{R}$ and $\varepsilon > 0$. That is, F_* differs from F only at discontinuity points of F. For a sequence $(\mu^n)_{n\in\mathbb{N}} \in \mathcal{M}_{\leqslant 1}$, which is bounded, the infimum $\inf_{n\in\mathbb{N}}\mu^n$ w.r.t. $\leqslant^{\text{st}}$ is then given by the lower semi-continuous envelope of the pointwise infimum of the survival functions $(\mu_0^n)_{n\in\mathbb{N}}$. Because the weak convergence of a sequence of subprobability measures coincides with the pointwise convergence of the related survival functions at every continuity point, the assertion follows. □

Let $(S, \mathcal{S}, \pi)$ be a σ-finite measure space. We denote the Borel σ-algebra of the weak topology by $\mathcal{B}(\mathcal{M}_{\leqslant 1})$ and the lattice of all equivalence classes of $\mathcal{S}$-$\mathcal{B}(\mathcal{M}_{\leqslant 1})$–measurable functions $S \to \mathcal{M}_{\leqslant 1}$ by $L^0_{\text{st}} = L^0(S, \mathcal{S}, \pi; \mathcal{M}_{\leqslant 1})$. An arbitrary element μ of L^0_{st} will be denoted in the form $\mu = (\mu_t)_{t\in S}$. On L^0_{st}, we consider the order relation $\leqslant^{L^0_{\text{st}}}$, given by $\mu \leqslant^{L^0_{\text{st}}} \nu$ if and only if $\mu_t \leqslant^{\text{st}} \nu_t$ for π almost all (a.a.) $t \in S$.

In the sequel, we consider a family $(L_n)_{n\in\mathbb{N}}$ of Dedekind σ-complete sublattices of $\mathcal{M}_{\leqslant 1}$, which correspond to a countable number of constraints, and a family $(B_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ of measurable sets, on which the constraints in terms of the family $(L_n)_{n\in\mathbb{N}}$ should be satisfied. Before we state the main result of this section, we list some possible choices for measurable spaces $(S, \mathcal{S}, \pi)$, Dedekind σ-complete lattices $L = L_n$, and measurable sets $B = B_n \in \mathcal{S}$ for $n \in \mathbb{N}$.

Example 1.

a. The measure space $(S, \mathcal{S}, \pi)$ can be, for example,
- $S = [0, T]$, $\mathcal{S} = \mathcal{B}([0, T])$, $\pi = \delta_0 + \lambda_{[0,T]}$, also with $[0, \infty)$ instead of $[0, T]$ and $e^{-\delta t}\mathrm{d}t$ instead of λ;
- $\Omega \times [0, T]$, $\mathcal{S}$ the σ-algebra of all predictable processes, and $\pi = \mathbb{P} \otimes (\delta_0 + \lambda_{[0,T]})$.

b. The following are possible choices for $L = L_n$.
- The simplest choice is $L = \mathcal{M}_{\leqslant 1}$ or $L = \{\mu \in \mathcal{M}_{\leqslant 1} \mid \mu(\mathbb{R}) = 1\}$.
- Another choice is $L = \{\mu \in \mathcal{M}_{\leqslant 1} \mid \underline{\mu} \leqslant^{\text{st}} \mu \leqslant^{\text{st}} \bar{\mu}\}$ with $\underline{\mu}, \bar{\mu} \in \mathcal{M}_{\leqslant 1}$. If $\underline{\mu} = \bar{\mu} =: \nu$, this results in $L = \{\nu\}$. Note that $\underline{\mu} \equiv 0$ is not excluded.
- Let $a, b \in \mathbb{R}$ with $a \leqslant b$. Then, $L = \{\mu \in \mathcal{M}_{\leqslant 1} \mid \operatorname{supp}\mu \subset [a, b]\}$ is Dedekind σ complete. In fact, a subprobability $\mu \in \mathcal{M}_{\leqslant 1}$ is an element of L if and only if its survival function μ_0 is constant on $(-\infty, a)$ and (b, ∞), a property that carries over to suprema and infima of countably many elements of L. The same holds true if the interval $[a, b]$ is replaced by $[a, \infty)$ or $(-\infty, b]$.
- Another possible choice is $L = \{\delta_x \mid x \in \mathbb{R}\}$ (Dirac measures).

c. Possible choices for $B = B_n$ are
- $B = \{0\}$ or $B = \{0\} \times \Omega$ in order to prescribe an initial condition,
- $B = [0, T]$ or $B = \Omega \times [0, T]$ in order to give a condition that should be satisfied for all times $t \in [0, T]$ and in all states $\omega \in \Omega$, and
- $B = A \times (t_1, t_2]$ in order to prescribe a condition on a certain event A during the time period $(t_1, t_2]$.

We consider the set

$$\mathcal{L} := \{\mu \in L^0_{\mathrm{st}} \,|\, \forall n \in \mathbb{N} : \pi(\{t \in S | \mu_t \notin L_n\} \cap B_n) = 0\}.$$

This is the set of all measurable flows $(\mu_t)_{t\in S}$ of subprobability measures such that for all $n \in \mathbb{N}$, $\mu_t \in L_n$ for π a.a. $t \in B_n$. The following theorem is the main result of this section.

Theorem 2.

a. *The lattice $\mathcal{L}$ is Dedekind super complete.*

b. *If $M \subset \mathcal{L}$ is a nonempty set, which is bounded above or below and directed upward or downward, then there exist sequences $(\bar{\mu}^n)_{n\in\mathbb{N}} \subset M$ and $(\underline{\mu}^n)_{n\in\mathbb{N}} \subset M$ with $\bar{\mu}^n \leqslant^{L^0_{\mathrm{st}}} \bar{\mu}^{n+1}$ and $\underline{\mu}^n \geqslant^{L^0_{\mathrm{st}}} \underline{\mu}^{n+1}$ for all $n \in \mathbb{N}$ and*

$$\bar{\mu}^n \to \sup M \in \mathcal{L} \quad \text{and} \quad \underline{\mu}^n \to \inf M \in \mathcal{L} \quad \text{weakly } \pi\text{-a.e. as } n \to \infty,$$

respectively.

Proof. Because every σ-finite measure can be transformed to a probability measure without changing the null sets, we may, without loss of generality, assume that $\pi(S) = 1$. By Remark 2 and because the lattices $(L_n)_{n\in\mathbb{N}}$ are Dedekind σ complete, $\mathcal{L}$ is Dedekind σ complete. Let $\Phi : \mathbb{R} \to (0, 1)$ be the cumulative distribution function of the standard normal distribution: that is,

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, \mathrm{d}y \quad \text{for all } x \in \mathbb{R}.$$

The map $S \to \mathbb{R}$, $t \longmapsto \int_{\mathbb{R}} \Phi(x)\, \mathrm{d}\mu_t(x)$ is $\mathcal{S}$-$\mathcal{B}(\mathbb{R})$ measurable for every $\mu \in L^0_{\mathrm{st}}$ because the bounded and continuous function $\Phi : \mathbb{R} \to (0, 1)$ induces a continuous (w.r.t. the weak topology) functional $\mathcal{M}_{\leqslant 1} \to \mathbb{R}$. Hence,

$$F : \mathcal{L} \to \mathbb{R}, \quad \mu \longmapsto \int_S \int_{\mathbb{R}} \Phi(x)\, \mathrm{d}\mu_t(x)\, \mathrm{d}\pi(t)$$

is well defined and strictly increasing because Φ is nonnegative and strictly increasing (see, e.g., theorem 1.A.8 in the book by Shaked and Shanthikumar [55]). The assertions now follow from Lemmas 2 and A.1. □

4. Submodular Mean Field Games with Markov Chains

Throughout this section, let $d \in \mathbb{N} \setminus \{1\}$ and $S := \{1, \ldots, d\}$ be a finite state space. We endow S with the natural order and identify elements of the set $\mathcal{P}(S)$ of all probability measures with their probability vectors according to

$$\mu \equiv (\mu_1, \ldots, \mu_d) := (\mu(\{1\}), \ldots, \mu(\{d\})), \quad \mu \in \mathcal{P}(S).$$

We consider probability vectors as row vectors. On $\mathcal{P}(S)$, we introduce a partial order $\preceq$ through

$$\mu \preceq \nu \quad \text{if and only if} \quad \sum_{i=1}^{l} \mu_i \geqslant \sum_{i=1}^{l} \nu_i \quad \text{for all} \quad l \in \{1, \ldots, d\}.$$

This corresponds to the usual stochastic order in terms of cumulative distribution functions when interpreting $S = \{1, \ldots, d\}$ as a subset of $\mathbb{R}$ with the natural order. As a consequence, we have

$$\sum_{i=1}^{d} c_i \mu_i \geqslant \sum_{i=1}^{d} c_i \nu_i \text{ whenever } \mu \preceq \nu \text{ and } S \ni i \longmapsto c_i \in \mathbb{R} \quad \text{is} \quad \text{nonincreasing;} \tag{2}$$

see, for instance, section 1.A.1 in the book by Shaked and Shanthikumar [55].

For $\mu, \nu \in \mathcal{P}(S)$, their greatest lower bound $\mu \wedge \nu$ and least upper bound $\mu \vee \nu$ are given by

$$(\mu \wedge \nu)_j := \max\left\{\sum_{k=1}^{j} \mu_k, \sum_{k=1}^{j} \nu_k\right\} - \max\left\{\sum_{k=1}^{j-1} \mu_k, \sum_{k=1}^{j-1} \nu_k\right\} \quad \text{and}$$

$$(\mu \vee \nu)_j := \min\left\{\sum_{k=1}^{j} \mu_k, \sum_{k=1}^{j} \nu_k\right\} - \min\left\{\sum_{k=1}^{j-1} \mu_k, \sum_{k=1}^{j-1} \nu_k\right\} \quad \text{for all } j \in \{1, \ldots, d\},$$

respectively, where we use the convention $\sum_{k=1}^{0} \mu_k := 0$ and $\sum_{k=1}^{0} \nu_k := 0$. Then, $(\mathcal{P}(S), \preceq)$ is a complete lattice.

We consider a fixed finite time horizon $T \in \mathbb{N}$ and a fixed initial distribution $\eta \in \mathcal{P}(S)$. Let L be the set of all flows

$$\mu : \{0, \ldots, T\} \to \mathcal{P}(S) \quad \text{with } \mu_0 = \eta,$$

and let $\leqslant^L$ be the partial order on L induced by $\preceq$: that is,

$$\mu \leqslant^L \nu \quad \text{if and only if} \quad \mu_t \preceq \nu_t \quad \text{for all } t \in \{0, \ldots, T\}.$$

The greatest lower bound $\mu \wedge^L \nu$ and the least upper bound $\mu \vee^L \nu$ of two elements $\mu, \nu \in L$ are then given by

$$(\mu \wedge^L \nu)_t := \mu_t \wedge \nu_t \quad \text{and} \quad (\mu \vee^L \nu)_t := \mu_t \vee \nu_t \quad \text{for all} \quad t \in \{0, \ldots, T\}.$$

Observe that $(L, \leqslant^L)$ is again a complete lattice.

Let Γ be a nonempty set; Γ represents the set of control actions for the representative player. Define the set $\mathcal{U}$ of Γ-valued open-loop strategies as the set of all mappings $u : \{0, \ldots, T-1\} \to \Gamma$.

Let $A(\gamma))_{\gamma \in \Gamma}$ be a family of transition matrices on S. Thus, for each $\gamma \in \Gamma$, $A(\gamma) = (a_{ij}(\gamma))_{i,j \in S}$ is a $d \times d$ matrix with nonnegative entries such that

$$\sum_{j=1}^{d} a_{ij}(\gamma) = 1 \quad \text{for all} \quad i \in S.$$

For $u \in \mathcal{U}$, we define the flow μ^u of laws of the controlled Markov chain recursively through

$$\mu_0^u := \eta \quad \text{and} \quad \mu_{t+1}^u := \mu_t^u A(u_t) \quad \text{for all } t \in \{0, \ldots, T-1\}, \tag{3}$$

where $\eta \in \mathcal{P}(S)$ is the fixed initial distribution, and we recall that elements of $\mathcal{P}(S)$ are identified as row vectors.

Let E be the subset of $\mathcal{U} \times L$ given by

$$E := \{(u, \mu^u) : u \in \mathcal{U}\},$$

and let $p : E \to L$ be the projection on the second component:

$$p(u, \mu) := \mu \quad \text{for all } (u, \mu) \in E.$$

Thus, $p(u, \mu^u) = \mu^u$ for all $u \in \mathcal{U}$.

Let $f : \{0, \ldots, T-1\} \times S \times \mathcal{P}(S) \times \Gamma \to \mathbb{R}$, $g : S \times \mathcal{P}(S) \to \mathbb{R}$ be functions, representing the running and terminal costs, respectively. Define a functional $J : E \times L \to \mathbb{R}$ according to

$$J((u, \mu^u), \mu) := \sum_{t=0}^{T-1} \sum_{i=1}^{d} f(t, i, \mu_t, u_t) \mu_{t,i}^u + \sum_{i=1}^{d} g(i, \mu_T) \mu_{T,i}^u,$$

where for $\mu \in L$, $t \in \{0, \ldots, T\}$, and $i \in S$, $\mu_{t,i}$ denotes the ith coordinate of μ_t.

As in Section 2, we define the best-response map $R : L \to 2^L$ according to

$$R(\mu) := \{p(\nu) : \nu \in \arg\min_E J(\cdot, \mu)\}.$$

The following conditions on the solution map and J will entail the assumptions of the general setup.

Assumption 4 (Sufficient Conditions). *Suppose that $\leqslant^{\mathcal{U}}$ is a partial order on $\mathcal{U}$ making $(\mathcal{U}, \leqslant^{\mathcal{U}})$ a complete lattice.*

1. *For every sequence $(u_n)_{n \in \mathbb{N}} \subseteq \mathcal{U}$,*

$$\inf_{n \in \mathbb{N}} \mu^{u_n} = \mu^{u^\wedge} \quad \text{with } u^\wedge = \inf_{n \in \mathbb{N}} u_n \quad \text{and}$$

$$\sup_{n \in \mathbb{N}} \mu^{u_n} = \mu^{u^\vee} \quad \text{with } u^\vee = \sup_{n \in \mathbb{N}} u_n.$$

2. *For all $\hat{\mu}, \check{\mu} \in L$ and $\hat{u}, \check{u} \in \mathcal{U}$ with $\hat{\mu} \leqslant^L \check{\mu}$, $\hat{u} \leqslant^{\mathcal{U}} \check{u}$,*

$$J((\check{u}, \mu^{\check{u}}), \check{\mu}) - J((\hat{u}, \mu^{\hat{u}}), \check{\mu}) \leqslant J((\check{u}, \mu^{\check{u}}), \hat{\mu}) - J((\hat{u}, \mu^{\hat{u}}), \hat{\mu}).$$

3. *Given any $\mu \in L$, we have for all sequences $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}$, with $u^\wedge := \inf_{n\in\mathbb{N}} u_n$ and $u^\vee := \sup_{n\in\mathbb{N}} u_n$,*

$$J((u^\wedge, \mu^{u^\wedge}), \mu) = \inf_{n\in\mathbb{N}} J((u_n, \mu^{u_n}), \mu) \quad \text{and}$$

$$J((u^\vee, \mu^{u^\vee}), \mu) = \sup_{n\in\mathbb{N}} J((u_n, \mu^{u_n}), \mu),$$

or else,

$$J((u^\wedge, \mu^{u^\wedge}), \mu) = \sup_{n\in\mathbb{N}} J((u_n, \mu^{u_n}), \mu) \quad \text{and}$$

$$J((u^\vee, \mu^{u^\vee}), \mu) = \inf_{n\in\mathbb{N}} J((u_n, \mu^{u_n}), \mu).$$

Proposition 1. *Given Assumption 4, the set E together with the pointwise lattice operations*

$$(u, \mu^u) \wedge^E (v, \mu^v) := (u \wedge^{\mathcal{U}} v, \mu^u \wedge^L \mu^v) \quad \text{and} \quad (u, \mu^u) \vee^E (v, \mu^v) := (u \vee^{\mathcal{U}} v, \mu^u \vee^L \mu^v),$$

for $u, v \in \mathcal{U}$, becomes a lattice, and the best-response map R, the projection p, and the cost functional J satisfy Assumption 2 and the alternative for Assumption 1 from Remark 1.

Proof. First observe that thanks to condition 1 in Assumption 4, the operations $\wedge^E$, $\vee^E$ are well defined in the sense that if $\nu, \bar{\nu} \in E$, then $\nu \wedge^E \bar{\nu}$ and $\nu \vee^E \bar{\nu}$ are again elements of E.

Because $\mu \wedge^L \bar{\mu} \leqslant^L \mu \vee^L \bar{\mu}$ for all $\mu, \bar{\mu} \in L$, we find that the projection p satisfies condition 1 in Assumption 2. Indeed, if $(u, \mu^u), (v, \mu^v) \in E$, then

$$\begin{aligned} p((u, \mu^u) \wedge^E (v, \mu^v)) &= \mu^u \wedge^L \mu^v = p((u, \mu^u)) \wedge^L p((v, \mu^v)) \leqslant p((u, \mu^u)) \vee^L p((v, \mu^v)) \\ &= \mu^u \vee^L \mu^v = p((u, \mu^u) \vee^E (v, \mu^v)). \end{aligned}$$

Again, thanks to condition 1 in Assumption 4, we have

$$\mu^u \leqslant^L \mu^v \quad \text{for all } u, v \in \mathcal{U} \text{ with } u \leqslant^{\mathcal{U}} v, \tag{4}$$

for in that situation, setting $u_1 := u$ and $u_n := v$, for $n \in \mathbb{N} \setminus \{1\}$, we find $\mu^u = \mu^u \wedge^L \mu^v \leqslant^L \mu^v$.

Let $(u, \mu^u), (v, \mu^v) \in E$, and let $\hat{\mu}, \check{\mu} \in L$ be such that $\hat{\mu} \leqslant^L \check{\mu}$. Set $\check{u} := u \vee^{\mathcal{U}} v$. Then, thanks to conditions 1 and 2 in Assumption 4,

$$\begin{aligned} J((u, \mu^u) \vee^E (v, \mu^v), \check{\mu}) - J((v, \mu^v), \check{\mu}) &= J((\check{u}, \mu^{\check{u}}), \check{\mu}) - J((v, \mu^v), \check{\mu}) \\ &\leqslant J((\check{u}, \mu^{\check{u}}), \hat{\mu}) - J((v, \mu^v), \hat{\mu}) \\ &= J((u, \mu^u) \vee^E (v, \mu^v), \hat{\mu}) - J((v, \mu^v), \hat{\mu}). \end{aligned}$$

This establishes the first inequality in condition 2 in Assumption 2. The second inequality in condition 2 in Assumption 2 is a consequence of condition 3 in Assumption 4. In fact, thanks to condition 3 in Assumption 4, we have for every $\mu \in L$ and all $(u, \mu^u), (v, \mu^v) \in E$,

$$\begin{aligned} &J((u, \mu^u) \vee^E (v, \mu^v), \mu) + J((u, \mu^u) \wedge^E (v, \mu^v), \mu) \\ &\quad = J((u \vee^{\mathcal{U}} v, \mu^u \vee^L \mu^v), \mu) + J((u \wedge^{\mathcal{U}} v, \mu^u \wedge^L \mu^v, \mu) \\ &\quad = \min\{J((u, \mu^u), \mu), J((v, \mu^v), \mu)\} + \max\{J((u, \mu^u), \mu), J((v, \mu^v), \mu)\} \\ &\quad = J((u, \mu^u), \mu) + J((v, \mu^v), \mu). \end{aligned}$$

Let $\mu \in L$ and $((u_n, \mu^{u_n}))_{n\in\mathbb{N}} \subset E$ be such that $J((u_n, \mu^{u_n}), \mu) \searrow \inf_{\nu \in E} J(\nu, \mu)$ as $n \to \infty$. Notice that this infimum exists in $[-\infty, \infty)$. Set

$$\hat{u} := \inf_{n\in\mathbb{N}} u_n \quad \text{and} \quad \check{u} := \sup_{n\in\mathbb{N}} u_n.$$

By condition 1 in Assumption 4, we have

$$\mu^{\hat{u}} = \inf_{n\in\mathbb{N}} \mu^{u_n} \quad \text{and} \quad \check{u} = \sup_{n\in\mathbb{N}} \mu^{u_n}.$$

By condition 3 in Assumption 4 (we only treat the first case there; the second is obtained by interchanging infima and suprema), we find that

$$J((\hat{u},\mu^{\hat{u}}),\mu)=\inf_{n\in\mathbb{N}} J((u_n,\mu^{u_n}),\mu) \quad \text{and} \quad J((\check{u},\mu^{\check{u}}),\mu)=\sup_{n\in\mathbb{N}} J((u_n,\mu^{u_n}),\mu).$$

It follows that $\inf_{\nu\in E} J(\nu,\mu)=J((\hat{u},\mu^{\hat{u}}),\mu)$, which shows that $(\hat{u},\mu^{\hat{u}})\in\arg\min_E J(\cdot,\mu)$ and thus, $\mu^{\hat{u}}\in R(\mu)$. In particular, the set of best-response distributions is nonempty.

Now, suppose that $(\mu_n)_{n\in\mathbb{N}}\subseteq R(\mu)$. For $n\in\mathbb{N}$, choose $u_n\in\arg\min_{u\in\mathcal{U}} J((u,\mu^u),\mu)$ such that $\mu^{u_n}=\mu_n$. Define $\hat{u}$ and $\check{u}$ in the same way. By condition 1 in Assumption 4, we have

$$\mu^{\hat{u}}=\inf_{n\in\mathbb{N}}\mu_n \quad \text{and} \quad \mu^{\check{u}}=\sup_{n\in\mathbb{N}}\mu_n,$$

and by condition 3 in Assumption 4 (in the first case there), we find again that

$$J((\hat{u},\mu^{\hat{u}}),\mu)=\inf_{n\in\mathbb{N}} J((u_n,\mu^{u_n}),\mu) \quad \text{and} \quad J((\check{u},\mu^{\check{u}}),\mu)=\sup_{n\in\mathbb{N}} J((u_n,\mu^{u_n}),\mu).$$

However, $u_n\in\arg\min_{u\in\mathcal{U}} J((u,\mu^u),\mu)$ for every $n\in\mathbb{N}$; hence

$$\inf_{n\in\mathbb{N}} J((u_n,\mu^{u_n}),\mu)=\sup_{n\in\mathbb{N}} J((u_n,\mu^{u_n}),\mu).$$

It follows that $\mu^{\hat{u}}\in R(\mu)$ as well as $\mu^{\check{u}}\in R(\mu)$. In particular, any monotone sequence in $R(\mu)$ has a limit in $R(\mu)$. We thus see that the alternative for Assumption 1 from Remark 1 is satisfied. □

By Proposition 1, Remark 1, and part (a) of Theorem 1, one immediately obtains the following.

Corollary 1. *Given Assumption 4, the set M of solutions to the finite state mean field game is nonempty and contains* $\inf M$ *as well as* $\sup M$.

The following example shows a family of simple two-state models where the assumptions of Proposition 1 are satisfied.

Example 2. Choose $d=2$, and set $\Gamma:=[0,1]$ (with the natural order $\leqslant$). Choose $p,q\in(0,1]$ with $p\leqslant q$, and define controlled transition matrices $A(\gamma)$ according to

$$A(\gamma)\doteq\begin{pmatrix}1-p\gamma & p\gamma\\ 1-q\gamma & q\gamma\end{pmatrix} \quad \text{for all } \gamma\in\Gamma.$$

With this choice, for all $\gamma\in\Gamma=[0,1]$ and all $\mu=(\mu_1,\mu_2)\in\mathcal{P}(S)=\mathcal{P}(\{1,2\})$,

$$\mu A(\gamma)=(1-\gamma(p+\mu_2(q-p)),\gamma(p+\mu_2(q-p))).$$

For $\mu,\bar{\mu}\in\mathcal{P}(S)$, we have that $\mu\preceq\bar{\mu}$ if and only if $\mu_2\leqslant\bar{\mu}_2$ and that

$$\{\mu\wedge\bar{\mu},\mu\vee\bar{\mu}\}=\{\mu,\bar{\mu}\}.$$

Therefore, if $\mu,\bar{\mu}\in L$, then for all $t\in\{0,\dots,T\}$,

$$\{(\mu\wedge^L\bar{\mu})_t,(\mu\vee^L\bar{\mu})_t\}=\{\mu_t,\bar{\mu}_t\}.$$

It also follows that

$$\mu A(\gamma)\preceq\bar{\mu}A(\tilde{\gamma}) \quad \text{whenever } \gamma\leqslant\tilde{\gamma} \text{ and } \mu\preceq\bar{\mu}.$$

In case $\mu=\bar{\mu}$, we have, for all $\gamma,\bar{\gamma}\in\Gamma$ and $(\gamma_n)_{\in\mathbb{N}}\subset\Gamma$,

$$\mu A(\gamma\wedge\bar{\gamma})=(\mu A(\gamma))\wedge(\mu A(\bar{\gamma})),\mu A(\gamma\vee\bar{\gamma})=\Big(\mu A(\gamma)\Big)\vee\Big(\mu A(\bar{\gamma})\Big),$$

$$\mu A\Big(\inf_{n\in\mathbb{N}}\gamma_n\Big)=\inf\{\mu A(\gamma):n\in\mathbb{N}\},\mu A(\sup_{n\in\mathbb{N}}\gamma_n)=\sup\{\mu A(\gamma):n\in\mathbb{N}\}.$$

We introduce a partial order $\leqslant^{\mathcal{U}}$ on $\mathcal{U}$ by

$$u \leqslant^{\mathcal{U}} \tilde{u} \quad \text{if and only if} \quad \mu^u \leqslant^L \mu^{\tilde{u}}.$$

Then, the greatest lower bound $u \wedge^{\mathcal{U}} v$ of two elements $u, v \in \mathcal{U}$ is defined as follows. Set $\hat{\mu} := \mu^u \wedge^L \mu^v$, and define, for $t \in \{0, \dots, T-1\}$,

$$(u \wedge^{\mathcal{U}} v)_t := \min\left\{u_t \cdot \frac{p + (\mu_t^u)_2(q-p)}{p + (\hat{\mu}_t)_2(q-p)},\, v_t \cdot \frac{p + (\mu_t^v)_2(q-p)}{p + (\hat{\mu}_t)_2(q-p)}\right\},$$

where we recall that $\mu_0^u = \eta = \mu_0^v$, hence also $\hat{\mu}_0 = \eta$. By induction, one checks that for every $t \in \{0, \dots, T-1\}$,

$$(u \wedge^{\mathcal{U}} v)_t \in [0,1], \mu_t^{u \wedge^{\mathcal{U}} v} = \hat{\mu}_t.$$

Indeed, the claim holds for $t = 0$. Now, suppose that it holds up to time t and that $\mu_{t+1}^u = \hat{\mu}_{t+1}$. Then, $\hat{\mu}_{t+1} = \mu_t^u A(u_t)$, and there exists $\tilde{\gamma} \in \{u_t, \tilde{u}_t\}$ such that $\hat{\mu}_{t+1} \preceq \hat{\mu}_t A(\tilde{\gamma})$. Then,

$$(\hat{\mu}_{t+1})_2 = u_t(p + (\mu_t^u)_2(q-p)) \leqslant \tilde{\gamma}(p + (\hat{\mu}_t)_2(q-p)),$$

and hence,

$$0 \leqslant u_t \cdot \frac{p + (\mu_t^u)_2(q-p)}{p + (\hat{\mu}_t)_2(q-p)} \leqslant \tilde{\gamma} \leqslant 1.$$

Moreover,

$$\left(\hat{\mu}_t A\left(u_t \cdot \frac{p + (\mu_t^u)_2(q-p)}{p + (\hat{\mu}_t)_2(q-p)}\right)\right)_2 = u_t \cdot \frac{p + (\mu_t^u)_2(q-p)}{p + (\hat{\mu}_t)_2(q-p)} \cdot (p + (\hat{\mu}_t)_2(q-p)) = (\hat{\mu}_{t+1})_2$$

because $\hat{\mu}_{t+1} = \mu_t^u A(u_t)$ by assumption. The case $\hat{\mu}_{t+1} = \mu_t^{\tilde{u}} A(\tilde{u}_t)$ is handled in the same way.

In analogy with the greatest lower bound, one defines the least upper bound $u \vee^{\mathcal{U}} \tilde{u}$. It follows that for all $u, \tilde{u} \in \mathcal{U}$,

$$\mu^{u \wedge^{\mathcal{U}} \tilde{u}} = \mu^u \wedge^L \mu \tilde{u},\; \mu^{u \vee^{\mathcal{U}} \tilde{u}} = \mu^u \vee^L \mu \tilde{u}.$$

Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}$. Set $\hat{\mu} := \inf_{n\in\mathbb{N}} \mu^{u_n}$, and define $\hat{u} \in \mathcal{U}$ by setting

$$\hat{u}_t := \inf\left\{u_n(t) \cdot \frac{p + (\mu_t^{u_n})_2(q-p)}{p + (\hat{\mu}_t)_2(q-p)} : n \in \mathbb{N}\right\}, \quad t \in \{0, \dots, T-1\};$$

in particular, $\hat{u}_0 = \inf_{n\in\mathbb{N}} u_n(0)$. By induction, one checks that $\mu^{\hat{u}} = \hat{\mu}$, hence the part of condition 1 in Assumption 4 regarding the greatest lower bound is satisfied. The upper bound part is analogous.

Regarding the costs, choose zero running costs $f \equiv 0$ and terminal costs g given by

$$g(i, m) := \phi(i) \cdot \psi(m_2), \quad i \in \{1, 2\},$$

where $\phi(2) < \phi(1)$ and $\psi : [0,1] \to \mathbb{R}$ is nondecreasing (but not necessarily continuous). Then, for $u \in \mathcal{U}$, $\mu \in L$,

$$J((u, \mu^u), \mu) = ((\phi(2) - \phi(1))(\mu_T^u)_2 + \phi(1)) \cdot \psi((\mu_T)_2).$$

Here, if $(\mu^{(n)})_{n\in\mathbb{N}} \subset L$ and $\hat{\mu} = \inf_{n\in\mathbb{N}} \mu^{(n)}$, $\check{u} = \sup_{n\in\mathbb{N}} \mu^{(n)}$, then

$$(\hat{\mu}_T)_2 = \inf\left\{(\mu_T^{(n)})_2 : n \in \mathbb{N}\right\},\; (\check{u}_T)_2 = \sup\left\{(\mu_T^{(n)})_2 : n \in \mathbb{N}\right\}.$$

The form of J and condition 1 in Assumption 4, therefore, imply that condition 3 in Assumption 4 holds.

In order to check the submodularity condition (i.e., condition 2 in Assumption 4), let $\hat{\mu}, \check{u} \in L$ and $\hat{u}, \check{u} \in \mathcal{U}$ be such that $\hat{\mu} \leqslant^L \check{u}$, $\hat{u} \leqslant^{\mathcal{U}} \check{u}$. Then,

$$J((\check{u}, \mu^{\check{u}}), \check{u}) - J((\hat{u}, \mu^{\hat{u}}), \check{u}) = ((\phi(2) - \phi(1))((\mu_T^{\check{u}})_2 - (\mu_T^{\hat{u}})_2)) \cdot \psi((\check{u}_T)_2),$$

$$J((\check{u}, \mu^{\check{u}}), \hat{\mu}) - J((\hat{u}, \mu^{\hat{u}}), \hat{\mu}) = ((\phi(2) - \phi(1))((\mu_T^{\check{u}})_2 - (\mu_T^{\hat{u}})_2)) \cdot \psi((\hat{\mu}_T)_2).$$

However, $\phi(2)-\phi(1)<0$, whereas $(\mu_T^{\check{u}})_2-(\mu_T^{\hat{u}})_2\geqslant 0$ by condition 1 in Assumption 4 because $\hat{u}\leqslant^{\mathcal{U}}\check{u}$ and $\psi((\hat{\mu}_T)_2)\leqslant\psi((\check{u}_T)_2)$ because $\hat{\mu}\leqslant^{L}\check{u}$ and ψ is nondecreasing. It follows that

$$J((\check{u},\mu^{\check{u}}),\check{u})-J((\hat{u},\mu^{\hat{u}}),\check{u})\leqslant J((\check{u},\mu^{\check{u}}),\hat{\mu})-J((\hat{u},\mu^{\hat{u}}),\hat{\mu}),$$

which is condition 2 in Assumption 4.

5. Submodular Mean Field Games with Singular Controls

In this section, we specialize to mean field games with singular controls and show that they can be embedded into the general setup given in Section 2. In the following, we consider MFGs with common noise, in which the representative player faces a convex optimization problem (see Section 5.1), and MFGs without common noise, in which the representative player faces a nonconvex optimization problem (see Section 5.2). In these two models, the operations, which are postulated in Assumption 2, can be constructed with different techniques. These operations can be explicitly constructed in the case in which the dynamics are given by controlled geometric Brownian motions and the costs are convex in the state variable. When the dynamics are nonlinear, the construction of such operations is provided by approximating singular controls via regular controls and exploiting the results in Dianetti et al. [25].

Throughout this section, we take measurable functions

$$f:[0,T]\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R},$$
$$g:\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R},$$
$$c:[0,T]\to[0,\infty),$$

satisfying the following conditions.

Assumption 5.

1. *For dt a.a. $t\in[0,T]$, the functions $f(t,\cdot,\mu)$ and $g(\cdot,\mu)$ are lower semicontinuous, and for some $p>1$ and all $(t,x,\mu)\in[0,T]\times\mathbb{R}\times\mathcal{P}(\mathbb{R})$,*

$$\kappa(|x|^p-1)\leqslant f(t,x,\mu)\leqslant K(1+|x|^p),\quad \kappa(|x|^p-1)\leqslant g(x,\mu)\leqslant K(1+|x|^p),$$

with constants $K,\kappa>0$.

2. *For dt a.a. $t\in[0,T]$, the functions $f(t,\cdot,\cdot)$ and g have decreasing differences in (x,μ); that is, for $\phi\in\{f(t,\cdot,\cdot),g\}$,*

$$\phi(\bar{x},\bar{\mu})-\phi(x,\bar{\mu})\leqslant\phi(\bar{x},\mu)-\phi(x,\mu),$$

for all $\bar{x},x\in\mathbb{R}$ and $\bar{\mu},\mu\in\mathcal{P}(\mathbb{R})$ with $\bar{x}\geqslant x$ and $\bar{\mu}\geqslant^{\mathrm{st}}\mu$.

3. *The cost c is nonincreasing and continuously differentiable with $c>0$.*

We point out that although conditions 1 and 3 are natural in order to treat the representative player minimization problem (see, e.g., the beginning of Section 5.1), condition 2 is a structural hypothesis, which will ensure that the submodularity conditions (in particular, condition 2 in Assumption 2) are satisfied. For this reason, sometimes we will also refer to condition 2 in Assumption 5 as to the submodularity condition.

Remark 2 (On the Lasry–Lions Monotonicity Condition). It is natural to compare the submodulartity condition (condition 2 in Assumption 5) with the so-called *Lasry–Lions monotonicity condition*

$$\int_{\mathbb{R}}(\phi(x,\bar{\mu})-\phi(x,\mu))d(\bar{\mu}-\mu)(x)\geqslant 0,\qquad \text{for any } \bar{\mu},\mu\in\mathcal{P}(\mathbb{R}),\tag{5}$$

which is typically related to the uniqueness of equilibria (see, e.g., p. 169 in the book by Carmona and Delarue [19]). As already observed in Dianetti et al. [25], there is no relation between the submodularity condition and (5). However, condition 2 in Assumption 5 implies that the map $\phi(\cdot,\bar{\mu})-\phi(\cdot,\mu)$ is decreasing for $\mu,\bar{\mu}\in\mathcal{P}(\mathbb{R})$ with $\mu\leqslant^{\mathrm{st}}\bar{\mu}$. Therefore, from (1), we deduce that

$$\int_{\mathbb{R}}(\phi(x,\bar{\mu})-\phi(x,\mu))d(\bar{\mu}-\mu)(x)\leqslant 0,\qquad \text{for any } \bar{\mu},\mu\in\mathcal{P}(\mathbb{R}) \text{ with } \mu\leqslant^{\mathrm{st}}\bar{\mu};$$

the latter, roughly speaking, is sort of an opposite version of the Lasry–Lions monotonicity condition (5).

5.1. Controlled Geometric Brownian Motion and Common Noise

5.1.1. Formulation of the Model. Let Assumption 5 be satisfied with $p=2$. Let $W=(W_t)_{t\in[0,T]}$ and $B=(B_t)_{t\in[0,T]}$ be two independent Brownian motions on a complete filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. Define the set of *admissible monotone controls* as the set $\mathcal{V}_\uparrow$ of all $\mathbb{F}$-adapted càdlàg, nondecreasing, square-integrable, and nonnegative processes $\xi=(\xi_t)_{t\in[0,T]}$ such that

$$\mathbb{E}\left[\int_0^T \xi_t^2 d\pi(t)\right]<\infty, \text{ where } \quad \pi:=dt+\delta_T. \tag{6}$$

Let $b\in\mathbb{R}$, σ, $\sigma^o\geqslant 0$, and $\mathbb{F}^o:=(\mathcal{F}_t^o)_{t\in[0,T]}$ denote the filtration generated by $\sigma^o B$ (which is trivial in the case of no common noise; i.e., for $\sigma^o=0$). Let x_0 be a square integrable $\mathcal{F}_0$-random variable. For each $\xi\in\mathcal{V}_\uparrow$, let $X^\xi=(X_t^\xi)_{t\in[0,T]}$ denote the unique strong solution to the linearly controlled geometric dynamics, given by

$$dX_t^\xi=X_t^\xi(b\,dt+\sigma dW_t+\sigma^o dB_t)+d\xi_t,\quad t\in[0,T],\quad X_{0-}^\xi=x_0. \tag{7}$$

For any $\mathcal{P}(\mathbb{R})$-valued $\mathbb{F}^o$-progressively measurable process $\mu=(\mu_t)_{t\in[0,T]}$, we introduce the cost functional

$$J(\xi,\mu):=\mathbb{E}\left[\int_0^T f(t,X_t^\xi,\mu_t)dt+g(X_T^\xi,\mu_T)+\int_{[0,T]}c_t d\xi_t\right],\quad \xi\in\mathcal{V}_\uparrow,$$

and consider the singular control problem $\inf_{\xi\in\mathcal{V}_\uparrow}J(\xi,\mu)$. We say that (X^μ,ξ^μ) is an *optimal pair* for the flow μ if $J(\xi^\mu,\mu)\leqslant J(\xi,\mu)$ for each admissible ξ and $X^\mu=X^{\xi^\mu}$.

Definition 2. A $\mathcal{P}(\mathbb{R})$-valued $\mathbb{F}^o$-progressively measurable process $\mu=(\mu_t)_{t\in[0,T]}$ is an equilibrium of the MFG with singular controls and common noise if

1. there exists an optimal pair (X^μ,ξ^μ) for μ and
2. $\mu_t=\mathbb{P}[X_t^\mu\in\cdot\,|\mathcal{F}_T^o]$ $\mathbb{P}$ almost surely (a.s.), for any $t\in[0,T]$.

We point out that similarly to remark 1 in Tchuendom [56], it can be shown that for any $\xi\in\mathcal{V}_\uparrow$, one has $\mathbb{P}[X_t^\xi\in\cdot\,|\mathcal{F}_T^o]=\mathbb{P}[X_t^\xi\in\cdot\,|\mathcal{F}_t^o]$ $\mathbb{P}$ a.s., for any $t\in[0,T]$. Therefore, any equilibrium μ as in Definition 2 is actually strong, in the sense that it is adapted to the filtration generated by the common noise B.

5.1.2. Optimal Controls and A Priori Estimates. Recalling that $c>0$, we enforce the following requirements.

Assumption 6. *For dt a.a. $t\in[0,T]$, the functions $f(t,\cdot,\mu)$ and $g(\cdot,\mu)$ are strictly convex.*

Under Assumption 6, by employing arguments such as those in the proof of theorem 8 in Menaldi and Taksar [46], it can be shown that for any process μ, there exists a unique optimal pair (X^μ,ξ^μ). Moreover, because the control, which constantly equals to zero, is suboptimal, the growth conditions in Assumption 6 imply that

$$\begin{aligned}\kappa\mathbb{E}\left[\int_0^T|X_t^\mu|^2dt+|X_T^\mu|^2\right]-\kappa(1+T)&\leqslant J(\xi^\mu,\mu)\\ &\leqslant J(0,\mu)\leqslant K\mathbb{E}\left[\int_0^T|X_t^0|^2dt+|X_T^0|^2\right]+K(1+T),\end{aligned}$$

so that for some constant $\bar{C}>0$ independent of μ, we have

$$\mathbb{E}\left[\int_0^T|X_t^\mu|^2dt+|X_T^\mu|^2\right]\leqslant\bar{C}.$$

Therefore, for a suitable generic constant $C>0$ (changing from line to line), we obtain

$$\begin{aligned}\mathbb{E}[|\xi_T^\mu|^2]&\leqslant\mathbb{E}\left[\left(X_T^\mu-x_0-\int_0^T X_t^\mu(b\,dt+\sigma dW_t+\sigma^o dB_t)\right)^2\right]\\ &\leqslant C\mathbb{E}\left[|X_T^\mu|^2+|x_0|^2+\int_0^T|X_t^\mu|^2dt\right]\leqslant C,\end{aligned}$$

and by a standard use of Grönwall's inequality, we conclude that

$$\mathbb{E}[|X_t^\mu|^2+|\xi_t^\mu|^2]\leqslant M,\qquad \text{for each } t\in[0,T], \tag{8}$$

for a constant $M>0$, which does not depend on μ.

5.1.3. The Control Set E and Its Operations. Define

$$E := \{(X^\xi, \xi) \mid \xi \in \mathcal{V}_\uparrow, X^\xi \text{ solution to (7)}\} \quad \text{and}$$
$$p(X^\xi, \xi)_t(A) := \mathbb{P}[X_t^\xi \in A \mid \mathcal{F}_T^o], \quad A \in \mathcal{B}(\mathbb{R}).$$

Because of (6), the set E is a subset of the space $\mathbb{L}_\pi^2$ of $\mathbb{R}^2$-valued progressively measurable processes ν such that $\|\nu\|_{\pi,2} := \mathbb{E}\left[\int_0^T |\nu_t|^2 d\pi(t)\right] < \infty$, endowed with the norm $\|\cdot\|_{\pi,2}$. Moreover, the lower semicontinuity properties of J in Assumptions 1 and 3 are satisfied, whereas the continuity of J w.r.t. μ holds by assuming f and g to be continuous in μ.

Observe that for each $\xi \in \mathcal{V}_\uparrow$, the solution to the stochastic differential equation (SDE) (7) is, $\mathbb{P}$ a.s., given by

$$X_t^\xi = \mathcal{E}_t\left[x_0 + \int_{[0,t]} \mathcal{E}_s^{-1} d\xi_s\right] \quad \text{with} \quad \mathcal{E}_t := \exp\left[\left(b - \frac{(\sigma^2 + (\sigma^o)^2)}{2}\right)t + \sigma W_t + \sigma^o B_t\right] \tag{9}$$

for each $t \in [0,T]$. Hence, defining the map $\Phi : \mathcal{V}_\uparrow \to \mathcal{V}_\uparrow$ by $\Phi_t(\xi) := \int_{[0,t]} \mathcal{E}_s^{-1} d\xi_s$, we have, $\mathbb{P}$ a.s.,

$$X_t^\xi = \mathcal{E}_t[x_0 + \Phi_t(\xi)], \quad \text{for each } t \in [0,T].$$

Moreover, for $\bar\xi, \xi \in \mathcal{V}_\uparrow$, $\bar\zeta := \Phi(\bar\xi)$, and $\zeta := \Phi(\xi)$, we define, $\mathbb{P}$ a.s., the controls

$$\xi_t^\wedge := \int_{[0,t]} \mathcal{E}_s d(\bar\zeta \wedge \zeta)_s \quad \text{and} \quad \xi_t^\vee := \int_{[0,t]} \mathcal{E}_s d(\bar\zeta \vee \zeta)_s, \quad \text{for each } t \in [0,T], \tag{10}$$

and obtain

$$X_t^{\bar\xi} \wedge X_t^\xi = \mathcal{E}_t[x_0 + \bar\zeta_t \wedge \zeta_t] = \mathcal{E}_t\left[x_0 + \int_{[0,t]} \mathcal{E}_s^{-1} d\xi_s^\wedge\right] = X_t^{\xi^\wedge} \quad \text{and}$$
$$X_t^{\bar\xi} \vee X_t^\xi = \mathcal{E}_t[x_0 + \bar\zeta_t \vee \zeta_t] = \mathcal{E}_t\left[x_0 + \int_{[0,t]} \mathcal{E}_s^{-1} d\xi_s^\vee\right] = X_t^{\xi^\vee}. \tag{11}$$

According to (10), we introduce the operations $\wedge^E, \vee^E : E \times E \to E$ via

$$(X^{\bar\xi}, \bar\xi) \wedge^E (X^\xi, \xi) := (X^{\xi^\wedge}, \xi^\wedge) \quad \text{and} \quad (X^{\bar\xi}, \bar\xi) \vee^E (X^\xi, \xi) := (X^{\xi^\vee}, \xi^\vee). \tag{12}$$

Note that in light of (11), the operations $\wedge^E, \vee^E$ satisfy condition 1 in Assumption 2.

5.1.4. The Submodularity Condition. Using the definition of $\xi^\vee$, the linearity of the integral, and that $\bar\zeta \vee \zeta - \bar\zeta = \zeta - \bar\zeta \wedge \zeta$, we obtain that for each $t \in [0,T]$,

$$\xi_t^\vee - \bar\xi_t = \int_{[0,t]} \mathcal{E}_s (d(\bar\zeta \vee \zeta)_s - d\bar\zeta_s) = \int_{[0,t]} \mathcal{E}_s (d\zeta_s - d(\bar\zeta \wedge \zeta)_s) = \xi_t - \xi_t^\wedge \quad \mathbb{P} \text{ a.s.} \tag{13}$$

Recalling the definition of the measure π in (6), for $\mu, \bar\mu \in L$, we define the order relation

$$\mu \leqslant^L \bar\mu \text{ if and only if } \mu_t \leqslant^{st} \nu_t, \ \mathbb{P} \text{ a.s., for } \pi \text{ a.a. } t \in [0,T]. \tag{14}$$

Now, let $\mu, \bar\mu$ be two $\mathcal{P}(\mathbb{R})$-valued, $\mathbb{F}^o$-progressively measurable processes with $\mu \leqslant^L \bar\mu$ and $\xi, \bar\xi \in \mathcal{V}_\uparrow$. Using (11) and (13), we find

$$\begin{aligned}
J(\xi^\vee, \bar\mu) - J(\bar\xi, \bar\mu) &= \mathbb{E}\left[\int_0^T (f(t, X_t^{\bar\xi} \vee X_t^\xi, \bar\mu_t) - f(t, X_t^{\bar\xi}, \bar\mu_t))dt\right] \\
&\quad + \mathbb{E}\left[g(X_T^{\bar\xi} \vee X_T^\xi, \bar\mu_T) - g(X_T^{\bar\xi}, \bar\mu_T) + \int_{[0,T]} c_t d(\xi^\vee - \bar\xi)_t\right] \\
&= \mathbb{E}\left[\int_0^T (f(t, X_t^\xi, \bar\mu_t) - f(t, X_t^{\bar\xi} \wedge X_t^\xi, \bar\mu_t))dt\right] \\
&\quad + \mathbb{E}\left[g(X_T^\xi, \bar\mu_T) - g(X_T^{\bar\xi} \wedge X_T^\xi, \bar\mu_T) + \int_{[0,T]} c_t d(\xi - \xi^\wedge)_t\right] \\
&= J(\xi, \bar\mu) - J(\xi^\wedge, \bar\mu).
\end{aligned} \tag{15}$$

Moreover, by using (11) and Assumption 5, we obtain that

$$\begin{aligned} J(\xi,\bar{\mu}) - J(\xi^\wedge,\bar{\mu}) &\leqslant \mathbb{E}\left[\int_0^T (f(t,X_t^\xi,\mu_t) - f(t,X_t^{\bar{\xi}}\wedge X_t^\xi,\mu_t))dt\right] \\ &\quad + \mathbb{E}\left[g(X_T^\xi,\mu_T) - g(X_T^{\bar{\xi}}\wedge X_T^\xi,\mu_T) + \int_{[0,T]} c_t d(\xi-\xi^\wedge)_t\right] \\ &= J(\xi,\mu) - J(\xi^\wedge,\mu). \end{aligned} \tag{16}$$

Note that (15) and (16) imply that condition 2 in Assumption 2 is satisfied, so that the operations $\wedge^E$, $\vee^E$ fulfill all the requirements of Assumption 2.

Moreover, taking $\xi \in \arg\min_{\mathcal{V}_\uparrow} J(\cdot,\mu)$ and $\bar{\xi} \in \arg\min_{\mathcal{V}_\uparrow} J(\cdot,\bar{\mu})$ and using (15) and (16), we find that $\xi^\wedge \in \arg\min_{\mathcal{V}_\uparrow} J(\cdot,\mu)$ and $\xi^\vee \in \arg\min_{\mathcal{V}_\uparrow} J(\cdot,\bar{\mu})$. Therefore, by the uniqueness of optimal controls, we conclude that $\xi^\wedge = \xi$ and $\xi^\vee = \bar{\xi}$, so that

$$X_t^\mu \leqslant X_t^{\bar{\mu}},\, \pi \ \text{ a.e., whenever } \mu \leqslant^{\mathrm{L}} \bar{\mu}. \tag{17}$$

5.1.5. The Lattice *L*. We move on to the identification of a suitable partially ordered set $(L, \leqslant^L)$. Thanks to the a priori estimate (8) and Chebyshev's inequality for conditional probabilities, we obtain (employing the convention $x/0 = \infty$ for any $x \geqslant 0$)

$$\begin{aligned} \mathbb{P}[X_t^\mu \leqslant x|\mathcal{F}_T^o] &\geqslant \left(1 - \frac{\mathbb{E}[|X_t^\mu|^2|\mathcal{F}_T^o]}{(x\vee 0)^2}\right)\vee 0 \geqslant \left(1 - \frac{\operatorname{ess\,sup}_\mu \mathbb{E}[|X_t^\mu|^2|\mathcal{F}_T^o]}{(x\vee 0)^2}\right)\vee 0 \\ &=: \mu_t^{\mathrm{Max}}((-\infty,x]), \end{aligned} \tag{18}$$

as well as

$$\mathbb{P}[X_t^\mu \leqslant x|\mathcal{F}_T^o] \leqslant \frac{\mathbb{E}[|X_t^\mu|^2|\mathcal{F}_T^o]}{(x\wedge 0)^2}\wedge 1 \leqslant \frac{\operatorname{ess\,sup}_\mu \mathbb{E}[|X_t^\mu|^2|\mathcal{F}_T^o]}{(x\wedge 0)^2}\wedge 1 =: \mu_t^{\mathrm{Min}}((-\infty,x]) \tag{19}$$

for any $\mathcal{P}(\mathbb{R})$-valued $\mathbb{F}^o$-progressively measurable flow μ. From (17), we see that the set $\{X^\mu | \mu$ is a $\mathcal{P}(\mathbb{R})$-valued $\mathbb{F}^o$-progressively measurable flow$\}$ and is directed downward (and upward). Therefore, by the monotone convergence theorem,

$$\operatorname*{ess\,sup}_\mu \mathbb{E}[|X_t^\mu|^2|\mathcal{F}_T^o] \in \mathbb{L}^1(\Omega;\mathbb{P}),$$

so that $\operatorname{ess\,sup}_\mu \mathbb{E}[|X_t^\mu|^2|\mathcal{F}_T^o] < \infty$ $\mathbb{P}$ a.s. We deduce that the $\mathbb{F}^o$-progressively measurable processes μ^{Min} and μ^{Max} are $\mathcal{P}(\mathbb{R})$ valued and that for all $\mathcal{P}(\mathbb{R})$-valued $\mathbb{F}^o$-progressively measurable flows μ,

$$\mu_t^{\mathrm{Min}} \leqslant^{\mathrm{st}} \mathbb{P}[X_t^\mu \in \cdot|\mathcal{F}_T^o] \leqslant^{\mathrm{st}} \mu_t^{\mathrm{Max}} \quad \mathbb{P} \text{ a.s.,} \quad \text{for all } t \in [0,T]. \tag{20}$$

We, therefore, consider the set L of all $\mathcal{P}(\mathbb{R})$-valued, $\mathbb{F}^o$-progressively measurable processes μ with

$$\mu_t^{\mathrm{Min}} \leqslant^{\mathrm{st}} \mu_t \leqslant^{\mathrm{st}} \mu_t^{\mathrm{Max}} \quad \mathbb{P} \text{ a.s.,} \quad \text{for } \pi \text{ a.a. } t \in [0,T],$$

endowed with the order relation $\leqslant^L$ defined in (14). Because the ordered set $(L, \leqslant^L)$ is a special instance of the lattice $\mathcal{L}$ considered in Section 3, which is, in addition, order bounded, it is a complete and Dedekind super complete lattice.

5.1.6. Existence and Approximation of Equilibria. For any $\mu \in L$, set $R(\mu)_t := \mathbb{P}[X_t^\mu \in \cdot|\mathcal{F}_T^o]$, for each $t \in [0,T]$. Thanks to (20), the best-reply map $R: L \to L$ is well defined, and the MFG equilibria of the MFG with singular controls correspond to processes $\mu \in L$ with $R(\mu) = \mu$.

We can now state and prove the main result of this subsection.

Theorem 3. *The set of solutions of the MFG with singular controls and common noise is a nonempty complete lattice. Moreover, if f and g are continuous in (x,μ), then*

1. *the learning procedure $\underline{\mu}^n$ defined inductively by $\underline{\mu}^0 = \inf L$ and $\underline{\mu}^{n+1} = R(\underline{\mu}^n)$ is nondecreasing in L and converges to the minimal MFG solution, and*
2. *the learning procedure $\bar{\mu}^n$ defined inductively by $\bar{\mu}^0 = \sup L$ and $\bar{\mu}^{n+1} = R(\bar{\mu}^n)$ is nonincreasing in L and converges to the maximal MFG solution.*

Proof. The fact that the set of MFG solutions is a nonempty complete lattice is a direct consequence of Theorem 1. We, therefore, just prove the convergence of the learning procedure in claim (1) (claim (2) can be proved analogously). Even if the sequential compactness in Assumption 3 is not satisfied, the arguments in the proof of Theorem 1 can be recovered as follows.

We first observe that thanks to (17) and the definition of R, the sequence $\underline{\mu}^n$ is nondecreasing in L. Hence, setting $(X^n, \xi^n) := (X^{\underline{\mu}^n}, \xi^{\underline{\mu}^n})$, again by (17) we have that $X_t^n \leqslant X_t^{n+1}$, $\mathbb{P} \otimes \pi$ almost everywhere (a.e.) for any $n \in \mathbb{N}$. Therefore, we can define the process $X_t := \sup_n X_t^n$, and by the monotone convergence theorem and the estimates in (8), we conclude that $X^n \to X$ in $\mathbb{L}^2_\pi$ as $n \to \infty$. Next, we define the control process ξ by setting

$$\xi_t := X_t - x_0 - \int_0^t X_s(bdt + \sigma dW_s + \sigma^o dB_s).$$

The convergence of X^n in $\mathbb{L}^2_\pi$ implies that $\xi^n \to \xi$ in $\mathbb{L}^2_\pi$ as $n \to \infty$, so that ξ is nondecreasing. Employing lemma 3.5 in Kabanov [40], we can take càdlàg versions of X and ξ, so that $(X, \xi) \in E$. After repeating the arguments from the proof of Theorem 1, the proof is complete. □

5.2. Nonconvex Case Without Common Noise

In this subsection, we treat a model of mean field games with singular controls and no common noise for a general drift and a not necessarily convex running cost. As a consequence, optimal controls are in general not unique. In comparison with the previous subsection, this case requires a more technical analysis, which makes use of a weak formulation of the problem in the spirit of Haussmann and Suo [37].

5.2.1. Model Formulation. Let $\sigma \geqslant 0$ be a constant and $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. In order to come up with a weak formulation of the problem, the initial value of the dynamics will be described through a fixed initial distribution $\nu_0 \in \mathcal{P}(\mathbb{R})$, satisfying $|\nu_0|^p := \int_{\mathbb{R}} |y|^p d\nu_0(y) < \infty$ with $p > 1$ from Assumption 5.

Definition 3. A tuple $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi)$ is said to be an admissible singular control if
1. $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions,
2. x_0 is an $\mathcal{F}_0$-measurable $\mathbb{R}$-valued random variable with $\mathbb{P} \circ x_0^{-1} = \nu_0$,
3. W is a standard $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$-Brownian motion, and
4. $\xi : \Omega \times [0,T] \to [0, \infty)$ is an $\mathbb{F}$-adapted nondecreasing càdlàg process.

We denote by E^w the set of admissible singular controls.

Again, because b is assumed to satisfy the usual Lipschitz continuity and growth conditions, for any $\rho \in E^w$ there exists a unique process $X^\rho : \Omega \times [0,T] \to \mathbb{R}$ solving the system's dynamics equation that now reads as

$$X_t^\rho = x_0 + \int_0^t b(t, X_t^\rho)dt + \sigma W_t + \xi_t, \quad t \in [0,T]. \tag{21}$$

Then, for a measurable flow of probability measures μ, we define the cost functional

$$J(\rho, \mu) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t^\rho, \mu_t)dt + g(X_T^\rho, \mu_T) + \int_{[0,T]} c_t d\xi_t\right], \quad \rho \in E^w,$$

and we say that $\rho \in E^w$ is an *optimal control* for the flow of measures μ if it solves the optimal control problem related to μ: that is, if $J(\rho, \mu) = \inf_{E^w} J(\cdot, \mu)$.

Definition 4. A measurable flow of probabilities μ is an MFG equilibrium if
1. there exists an optimal control $\rho \in E^w$ for μ and
2. $\mu_t = \mathbb{P} \circ (X_t^\rho)^{-1}$ for any $t \in [0,T]$.

5.2.2. Reformulation via Control Rules and Preliminary Remarks. In order to have a topology on the space of admissible controls, we reformulate the problem in terms of control rules. We introduce the following canonical space $(\Omega, \mathcal{F})$ by

$$\Omega := \mathbb{R} \times \mathcal{C} \times \mathcal{D} \times \mathcal{D}_\uparrow, \quad \mathcal{F} := \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathcal{C}) \otimes \mathcal{B}(\mathcal{D}) \otimes \mathcal{B}(\mathcal{D}_\uparrow). \tag{22}$$

We define the set of control rules as

$$E := \{\nu^\rho \,|\, \rho \in E^w\}, \quad \text{where} \quad \nu^\rho := \mathbb{P} \circ (x_0, W, X^\rho, \xi)^{-1} \text{ for } \quad \rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi) \in E^w,$$

And with a slight abuse of notation, we set $J(\nu^\rho, \mu) := J(\rho, \mu)$. In this way, E is naturally defined as a subspace of the topological space $\mathbb{P}(\Omega)$.

Remark 3 (Existence of Optimal Controls). Under the standing assumptions, it is shown in Haussmann and Suo [37] that for each measurable flow of probabilities μ, $J(\cdot, \mu)$ is lower semicontinuous, and the set $\arg\min_E J(\cdot, \mu) \subset E$ is nonempty (see theorems 3.6 and 3.8 in Haussmann and Suo [37]). Also, one can show that (see theorem 3.7 in Haussmann and Suo [37]) for each sequence $(\nu_n)_{n\in\mathbb{N}} \subset \arg\min_E J(\cdot, \mu)$, we can find an admissible singular control $\nu \in \arg\min_E J(\cdot, \mu)$ such that, up to a subsequence, ν_n converges weakly to ν in $\mathcal{P}(\Omega)$.

Now, for any measurable flow of measures μ, if $\rho \in \arg\min_{E^w} J(\cdot, \mu)$, we can repeat (with minor modifications) the arguments leading to (8) in order to get a priori estimates on the moments of optimally controlled trajectories; namely, we have

$$\mathbb{E}^{\mathbb{P}}[|X_t^\rho|^p + (\xi_T)^p] \leqslant M, \quad \text{for any} \quad t \in [0, T] \quad \text{and} \quad \rho \in \arg\min_{E^w} \; J(\cdot, \mu), \tag{23}$$

with a constant $M > 0$ independent of the flow of measures μ. Therefore, following computations similar to those leading to (18) and (19) (see also lemma 3.4 in Nendel [50]), we can find μ^{Min}, $\mu^{\text{Max}} \in \mathcal{P}(\mathbb{R})$ such that for any flow of measures μ, one has

$$\mu^{\text{Min}} \leqslant^{\text{st}} \mathbb{P} \circ (X_t^\rho)^{-1} \leqslant^{\text{st}} \mu^{\text{Max}}, \quad \text{for any} \quad t \in [0, T] \text{ and } \rho \in \arg\min_{E^w} J(\cdot, \mu). \tag{24}$$

We thus define the set of feasible flows of measures L as the set of all equivalence classes (w.r.t. the measure $\pi := dt + \delta_T$ on the interval $[0, T]$) of measurable flows of probabilities $\mu : [0, T] \to \mathcal{P}(\mathbb{R})$ with $\mu_t \in [\mu^{\text{Min}}, \mu^{\text{Max}}]$ for π a.a. $t \in [0, T]$. On L, we consider the order relation $\leqslant^L$ given by $\mu \leqslant^L \nu$ if and only if $\mu_t \leqslant^{\text{st}} \bar{\mu}_t$, for π a.a. $t \in [0, T]$, with the lattice structure given by

$$(\mu \wedge^L \bar{\mu})_t := \mu_t \wedge^{\text{st}} \bar{\mu}_t \quad \text{and} \quad (\mu \vee^L \bar{\mu})_t := \mu_t \vee^{\text{st}} \bar{\mu}_t \quad \text{for} \quad \pi \text{ a.a.} \quad t \in [0, T].$$

Again, this is a particular instance of the lattice $\mathcal{L}$ considered in Section 3, and it is, by definition, norm bounded. As a consequence, $(L, \leqslant^L)$ is a complete and Dedekind super complete lattice.

Next, we can define the set

$$E^{M,w} := \{\nu \in E^w | (23) \text{ holds } \} \quad \text{and} \quad \mathrm{E}^{\mathrm{M}} := \{\nu^\rho \mid \rho \in \mathrm{E}^{\mathrm{M,w}}\},$$

so that $\arg\min_E J(\cdot, \mu) \subset E^M$ for any flow μ. We observe that because of the Meyer–Zheng tightness criteria (see theorem 4 on p. 360 in Meyer and Zheng [47]), the set E^M is a relatively compact subset of $\mathcal{P}(\Omega)$. Moreover, the projection map

$$p : E \to L \quad \text{with} \quad p(\nu^\rho) := \mathbb{P} \circ (X^\rho)^{-1}, \quad \text{for } \rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, \mathrm{W}, \xi) \in E^w,$$

satisfies the conditions in Assumptions 1 and 3.

Let 2^L be the set of all subsets of L. Then, thanks to (24), the best-response correspondence $R : L \to 2^L$, given by $R(\mu) := \{p\nu | \nu \in \arg\min_E J(\cdot, \mu)\}$ for $\mu \in L$ is well defined. The flow of measures $\mu^* \in L$ is a solution to the mean field game with singular controls if $\mu^* \in R(\mu^*)$.

5.2.3. Existence and Approximation of Solutions. In order to employ the results from Section 2, we begin by providing the following technical result.

Lemma 3. *There exist two operations $\wedge^E, \vee^E : E^M \times E^M \to E$ satisfying Assumption 2.*

Proof. The argument exploits an approximation scheme of the singular controls through regular controls and the results derived in Dianetti et al. [25]. We divide the proof in four steps.

Step 1. For $i = 1, 2$, take control rules $\nu_i = \nu^{\rho_i} \in E^M$ with $\rho_i = (\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, x_0^i, W^i, \xi^i) \in E^{M,w}$. Without loss of generality, we can assume that the controls ρ_1, ρ_2 are defined on a same stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W)$; that is, $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, x_0^i, W^i) = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W)$, for $i = 1, 2$.

Introduce a Wong–Zakai-type approximation of ξ^i by defining the sequences of processes $(\xi^{i,n})_{n\in\mathbb{N}}$ through

$$\xi_t^{i,n} := \begin{cases} n\int_{t-1/n}^{t} \xi_s^i ds, & t\in[0,T), \\ \xi_T^i, & t=T, \end{cases} \tag{25}$$

for each $n\in\mathbb{N}$. Recall that processes are always (implicitly) assumed to be equal to zero for negative times. Further, note that because $\mathbb{E}^{\mathbb{P}}[|\xi_T^i|^p]<\infty$ (recall that $\xi^i\in E^{M,w}$ by assumption), the processes $\xi^{i,n}$ are Lipschitz continuous on the time interval $[0,T)$. However, they may have discontinuities at time T. Moreover, for each $i=1,2$ and all $n\in\mathbb{N}$, denote by $X^{i,n}$ the solution to the controlled SDE

$$X_t^{i,n} = x_0 + \int_0^t b(s, X_s^{i,n})ds + \sigma W_t + \xi_t^{i,n}, \quad t\in[0,T].$$

Next, given that the processes $\xi^{i,n}$ have Lipschitz paths and are nondecreasing, we can find $\mathbb{F}$-adapted processes $u^{i,n}:\Omega\times[0,T]\to[0,\infty)$ such that

$$\xi_t^{i,n} = \int_0^t u_s^{i,n} ds, \quad t\in[0,T).$$

Observing that the processes $u^{i,n}$ can be regarded as regular controls, we wish to employ the results from Dianetti et al. [25] in order to construct $\rho^\wedge$, $\rho^\vee$. However, we need to take care of possible discontinuities at time T.

As in lemma 2.10 in Dianetti et al. [25], for each $n\in\mathbb{N}$, we find two $\mathbb{F}$-adapted $[0,\infty)$-valued processes $u^{\wedge,n}, u^{\vee,n}$ such that defining $\mathbb{P}$ a.s.,

$$\xi_t^{\wedge,n} := \int_0^t u_s^{\wedge,n} ds \quad \text{and} \quad \xi_t^{\vee,n} := \int_0^t u_s^{\vee,n} ds, \quad \text{for each } t\in[0,T), \tag{26}$$

we have for each $t\in[0,T)$, $\mathbb{P}$ a.s.,

$$\begin{aligned} X_t^{1,n}\wedge X_t^{2,n} &= x_0 + \int_0^t b(s, X_s^{1,n}\wedge X_s^{2,n})ds + \sigma W_t + \xi_t^{\wedge,n} \quad \text{and} \\ X_t^{1,n}\vee X_t^{2,n} &= x_0 + \int_0^t b(s, X_s^{1,n}\vee X_s^{2,n})ds + \sigma W_t + \xi_t^{\vee,n}. \end{aligned} \tag{27}$$

This suggests that we define the processes $\xi^{\wedge,n}$ and $\xi^{\vee,n}$ at time T by setting, $\mathbb{P}$ a.s.,

$$\begin{aligned} \xi_T^{\wedge,n} &:= X_T^{1,n}\wedge X_T^{2,n} - x_0 - \int_0^T b(s, X_s^{1,n}\wedge X_s^{2,n})ds - \sigma W_T \quad \text{and} \\ \xi_T^{\vee,n} &:= X_T^{1,n}\vee X_T^{2,n} - x_0 - \int_0^T b(s, X_s^{1,n}\vee X_s^{2,n})ds - \sigma W_T. \end{aligned}$$

We define

$$\begin{aligned} \rho^{\wedge,n} &:= (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi^{\wedge,n}), \\ \rho^{\vee,n} &:= (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi^{\vee,n}), \\ \rho^{i,n} &:= (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi^{i,n}), \quad \text{for } i=1,2, \end{aligned}$$

so that by virtue of (27) and the definition of $\xi_T^{\wedge,n}$ and $\xi_T^{\vee,n}$, we obtain, $\mathbb{P}$ a.s.,

$$X_t^{1,n}\wedge X_t^{2,n} = X_t^{\rho^{\wedge,n}} \quad \text{and} \quad X_t^{1,n}\vee X_t^{2,n} = X_t^{\rho^{\vee,n}}, \quad \text{for any} \quad t\in[0,T]. \tag{28}$$

Moreover, we observe that the processes $\xi^{\wedge,n}$ and $\xi^{\vee,n}$ are nondecreasing.

Step 2. In this step, we prove that

$$J(\rho^{\wedge,n},\mu) + J(\rho^{\vee,n},\mu) = J(\rho^{1,n},\mu) + J(\rho^{2,n},\mu). \tag{29}$$

This is again done by adapting arguments from Dianetti et al. [25], taking care of possible discontinuities of the processes $\xi^{i,n}$, $\xi^{\wedge,n}$, $\xi^{\vee,n}$ at time T.

For a generic admissible control $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi) \in E^w$, using integration by parts and the controlled SDE (21), we rewrite the cost functional as

$$\begin{aligned} J(\xi,\mu) &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t^\rho, \mu_t)dt + g(X_T^\rho, \mu_T) + c_T\xi_T - \int_0^T \xi_t c_t' dt\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T (f(t, X_t^\rho, \mu_t) - c_T b(t, X_t^\rho) - \xi_t c_t')dt + g(X_T^\rho, \mu_T) + c_T X_T^\rho\right] - c_T\mathbb{E}^{\mathbb{P}}[x_0] \\ &= G^1(\rho,\mu) - G^2(\rho,\mu) + H(\rho,\mu) - c_T\mathbb{E}^{\mathbb{P}}[x_0], \end{aligned} \tag{30}$$

where we have set

$$\begin{aligned} G^1(\rho,\mu) &:= \mathbb{E}^{\mathbb{P}}\left[\int_0^T (f(t, X_t^\rho, \mu_t) - c_T b(t, X_t^\rho))dt\right], \\ G^2(\rho,\mu) &:= \mathbb{E}^{\mathbb{P}}\left[\int_0^T \xi_t c_t' dt\right], \\ H(\rho,\mu) &:= \mathbb{E}^{\mathbb{P}}[g(X_T^\rho, \mu_T) + c_T X_T^\rho]. \end{aligned}$$

Observing that the functional G^1 depends on the control only on the interval $[0,T)$, thanks to the construction of $u^{\wedge,n}$, $u^{\vee,n}$ provided in Step 1, we can repeat the arguments in the proof of lemma 2.11 in Dianetti et al. [25] in order to come up with

$$G^1(\rho^{\wedge,n},\mu) + G^1(\rho^{\vee,n},\mu) = G^1(\rho^{1,n},\mu) + G^1(\rho^{2,n},\mu). \tag{31}$$

Moreover, from the definition of $u^{\wedge,n}$ and $u^{\vee,n}$ in Step 1, as in the proof of lemma 2.11 in Dianetti et al. [25], we see that

$$\xi_t^{\wedge,n} + \xi_t^{\vee,n} = \int_0^t (u_s^{\wedge,n} + u_s^{\vee,n})ds = \int_0^t (u_s^{1,n} + u_s^{2,n})ds = \xi_t^{1,n} + \xi_t^{2,n}, \quad \text{for each } t \in [0,T),$$

so that

$$G^2(\rho^{\wedge,n},\mu) + G^2(\rho^{\vee,n},\mu) = G^2(\rho^{1,n},\mu) + G^2(\rho^{2,n},\mu). \tag{32}$$

Finally, we easily find that

$$H(X_T^{\rho^{\wedge,n}},\mu) + H(X_T^{\rho^{\vee,n}},\mu) = H(X_T^{\rho^{1,n}},\mu) + H(X_T^{\rho^{2,n}},\mu). \tag{33}$$

Therefore, adding (31), (32), and (33) and using the representation in (30), we obtain (29).

Step 3. Set $X^i := X^{\rho_i}$, $i = 1, 2$, and define the right-continuous processes $\xi^\wedge$, $\xi^\vee$ by setting

$$\begin{aligned} \xi_t^\wedge &:= X_t^1 \wedge X_t^2 - x_0 - \int_0^t b(s, X_s^1 \wedge X_s^2)ds - \sigma W_t, \\ \xi_t^\vee &:= X_t^1 \vee X_t^2 - x_0 - \int_0^t b(s, X_s^1 \vee X_s^2)ds - \sigma W_t. \end{aligned} \tag{34}$$

The aim of this step is to prove that the controls $\rho^\wedge := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi^\wedge)$ and $\rho^\vee := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi^\vee)$ are admissible and that the control rules

$$\nu_1 \wedge^E \nu_2 := \nu^{\rho^\wedge} \quad \text{and} \quad \nu_1 \vee^E \nu_2 := \nu^{\rho^\vee}$$

satisfy the conditions in Assumption 2.

From (25), we immediately see that, $\mathbb{P}$ a.s.,

$$\begin{cases} \xi_t^{i,n} \to \xi_t^i \text{ as } n \to \infty \text{ for all continuity points } t \in [0,T) \quad \text{of} \quad \xi^i, \\ \xi_T^{i,n} \to \xi_T^i \text{ as } n \to \infty. \end{cases} \tag{35}$$

Therefore, using (35) and Grönwall's inequality, we deduce that, $\mathbb{P}$ a.s.,

$$\begin{cases} X_t^{i,n} \to X_t^i \text{ as } \quad n \to \infty \text{ for all continuity points } \quad t \in [0,T) \quad \text{of} \quad X^i, \\ X_T^{i,n} \to X_T^i \text{ as } \ n \to \infty. \end{cases} \tag{36}$$

This allows us to take limits in (28) in order to conclude that, $\mathbb{P}$ a.s., for π a.a. $t \in [0,T]$, we have

$$X_t^{1,n} \wedge X_t^{2,n} \to X_t^1 \wedge X_t^2, \quad X_t^{1,n} \vee X_t^{2,n} \to X_t^1 \vee X_t^2, \quad \xi_t^{\wedge,n} \to \xi_t^{\wedge}, \quad \xi_t^{\vee,n} \to \xi_t^{\vee}. \tag{37}$$

Given that the processes $\xi^{\wedge,n}$ and $\xi^{\vee,n}$ are nonnegative and nondecreasing and also, the limit processes $\xi^{\wedge}$ and $\xi^{\vee}$ are nonnegative and nondecreasing, hence $\rho^{\wedge}$ and $\rho^{\vee}$ are admissible. Moreover, by definition of $\xi^{\wedge}$ and $\xi^{\vee}$, we have

$$X_t^{\rho^{\wedge}} = X_t^1 \wedge X_t^2 \leqslant X_t^1 \vee X_t^2 \leqslant X_t^{\rho^{\vee}}, \mathbb{P} \text{ a.s.,} \quad \text{for each } \ t \in [0,T],$$

which proves that $\nu_1 \wedge^E \nu_2$ and $\nu_1 \vee^E \nu_2$ satisfy condition 1 in Assumption 2.

Step 4. We conclude by proving that $\nu_1 \wedge^E \nu_2$ and $\nu_1 \vee^E \nu_2$ satisfy condition 2 in Assumption 2. We begin by observing that for a generic constant $C > 0$, by Grönwall's inequality, we have

$$|X_t^{i,n}|^p \leqslant C\left(1 + |x_0|^p + \sigma^p \sup_{s \in [0,T]} |W_s|^p + |\xi_T^{i,n}|^p\right),$$

so that by definition of $\xi^{i,n}$, we obtain

$$\sup_n \sup_{t \in [0,T]} |X_t^{i,n}|^p \leqslant C\left(1 + |x_0|^p + \sigma^p \sup_{s \in [0,T]} |W_s|^p + |\xi_T^i|^p\right) \in \mathbb{L}^1(\Omega; \mathbb{P}), \tag{38}$$

where the integrability condition of the right-hand side follows from the fact that $\nu_1, \nu_2 \in E^M$. Therefore, thanks to the convergences in (35) and (36) and the estimate (38), the growth conditions on f and g allow us to employ the dominated convergence theorem in order to come up with

$$\begin{aligned} J(\rho_i, \mu) &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t^{\rho_i}, \mu_t) dt + g(X_T^{\rho_i}, \mu_T) + \int_{[0,T]} c_t d\xi_t^i\right] \\ &= \lim_n \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t^{\rho^{i,n}}, \mu_t) dt + g(X_T^{\rho^{i,n}}, \mu_T) + \int_{[0,T]} c_t d\xi_t^{i,n}\right] = \lim_n J(\rho^{i,n}, \mu). \end{aligned} \tag{39}$$

Now, an integration by parts together with the limit behavior in (37) and Fatou's lemma yields the estimate

$$\begin{aligned} J(\rho^{\wedge}, \mu) &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t^{\rho^{\wedge}}, \mu_t) dt + g(X_T^{\rho^{\wedge}}, \mu_T) + c_T \xi_T^{\wedge} - \int_0^T \xi_t^{\wedge} c_t' dt\right] \\ &\leqslant \liminf_n \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t^{\rho^{\wedge,n}}, \mu_t) dt + g(X_T^{\rho^{\wedge,n}}, \mu_T) + c_T \xi_T^{\wedge,n} - \int_0^T \xi_t^{\wedge,n} c_t' dt\right] \\ &= \liminf_n J(\rho^{\wedge,n}, \mu). \end{aligned} \tag{40}$$

Similarly, it follows that

$$J(\rho^{\vee}, \mu) \leqslant \liminf_n J(\rho^{\vee,n}, \mu). \tag{41}$$

Finally, exploiting (39), (40), and (41), we can take limits in (29) in order to obtain condition 2 in Assumption 2. □

Thanks to Lemma 3 and Remark 3, we see that Assumptions 1 and 2 are satisfied. As a consequence of Theorem 1, we have the following result.

Theorem 4. *The set of mean field game equilibria* $\mathcal{M}$ *is nonempty with* $\inf \mathcal{M} \in \mathcal{M}$ *and* $\sup \mathcal{M} \in \mathcal{M}$. *Moreover, if* f *and* g *are continuous in* (x, μ), *then*

1. *the learning procedure* $\underline{\mu}^n$ *defined inductively by* $\underline{\mu}^0 = \inf L$ *and* $\underline{\mu}^{n+1} = \inf R(\underline{\mu}^n)$ *is nondecreasing in L and converges to the minimal MFG solution and*
2. *the learning procedure* $\bar{\mu}^n$ *defined inductively by* $\bar{\mu}^0 = \sup L$ *and* $\bar{\mu}^{n+1} = \sup R(\bar{\mu}^n)$ *is nonincreasing in L and converges to the maximal MFG solution.*

5.3. Remarks and Extensions

The previous arguments can be easily adapted in order to cover many classical settings, which typically arise in the literature on stochastic singular control, such as, for example, MFGs where the optimization problem concerns an infinite time horizon discounted criterion or involves controls of bounded variation rather than just monotone. A similar setting has been, for example, considered in Guo and Xu [35]. In the following, we illustrate a few specific settings of interest.

Remark 4 (Controlled Ornstein–Uhlenbeck Process and Common Noise). We underline that the results of Section 5.1 can also be obtained if the underlying dynamics are given by a controlled Ornstein–Uhlenbeck process: that is, if the state process evolves according to

$$dX_t^{\xi} = \theta(\lambda - X_t^{\xi})dt + \sigma dW_t + \sigma^o dB_t + d\xi_t,\ t \in [0,T], \quad X_{0-}^{\xi} = x_0,$$

with $\kappa,\ \lambda \in \mathbb{R},\ \sigma,\ \sigma^o \geqslant 0$. In this case, the state process can be explicitly written as

$$X_t^{\xi} = e^{-\theta t}\left(x_0 + \lambda(e^{\theta t} - 1) + \int_0^t e^{\theta s}(\sigma dW_s + \sigma^o dB_s) + \int_{[0,t]} e^{\theta s} d\xi_s\right),$$

and for $\xi,\ \bar{\xi} \in \mathcal{V}_{\uparrow}$, we have $X^{\xi} \wedge X^{\bar{\xi}} = X^{\xi^{\wedge}}$ and $X^{\xi} \vee X^{\bar{\xi}} = X^{\xi^{\vee}}$ by setting

$$\xi_t^{\wedge} := \int_{[0,t]} e^{-\theta s} d(\zeta \wedge \bar{\zeta})_s, \quad \xi_t^{\vee} := \int_{[0,t]} e^{-\theta s} d(\zeta \vee \bar{\zeta})_s, \quad \zeta := \int_{[0,t]} e^{\theta s} d\xi_s, \quad \bar{\zeta} := \int_{[0,t]} e^{\theta s} d\bar{\xi}_s.$$

Therefore, one can introduce, as in (12), operations that satisfy all the requirements from Assumption 2.

Remark 5 (Mean Field-Dependent Dynamics and Relation to Campi et al. [14]). The theory from Section 5.1 also allows us to cover problems, where the drift of the underlying state process depends in an increasing way (w.r.t. first-order stochastic dominance) on the mean field, in such a way that (8) holds true. This could be, for example, achieved if b in (7) is replaced by a bounded increasing function of $(\mu_t)_{t \in [0,T]}$.

Another example is given by the two-dimensional MFG of finite fuel capacity expansion considered in Campi et al. [14]. Therein, the mean of a uniformly bounded purely controlled process affects in a nondecreasing way the drift of an uncontrolled Itô diffusion, and there is no mean field dependence in the profit functional. We refer to remark 3.15 in Campi et al. [14] for additional details on how the existence of a mean field equilibrium for the problem considered in that paper can be indeed achieved via our lattice-theoretic techniques.

6. Submodular Mean Field Games with Reflecting Boundary Conditions

In this section, we consider an MFG model with reflecting boundary conditions, in which the state process of the representative player is forced to remain in a certain interval of the state space. These types of models were recently introduced in Bayraktar et al. [6] (see also Bayraktar et al. [5]), motivated by applications to queueing systems consisting of many strategic servers that are weakly interacting. Also, a particular setting in the same class of models is studied in Graber and Mouzouni [34], motivated by a model for the production of exhaustible resources. Here, we consider a version of the model in Bayraktar et al. [6] with submodular cost, which we solve through the results of Section 2.

6.1. Formulation of the Model

Fix $M > 0$ and $x_0 \in [0,M]$. Consider the set L_M of all measurable functions $\mu : [0,T] \to \mathcal{P}([0,M])$ with $\mu_0 = \delta_{x_0}$, endowed with the lattice structure coming from the order relation $\leqslant^L$ of $\pi := \delta_0 + dt + \delta_T$-pointwise first-order stochastic dominance. As in the previous section, this leads to a complete lattice $(L_M, \leqslant^L)$.

Next, we introduce the minimization problem. For technical reasons (i.e., in order to gain compactness of the set of controls), we do so by using relaxed controls, although we work with assumptions under which strict optimal controls always exist. For a compact control set $A \subset \mathbb{R}$ and a Lipschitz continuous function $b : [0,T] \times \mathbb{R} \to \mathbb{R}$, we define the set of *admissible relaxed controls* as the set E^w of tuples $\rho := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \lambda, v, X)$ such that

1. $W = (W_t)_{t \in [0,T]}$ is a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, satisfying the usual conditions;

2. λ is a $\mathcal{P}(A)$-valued progressively measurable process; and
3. the couple (v, X) is a solution to the controlled reflected SDE in the domain $(0,M)$:

$$\begin{cases} dX_t = (b(t,X_t) + \int_A a\lambda_t(da))dt + \sigma dW_t + dv_t,\ t \in [0,T], \quad X_0 = x_0, \\ X_t \in [0,M], \quad \int_0^t \mathbb{1}_{\{X_s \in (0,M)\}} d|v|_s = 0,\ \text{ for any } t \in [0,T],\ \mathbb{P} \text{ a.s.,} \end{cases} \tag{42}$$

where $|v|$ denotes the total variation of v. Moreover, we define the set of *admissible strict controls* $E^{w,s}$ as the set of elements $\rho := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \lambda, v, X) \in E^w$ such that $\lambda_t = \delta_{\alpha_t}$ $\mathbb{P} \otimes dt$ a.e. in $\Omega \times [0,T]$, for some A-valued progressively measurable process α.

We consider functions f, g, and c as in the beginning of Section 5 satisfying Assumption 5 and a lower semicontinuous function $l : [0,T] \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$, which is convex in a. For $\mu \in L_M$ and $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \lambda, v, X) \in E^w$, we define the cost functional

$$J(\rho,\mu) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T (f(t,X_t,\mu_t) + \int_A l(t,X_t,a)\lambda_t(da))dt + g(X_T,\mu_T) + \int_0^T c_t d|v|_t\right].$$

We say that $\rho \in E^w$ is an *optimal singular control* for the flow of measures μ if $J(\rho,\mu) = \inf_{E^w} J(\cdot,\mu)$.

We are interested in the following notion of equilibrium.

Definition 5. A flow of probabilities $\mu \in L_M$ is an MFG equilibrium if
1. there exists a strict optimal control $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \lambda, v, X) \in E^{w,s}$ for μ and
2. $\mu_t = \mathbb{P} \circ (X_t)^{-1}$ for any $t \in [0,T]$.

6.2. Reformulation via Control Rules and Preliminary Results

In order to have a topology on the space of admissible controls, we reformulate the problem in terms of control rules.

Introduce the canonical space $(\Omega, \mathcal{F})$, where

$$\Omega := \mathcal{C} \times \Lambda \times \mathcal{V} \times \mathcal{D}, \quad \mathcal{F} := \mathcal{B}(\mathcal{C}) \otimes \mathcal{B}(\Lambda) \otimes \mathcal{B}(\mathcal{V}) \otimes \mathcal{B}(\mathcal{D}).$$

Define the set of relaxed control rules

$$E := \{\nu^\rho \,|\, \rho \in E^w\} \quad \text{with} \quad \nu^\rho := \mathbb{P} \circ (W, \lambda, v, X)^{-1},\ \text{for } \rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \lambda, v, X) \in E^w,$$

and with a slight abuse of notation, we set $J(\nu^\rho, \mu) := J(\rho,\mu)$. The set of strict control rules is defined as $E^s := \{\nu^\rho \,|\, \rho \in E^{w,s}\}$. In this way, E is naturally defined as a subspace of the topological space $\mathbb{P}(\Omega)$. For any $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \lambda, v, X) \in E^w$, the controlled SDE (42) together with $X_t \in [0,M]$ implies the estimate

$$\mathbb{E}^{\mathbb{P}}[|v|_T^p] \leqslant K < \infty \tag{43}$$

with a constant $K > 0$. Moreover, because A is compact, so are Λ and $\mathcal{P}(\Lambda)$. This, together with (43), allows to use the Meyer–Zheng tightness criteria (see theorem 4 on p. 360 in Meyer and Zheng [47]) to show that the set E is a relatively compact subset of $\mathcal{P}(\Omega)$. Moreover, the projection map

$$p : E \to L_M \quad \text{with} \quad p(\nu^\rho) := \mathbb{P} \circ (X^\rho)^{-1},\ \text{for } \rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \lambda, v, X) \in E^w,$$

satisfies the conditions in Assumptions 1 and 3.

Lemma 4.
1. *For any $\mu \in L_M$, the set* $\arg\min_E J(\cdot,\mu)$ *is nonempty.*
2. *If $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, X, \lambda, v) \in E^w$, there exists a control $\hat{\rho} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, X, \hat{\lambda}, v) \in E^{w,s}$ such that $J(\hat{\rho},\mu) \leqslant J(\rho,\mu)$, for any $\mu \in L_M$.*

Proof. We begin by proving claim (1). In order to do so, take a minimizing sequence $(\nu_n)_n \subset E$ (i.e., $\lim_n J(\nu_n,\mu) = \inf_E J(\cdot,\mu)$) and controls $\rho_n = (\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbb{P}^n, W^n, \lambda^n, v^n, X^n) \in E^w$ with $\nu_n = \nu^{\rho_n}$. Because the set $E \subset \mathcal{P}(\Omega)$ is relatively compact, we can find a limit point $\nu_* \in \mathcal{P}(\mathcal{C} \times \Lambda \times \mathcal{V} \times \mathcal{D})$ and a subsequence (not relabeled) such that $\nu_n \to \nu_*$ weakly. Up to using a Skorokhod representation theorem for separable spaces (see theorem 3 in Dudley [27]), we can assume that there exists a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which the processes $(W^n, \lambda^n, v^n, X^n)$ are defined together with a process (W,λ,v,X), such that

$$(W^n,\lambda^n,v^n,X^n)\to(W,\lambda,v,X),\ \mathbb{P}\ \text{a.s., in}\ \mathcal{C}\times\Lambda\times\mathcal{V}\times\mathcal{D}\quad\text{as}\quad n\to\infty,$$
$$\nu_n=\mathbb{P}\circ(W^n,\lambda^n,v^n,X^n)^{-1}\ \text{and}\ \mathbb{P}\circ(W,\lambda,\mathrm{v},\mathrm{X})^{-1}=\nu_*. \tag{44}$$

Also, this convergence allows to show that X is a solution to the SDE $X_t=x_0+\int_0^t(b(s,X_s)+\int_A a\lambda_s(da))ds+\sigma W_t+v_t$, $t\in[0,T]$, $\mathbb{P}$ a.s. Moreover, by the Lipschitz continuity of the Skorokhod map (see lemma 2.1 in Bayraktar et al. [6]), we see that the couple $(v,\ X)$ solves the controlled reflected SDE (42). Therefore, defining $\rho_*=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,\lambda,v,X)$ with $\mathbb{F}$ being the (extended) filtration generated by (W,λ,v,X), we have that $\rho_*\in E^w$ and $\nu_*:=\nu^{\rho_*}$. Moreover, using the convergence in (44) and exploiting the lower semicontinuity of the costs f, g, l and the fact that c is nondecreasing, by Fatou's lemma we obtain that

$$J(\nu_*,\mu)\leqslant\liminf_n J(\nu_n,\mu)=\inf_E J(\cdot,\mu),$$

which completes the proof of claim (1).

We conclude by proving claim (2). Take $\rho=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,X,\lambda,v)\in E^w$, set $\alpha_t:=\int_A a\lambda_t(da)$, $\hat\lambda:=\delta_{\alpha_t}(da)dt$, and consider the control $\hat\rho=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,X,\hat\lambda,v)\in E^{w,s}$. First of all, we see that, $\mathbb{P}$ a.s., X solves the equation

$$X_t=x_0+\int_0^t(b(s,X_s)+\alpha_s)ds+\sigma W_t+v_t,\quad t\in[0,T].$$

Finally, by convexity of l, we can use Jensen's inequality obtaining

$$\begin{aligned}J(\hat\rho,\mu)&=\mathbb{E}^{\mathbb{P}}\Big[\int_0^T(f(t,X_t,\mu_t)+l(t,X_t,\alpha_t))dt+g(X_T,\mu_T)+\int_0^T c_t d|v|_t\Big]\\&\leqslant\mathbb{E}^{\mathbb{P}}\Big[\int_0^T(f(t,X_t,\mu_t)+\int_A l(t,X_t,a)\lambda_t(da))dt+g(X_T,\mu_T)+\int_0^T c_t d|v|_t\Big]\\&=J(\rho,\mu),\end{aligned}$$

which completes the proof of the lemma. □

6.3. Existence and Approximation of Equilibria

We begin by observing that a *relaxed MFG equilibrium* can now be seen as a fixed point of the best-response map

$$R:L_M\to L_M\quad\text{with}\quad R(\mu):=p\Big(\arg\min_E J(\cdot,\mu)\Big),\ \text{for}\ \mu\in L_M. \tag{45}$$

We move on by constructing operations $\wedge^E,\vee^E:E\times E\to E$ satisfying Assumption 2. For $\nu=\nu^\rho,\bar\nu=\nu^{\bar\rho}\in E$ with $\rho=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,\lambda,v,X)$, $\bar\rho=(\bar\Omega,\bar{\mathcal{F}},\bar{\mathbb{F}},\bar{\mathbb{P}},\bar W,\bar\lambda,\bar v,\bar X)\in E^w$, we can, without loss of generality (see, e.g., the proof of lemma 3.4 in Dianetti et al. [25]), assume these controls to be defined on the same stochastic basis: that is, $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W)=(\bar\Omega,\bar{\mathcal{F}},\bar{\mathbb{F}},\bar{\mathbb{P}},\bar W)$. Hence, define

$$\rho^\wedge:=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,\alpha^\wedge,v^\wedge,X\wedge\bar X)\quad\text{and}\quad\rho^\vee:=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,\alpha^\vee,v^\vee,X\vee\bar X),$$

where

$$\lambda_t^\lambda:=\lambda_t\,\mathbb{1}_{\{X_t\leqslant\bar X_t\}}+\bar\lambda_t\,\mathbb{1}_{\{X_t>\bar X_t\}},\quad v_t^\wedge:=\int_0^t\Big(\mathbb{1}_{\{X_s<\bar X_s\}}dv_s+\mathbb{1}_{\{X_s\geqslant\bar X_s\}}d\bar v_s\Big),$$
$$\lambda_t^\vee:=\bar\lambda_t\,\mathbb{1}_{\{X_t\leqslant\bar X_t\}}+\lambda_t\,\mathbb{1}_{\{X_t>\bar X_t\}},\quad v_t^\vee:=\int_0^t\Big(\mathbb{1}_{\{X_s<\bar X_s\}}d\bar v_s+\mathbb{1}_{\{X_s\geqslant\bar X_s\}}dv_s\Big).$$

Indeed, by the Meyer–Itô formula for continuous semimartingales (see, e.g., theorem 68 on p. 213 in the book by Protter [52]), we find

$$\begin{aligned}
X_t \wedge \bar{X}_t &= X_t + 0 \wedge (\bar{X}_t - X_t) \\
&= X_t + \int_0^t \mathbb{1}_{\{\bar{X}_s - X_s \leqslant 0\}} d(\bar{X} - X)_s - \frac{1}{2} L_t^0(\bar{X} - X) \\
&= x_0 + \sigma W_t \\
&\quad + \int_0^t \Big(\mathbb{1}_{\{X_s < \bar{X}_s\}} (b(s, X_s) + \int_A a \lambda_s(da)) + \mathbb{1}_{\{X_s \geqslant \bar{X}_s\}} \Big(b(s, \bar{X}_s) + \int_A a \bar{\lambda}_s(da) \Big) \Big) ds \\
&\quad + \int_0^t \Big(\mathbb{1}_{\{X_s < \bar{X}_s\}} dv_s + \mathbb{1}_{\{X_s \geqslant \bar{X}_s\}} d\bar{v}_s \Big) - \frac{1}{2} L_t^0(\bar{X} - X),
\end{aligned}$$

where $L_t^0(\bar{X} - X)$ is the local time of $\bar{X} - X$ at zero (see, e.g., chapter 4 in Protter [52]). We denote by $[\bar{X} - X, \bar{X} - X]$ the quadratic variation of the process $\bar{X} - X$ (see, e.g., p. 66 in Protter [52]). Because $\bar{X} - X$ is a process of bounded variation, we have $[\bar{X} - X, \bar{X} - X] = 0$. Therefore, using the characterization of local times (see, e.g., corollary 3 on p. 225 in Protter [52]), we obtain that

$$L_t^0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leqslant \bar{X}_s - X_s \leqslant \varepsilon\}} d[\bar{X} - X, \bar{X} - X]_s = 0,$$

and we conclude that

$$X_t \wedge \bar{X}_t = x_0 + \int_0^t (b(s, X_s \wedge \bar{X}_s) + \int_A a \lambda_s^\wedge (da)) ds + \sigma W_t + v_t^\wedge.$$

In the same way, the process $X \vee \bar{X}$ solves the SDE controlled by $\lambda^\vee$ with reflection $v^\vee$. Finally, $X_t \wedge \bar{X}_t$, $X_t \vee \bar{X}_t \in [0, M]$, and it can be easily verified that the support of the random measures $|v^\wedge|$ or $|v^\vee|$ is contained in the set of times at which $X_t \wedge \bar{X}_t \in \{0, M\}$ or $X_t \vee \bar{X}_t \in \{0, M\}$, respectively. This proves that $\rho^\wedge$, $\rho^\vee \in E$, so that defining $\nu \wedge^E \bar{\nu} := \nu^{\rho^\wedge}, \nu \vee^E \bar{\nu} := \nu^{\rho^\vee}$, we have $\nu \wedge^E \bar{\nu}$, $\nu \vee^E \bar{\nu} \in E$.

Moreover, one readily verifies that $|v^\wedge|_t + |v^\vee|_t \leqslant |v|_t + |\bar{v}|_t$. This together with the fact that $c' \leqslant 0$, in turn, yields the estimate

$$J(\nu \vee^E \bar{\nu}, \bar{\mu}) - J(\bar{\nu}, \bar{\mu}) \leqslant J(\nu \vee^E \bar{\nu}, \mu) - J(\bar{\nu}, \mu) \leqslant J(\nu, \mu) - J(\nu \wedge^E \bar{\nu}, \mu).$$

Hence, Assumption 2 is satisfied.

We can now state the main result of this section.

Theorem 5. *The set of mean field game equilibria $\mathcal{M}$ is a nonempty with* $\inf \mathcal{M} \in \mathcal{M}$ *and* $\sup \mathcal{M} \in \mathcal{M}$. *Moreover, if f and g are continuous in* (x, μ), *then*

1. *the learning procedure $\underline{\mu}^n$ defined inductively by $\underline{\mu}^0 = \inf L$ and $\underline{\mu}^{n+1} = \inf R(\underline{\mu}^n)$ is nondecreasing in L and converges to the minimal MFG solution, and*

2. *the learning procedure $\bar{\mu}^n$ defined inductively by $\bar{\mu}^0 = \sup L$ and $\bar{\mu}^{n+1} = \sup R(\bar{\mu}^n)$ is nonincreasing in L and converges to the maximal MFG solution.*

Proof. For relaxed MFG equilibria as in (45), the result follows from the general Theorem 1. Thanks to Lemma 4, this allows us to obtain the result for MFG equilibria as in Definition 5. □

7. Supermodular Mean Field Games with Optimal Stopping

In this section, we adapt the general results of Section 2 to an MFG, where the representative agent faces an optimal stopping maximization problem. In particular, we introduce and solve a version of the model discussed in Bouveret et al. [13], to which we add a common noise (see Example 3 for details). Our formulation also includes a particular case of the model studied in Carmona et al. [21] (see Example 4).

7.1. Formulation of the Model

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, satisfying the usual conditions. For $0 < T < \infty$, let $\mathcal{T}$ denote the set of $\mathbb{F}$-stopping times satisfying $\tau \leqslant T$ $\mathbb{P}$ a.s. Let $Z = (Z)_{t \in [0,T]}$ and $B = (B)_{t \in [0,T]}$ be progressively measurable stochastic processes, taking values in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ for $d_1, d_2 \in \mathbb{N}$, respectively. Set $X := (Z, B)$ and $d := d_1 + d_2$. Assume that the process B has independent increments, and denote by $\mathbb{F}^B$ the right-continuous extension of the filtration generated by B, augmented by the $\mathbb{P}$-null sets. The process B represents a common noise, and it can also be deterministic (in this case, $\mathbb{F}^B$ is the trivial filtration). For any $t \in [0,T]$, denote by $\mathcal{F}^B_{t,T}$ the σ-field generated by the family of increments $\{B^i_{s_2} - B^i_{s_1} \,|\, t \leqslant s_1 < s_2 \leqslant T,\ i = 1, \dots, d_2\}$. We assume that for any $t \in [0,T]$, the σ-fields $\mathcal{F}_t$ and $\mathcal{F}^B_{t,T}$ are independent.

Denote by $L^{\text{pr.}}(\Omega \times [0,T]; \mathcal{M}_{\leqslant 1}(\mathbb{R}))$ the set of all processes taking values in the set of subprobability measures $\mathcal{M}_{\leqslant 1}(\mathbb{R})$, which are $\mathbb{F}^B$-progressively measurable. Consider two measurable functions

$$f : [0,T] \times \mathbb{R}^d \times \mathcal{M}_{\leqslant 1}(\mathbb{R}) \to \mathbb{R} \quad \text{and} \quad g : [0,T] \times \mathbb{R}^d \to \mathbb{R}.$$

Next, for $m \in L^{\text{pr.}}(\Omega \times [0,T]; \mathcal{M}_{\leqslant 1}(\mathbb{R}))$, we define the profit functional

$$J(\tau, m) := \mathbb{E}\left[\int_0^\tau f(t, X_t, m_t)dt + g(\tau, X_\tau)\right], \quad \tau \in \mathcal{T}, \tag{46}$$

and consider the optimal stopping problem, parametrized by m, which consists of maximizing the profit functional $J(\cdot, m)$. For a process m, we say that the stopping time τ^m is optimal for m if $\tau^m \in \arg\max_{\mathcal{T}} J(\cdot, m)$.

We next consider a continuous function $\psi : \mathbb{R}^d \to \mathbb{R}$ and the following notion of solution.

Definition 6. A process $m \in L^{\text{pr.}}(\Omega \times [0,T]; \mathcal{M}_{\leqslant 1}(\mathbb{R}))$ is an MFG equilibrium if

$$m_t(A) = \mathbb{P}[\psi(X_t) \in A, t < \tau^m \,|\, \mathcal{F}^B_t], \quad \text{for all A} \in \mathcal{B}(\mathbb{R}),\ t \in [0,T],\ \mathbb{P} \text{ a.s.},$$

for some $\tau^m \in \arg\max_{\mathcal{T}} J(\cdot, m)$.

The function ψ allows us to embed in one single model different formulations of the MFG problem with optimal stopping; we refer to Examples 3 and 4 for further details.

7.2. Reformulation and Preliminary Results

In order to prove the existence and approximation of the equilibria of the MFG, we embed the problem in terms of the general formulation of Section 2.

Consider the set $E := \mathcal{T}$, endowed with the lattice structure $\wedge, \vee$ arising from the order relation $\leqslant$ given by the $\mathbb{P}$ a.s. pointwise order ($\tau \leqslant \bar{\tau}$ if and only if $\tau \leqslant \bar{\tau}$ $\mathbb{P}$ a.s.). The lattice E is complete, so that it is compact in the interval topology; see the appendix. The lattice structure on E allows us to directly use some of the results in Vives [59] (see, in particular, Remark 7).

For $\tau \in E$, we define the $\mathcal{M}_{\leqslant 1}(\mathbb{R})$-valued process $p\tau$ by setting, $\mathbb{P}$ a.s.,

$$(p\tau)_t(A) := \mathbb{P}[\psi(X_t) \in A, t < \tau \,|\, \mathcal{F}^B_t], \quad \text{for all } A \in \mathcal{B}(\mathbb{R}) \text{ and } \quad t \in [0,T]. \tag{47}$$

Note that $(p\tau)_t(y, \infty) \leqslant \mathbb{P}[\psi(X_t) > y \,|\, \mathcal{F}^B_t] =: \mu^\psi_t(y, \infty)$ $\mathbb{P}$ a.s., for $y \in \mathbb{R}$, so that

$$(p\tau)_t \leqslant^{\text{st}} \mu^\psi_t, \mathbb{P} \text{ a.s.}, \quad \text{for each } t \in [0,T]. \tag{48}$$

For m, $\bar{m} \in L^{\text{pr.}}(\Omega \times [0,T]; \mathcal{M}_{\leqslant 1}(\mathbb{R}))$, we define the order relation

$$m \leqslant^L \bar{m} \Leftrightarrow m_t \leqslant^{\text{st}} \bar{m}_t \, \mathbb{P} \text{ a.s.}, \quad \text{for} \quad dt \text{ a.a. } t \in [0,T],$$

and introduce the set of feasible distributions as

$$L := \{m \in L^{\text{pr.}}(\Omega \times [0,T]; \mathcal{M}_{\leqslant 1}(\mathbb{R})) \,|\, m \leqslant^L \mu^\psi\},$$

endowed with the order relation $\leqslant^L$. Thanks to the results in Section 3, the lattice $(L, \leqslant^L)$ is complete and Dedekind super complete (see, in particular, Example 1).

Observe that, from the definition of p, we have the following monotonicity properties:

$$p(\tau \wedge \bar{\tau}) \leqslant^L p\tau \wedge^L p\bar{\tau} \leqslant^L p\tau \vee^L p\bar{\tau} \leqslant^L p(\tau \vee \bar{\tau}), \quad \text{for each } \tau, \bar{\tau} \in \text{E}. \tag{49}$$

The following assumption will ensure that the projection p takes values in L and will give the necessary integrability of the payoffs in order to gain the continuity of the functional J.

Assumption 7.

1. *The processes Z and B are continuous.*
2. *The functions f, g are nonnegative, g is continuous, and*

$$\mathbb{E}\left[\sup_{t\in[0,T]}(f(t,X_t,\mu_t^{\psi})+g(t,X_t))\right]<\infty.$$

Lemma 5. *The map $p:E\longrightarrow L$ as in (47) is well defined (i.e., $p\tau\in L$ for any $\tau\in E$).*

Proof. Take $\tau\in E$. In light of (48), we only need to prove that the process $p\tau$ is $\mathbb{F}^B$-progressively measurable.

We first show that for each $\tau\in\mathcal{T}$ and $t\in[0,T]$, $\mathbb{P}$ a.s., we have

$$(p\tau)_t(A)=\mathbb{P}[\psi(X_t)\in A,t<\tau\,|\,\mathcal{F}_t^B]=\mathbb{P}[\psi(X_t)\in A,t<\tau\,|\,\mathcal{F}_T^B],\quad \text{for all } A\in\mathcal{B}(\mathbb{R}). \tag{50}$$

This can be shown similarly to remark 1 in Tchuendom [56]. Indeed, for any $t\in[0,T]$, the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{t,T}^B$ are independent, so that the random variables $Y_t^A:=\mathbb{1}_{\{\psi(X_t)\in A\}}\,\mathbb{1}_{\{t<\tau\}}$, $A\in\mathcal{B}(\mathbb{R})$, are independent from $\mathcal{F}_{t,T}^B$. Also, by assumption, the σ-fields $\mathcal{F}_t^B$ and $\mathcal{F}_{t,T}^B$ are independent. It thus follows that for any $A\in\mathcal{B}(\mathbb{R})$, one has

$$\mathbb{P}[\psi(X_t)\in A,t<\tau\,|\,\mathcal{F}_T^B]=\mathbb{E}[Y_t^A\,|\,\mathcal{F}_t^B\vee\mathcal{F}_{t,T}^B]=\mathbb{E}[Y_t^A\,|\,\mathcal{F}_t^B]=\mathbb{P}[\psi(X_t)\in A,t<\tau\,|\,\mathcal{F}_t^B],\ \mathbb{P}\ \text{ a.s.},$$

which proves (50).

We can now prove that the process $p\tau$ is right-continuous $\mathbb{P}$ a.s. Indeed, for $\phi\in\mathcal{C}_b(\mathbb{R})$, $t\in[0,T]$, and a sequence $(s_n)_n\subset[0,T]$ converging to t with $s_n\geqslant t$, we have

$$\begin{aligned}\lim_n\int_{\mathbb{R}}\phi(y)(p\tau)_{s_n}(dy)&=\lim_n\mathbb{E}[\phi(\psi(X_{s_n}))\,\mathbb{1}_{\{s_n<\tau\}}\,|\,\mathcal{F}_T^B]\\&=\mathbb{E}[\phi(\psi(X_t))\,\mathbb{1}_{\{t<\tau\}}\,|\,\mathcal{F}_T^B]\\&=\int_{\mathbb{R}}\phi(y)(p\tau)_t(dy),\quad \mathbb{P}\ \text{ a.s.},\end{aligned}$$

where the convergence follows by the dominated convergence theorem for conditional expectations and using the right continuity of $(\phi(\psi(X_s))\,\mathbb{1}_{\{s<\tau\}})_{s\in[0,T]}$ deriving from Assumption 7. Therefore, $(p\tau)_{s_n}$ weakly converges to $(p\tau)_t$, $\mathbb{P}$ a.s., as $n\longrightarrow\infty$, proving the right continuity of $p\tau$.

Finally, because the process $p\tau$ is $\mathbb{F}^B$ adapted and right continuous, it is $\mathbb{F}^B$-progressively measurable, completing the proof of the lemma. □

7.3. Existence and Approximation of Equilibria

We enforce the following structural condition.

Assumption 8. *For each $(t,x)\in[0,T]\times\mathbb{R}^d$, the function $f(t,x,\cdot)$ is increasing (i.e., $f(t,x,m)\leqslant f(t,x,\bar m)$ for any $m,\bar m\in\mathcal{M}_{\leqslant 1}(\mathbb{R})$ with $m\leqslant^{\text{st}}\bar m$).*

From Assumption 8, for $m,\bar m\in L$ with $m\leqslant^L\bar m$ and $\tau,\bar\tau\in E$, we have

$$\begin{aligned}J(\bar\tau,\bar m)-J(\bar\tau\wedge\tau,\bar m)&=\mathbb{E}\left[\int_{\bar\tau\wedge\tau}^{\bar\tau}f(t,X_t,\bar m_t)dt+g(\bar\tau,X_{\bar\tau})-g(\bar\tau\wedge\tau,X_{\bar\tau\wedge\tau})\right]\\&\geqslant\mathbb{E}\left[\int_{\bar\tau\wedge\tau}^{\bar\tau}f(t,X_t,m_t)dt+g(\bar\tau,X_{\bar\tau})-g(\bar\tau\wedge\tau,X_{\bar\tau\wedge\tau})\right]\\&=\mathbb{E}\left[\int_{\tau}^{\tau\vee\bar\tau}f(t,X_t,m_t)dt+g(\tau\vee\bar\tau,X_{\tau\vee\bar\tau})-g(\tau,X_\tau)\right]\\&=J(\tau\vee\bar\tau,m)-J(\tau,m),\end{aligned}$$

which reads as

$$J(\bar\tau,\bar m)-J(\bar\tau\wedge\tau,\bar m)\geqslant J(\bar\tau,m)-J(\bar\tau\wedge\tau,m)=J(\tau\vee\bar\tau,m)-J(\tau,m). \tag{51}$$

Remark 6. It is worth observing that the first inequality in (51) corresponds to the fact that the functional $J: E\times L \to \mathbb{R}$ has increasing differences, whereas the second equality in (51) implies that the functionals $J(\cdot, m): L\to\mathbb{R}$, $m\in L$, are supermodular. In this case, the game is said to be supermodular, and we refer to Vives [59] for further details.

We consider the best-response maps

$$\hat{R}(m) := \arg\min_E J(\cdot, m) \subset E, \quad R(m) := p(\hat{R}(m)) \subset L, \quad m \in L. \tag{52}$$

Combining Assumptions 7 and 8 together with the definition of L, we obtain that for any $m\in L$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} (f(t,X_t,m_t)+g(t,X_t))\right] \leqslant \mathbb{E}\left[\sup_{t\in[0,T]} (f(t,X_t,\mu_t^{\psi})+g(t,X_t))\right] < \infty. \tag{53}$$

This estimate, together with Assumption 7, allows us to show that the functionals $J(\cdot,m): L\to\mathbb{R}$, $m\in L$, are continuous in the interval topology on E. Therefore, arguing as in lemma 3.1 in Vives [59], for any $m\in L$ the set $\hat{R}(m)$ is nonempty so that thanks to Lemma 5, the best-reply map $R: L\to 2^L$ is well defined. Moreover, $m\in L$ is an MFG equilibrium if and only if $m\in R(m)$.

Remark 7. We observe that even if condition 2 in Assumption 1 is not satisfied, the same conclusions as in Lemma 1 can be deduced as follows. Thanks to the lattice structure on E and to the supermodularity property in (51), we can employ lemma 3.1 in Vives [59] in order to obtain the following.

1. The set $\hat{R}(m)$ is a lattice (i.e., for every $\tau_1,\tau_2\in\hat{R}(m)$, one has $\tau_1\wedge\tau_2$, $\tau_1\vee\tau_2\in\hat{R}(m)$).
2. For all $m,\bar{m}\in L$ with $m\leqslant\bar{m}$, $\inf_E\hat{R}(m)\leqslant\inf_E\hat{R}(\bar{m})$ and $\sup_E\hat{R}(m)\leqslant\sup_E\hat{R}(\bar{m})$.
3. For every $m\in L$, $\inf_E\hat{R}(m)\in\hat{R}(m)$ and $\sup_E\hat{R}(m)\in\hat{R}(m)$.

Therefore, because of the monotonicity of the projection p (see (49)), for any $m\in L$, we have

$$\inf R(m) = p\left(\inf_E \hat{R}(m)\right)\in R(m) \quad \text{and} \quad \sup R(m) = p\left(\sup_E \hat{R}(m)\right)\in R(m), \tag{54}$$

so that the assertions of Lemma 1 hold.

Now, we provide the main result of this section. We underline that some of the conditions in Assumption 3 are not satisfied (in particular, the continuity-like property of p for monotone sequences of stopping times is not satisfied). As a consequence, we obtain a result that is less general than that of Theorem 1 (see Remark 8), and some of the arguments in the proof of that theorem need to be adapted in order to prove existence and approximation of MFG sulutions.

Theorem 6. *The set of MFG equilibria M is nonempty with $\inf M\in M$ and $\sup M\in M$. Moreover, if f is continuous in m, we have that the learning procedure m^n defined inductively by $m^0=\inf L$ and $m^{n+1}=\inf R(m^n)$ is nondecreasing in L, and it converges to the minimal MFG solution.*

Proof. The existence and the lattice structure of equilibria follow by Tarski's fixed point theorem because the maps $\inf R$ and $\sup R$ are nondecreasing; see Remark 7.

We prove the convergence of the learning procedure $(m^n)_n$. Setting, for $n\geqslant 1$, $\tau_n := \inf_E\hat{R}(m^{n-1})$, by Remark 7, we have that $\tau_n\leqslant\tau_{n+1}$, $m^n\leqslant^L m^{n+1}$, and $m^n=p\tau_n$ for any $n\geqslant 1$. By the completeness of the lattices E and L, we can define $\tau_* := \sup_E\{\tau_n \mid n\geqslant 1\}$ and $m^* := \sup_n m^n$, and we have

$$\tau_n\to\tau_* \, \mathbb{P} \text{ a.s.} \quad \text{and} \quad m_t^n\to m_t^* \text{ weakly } \mathbb{P}\otimes dt \text{ a.e.,} \quad \text{as} \quad n\to\infty. \tag{55}$$

By definition of m^n and τ_n, for any $n\geqslant 1$, we have $J(\tau_n, m^{n-1})\geqslant J(\tau, m^{n-1})$ for any $\tau\in E$. Therefore, taking limits as $n\to\infty$ (justified by the integrability in (53) and the convergence in (55)), we obtain $J(\tau_*, m^*)\geqslant J(\tau, m^*)$ for any $\tau\in E$, so that

$$\tau_*\in\hat{R}(m^*). \tag{56}$$

Moreover, the sequence $(\tau_n)_n$ increasingly converges to τ_*, $\mathbb{P}$ a.s., as $n\to\infty$. Therefore, using the dominated convergence theorem for conditional expectations and exploiting the left continuity of the map $\mathbb{1}_{\{t<\cdot\}}$, we find that, $\mathbb{P}$ a.s.,

$$\begin{aligned}(p\tau_*)_t(y) &= \mathbb{E}[\mathbb{1}_{\{\psi(X_t)>y\}}\,\mathbb{1}_{\{t<\tau_*\}}\,|\,\mathcal{F}_T^{\mathrm{B}}]\\ &= \lim_n \mathbb{E}[\mathbb{1}_{\{\psi(X_t)>y\}}\,\mathbb{1}_{\{t<\tau_n\}}\,|\,\mathcal{F}_T^{\mathrm{B}}]\\ &= \lim_n (p\tau_n)_t(y) = \lim_n m_t^n(y), \quad \text{for any } (t,y)\in[0,T]\times\mathbb{R}.\end{aligned}$$

The latter, thanks to the convergence in (55), in turn implies that $p\tau_* = \sup_n m^n = m^*$. This, together with (56), gives that $m^* \in R(m^*)$, so that m^* is an MFG solution.

The fact that m^* is the minimal MFG solution follows as in the proof of the general Theorem 1, and this completes the proof of the theorem. □

7.4. Comments and Examples

Remark 8. We point out that because the function $\mathbb{1}_{\{t<\cdot\}}$ is not right continuous, the learning procedure $(m^n)_n \subset L$, which is defined inductively by $m^0 := \sup L$ and $m^{n+1} := \sup R(m^n)$, cannot be shown to converge to an MFG equilibrium.

Example 3 (MFGs of Optimal Stopping with Interaction of Scalar Type). As an example, we may consider an MFG in which the state variable Z evolves according to the SDE

$$dZ_t = b(t,Z_t)dt + \sigma(t,Z_t)dW_t + \sigma^o(t,Z_t)dB_t,\ t\in[0,T],$$

for functions $(b,\sigma,\sigma^o): [0,T]\times\mathbb{R}^{d_1} \to\times \mathbb{R}^{d_1}\times\mathbb{R}^{d_1\times d_1}\times\mathbb{R}^{d_1\times d_2}$ satisfying the usual Lipschitz conditions and for $\mathbb{F}$-adapted independent Brownian motions W and B taking values in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Moreover, one may consider a running profit function f, which enjoys a scalar nondecreasing dependence on the measure; that is, f is given by $f(t,x,m) := \bar{f}(t,x,\langle\phi,m\rangle)$, where $\bar{f}(t,x,\cdot)$ is nondecreasing, $\phi:\mathbb{R}\to[0,\infty)$ is nondecreasing, and $\langle\phi,m\rangle := \int_{[0,\infty)} \phi(x)\,dm(x)$.

Such a setting resembles the one considered in Bouveret et al. [13], even if several differences arise between the problem in Bouveret et al. [13] and ours. First, in Bouveret et al. [13], no common noise is considered, and a nondegeneracy condition on the volatility matrix is needed in order to employ results from PDE theory. These requirements are not needed for our lattice-theoretic approach to work. Second, in Bouveret et al. [13]—in order to establish uniqueness of the MFG equilibrium—a suitable antimonotonicity property is imposed on the dependence of the running profit function with respect to the measure variable (see assumption 8 therein), whereas in the setting of this example, we need that the function f is nondecreasing with respect to its third argument. Third, a convergence result is established in Bouveret et al. [13] for potential games, whereas the potential structure is not needed for our learning procedure to work.

Example 4 (MFGs of Timing with Common Noise). A particular example is when $Z_t = (t,\bar{Z}_t)$ and $\psi(t,\bar{z},b) = t$, for $(t,\bar{z},b)\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$. In this case, the fixed point condition in Definition 6 reduces to an identity on the space of $\mathcal{P}([0,T])$-valued random variables. In other words, an equilibrium is an $\mathcal{F}_T^B$-adapted $\mathcal{P}([0,T])$-valued random variable m such that, $\mathbb{P}$ a.s.,

$$m_t = \mathbb{P}[t<\tau^m\,|\,\mathcal{F}_T^B], \quad \text{for any } \ t\in[0,T] \text{ and some } \ \tau^m \in \arg\min_E \ J(\cdot,m).$$

This example corresponds to a particular case of the MFG of timing with common noise discussed in Carmona et al. [21].

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Appendix. Lattice-Theoretic Preliminaries

In this section, we collect some notions and preliminaries for lattices. Throughout, we consider a fixed lattice L (i.e., a partially ordered set (poset) in which every finite nonempty subset has a least upper bound and a greatest lower bound). We start with the following definition.

Definition A.1.

a. We say that L is *Dedekind σ complete* if every countable nonempty subset that is bounded above or below has a least upper bound or a greatest lower bound, respectively. We say that L is *Dedekind complete* if every nonempty subset that is bounded above or below has a least upper bound or a greatest lower bound, respectively. We say that L is *Dedekind super complete* if every nonempty subset that is bounded above or below has a countable subset with the same least upper bound or greatest lower bound, respectively. We say that L is *complete* if every nonempty subset of L has a least upper bound and a greatest lower bound.

b. We say that a set $M \subset L$ is *directed upward* or *directed downward* if for all $x, y \in M$, there exists some $z \in M$ with $x \vee y \leqslant z$ or $x \wedge y \geqslant z$, respectively.

Definition A.2. We say that a map $F : L \to \mathbb{R}$ is *strictly increasing* if

i. $F(x) \leqslant F(y)$ for all $x, y \in L$ with $x \leqslant y$ and

ii. for all $x, y \in L$ with $x \leqslant y$ and $F(x) = F(y)$, it follows that $x = y$.

The following lemma is a special case of lemma A.3 in Nendel [50] and gives a sufficient condition for a Dedekind σ-complete lattice to be Dedekind super complete. For the proof, we refer to Nendel [50].

Lemma A.1. *Let L be a Dedekind σ-complete lattice. If there exists a strictly increasing map $F : L \to \mathbb{R}$, then L is Dedekind super complete.*

A fundamental result by Birkhoff [11] (see section X.12, theorem X.20) and Frink [29] is that completeness of the lattice L corresponds to the compactness of L w.r.t. the so-called interval topology, whose definition we briefly recall here.

Definition A.3. The *interval topology* on L is the smallest topology τ on L such that all closed intervals of the form

$$(-\infty, a] := \{x \in L \,|\, x \leqslant a\} \quad \text{and} \quad [a, \infty) := \{x \in L \,|\, x \geqslant a\}, \quad \text{for } a \in L$$

are closed w.r.t. τ.

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