Time discretisation and rate of convergence for the optimal control of continuous-time stochastic systems with delay*

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Abstract

We study a semi-discretisation scheme for stochastic optimal control problems whose dynamics are given by controlled stochastic delay (or functional) differential equations with bounded memory. Performance is measured in terms of expected costs. By discretising time in two steps, we construct a sequence of approximating finite-dimensional Markovian optimal control problems in discrete time. The corresponding value functions converge to the value function of the original problem, and we derive an upper bound on the discretisation error or, equivalently, a worst-case estimate for the rate of convergence.

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1 Introduction

The object of this study is an approximation scheme for stochastic control systems with time delay in the dynamics. The control problems we wish to approximate are characterised as follows: The system dynamics are given by a controlled stochastic delay (or functional) differential equation (SDDE or SFDE), and the performance criterion is a cost functional

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of evolutional type over a finite deterministic time horizon. A control problem of this kind is generally infinite-dimensional in the sense that the corresponding value function lives on an infinite-dimensional function space. For simplicity, there will be neither state constraints nor state-dependent control constraints.

Our approximation scheme is based on a time discretisation of Euler-Maruyama type. This semi-discretisation procedure yields a sequence of finite-dimensional optimal control problems in discrete time. Under quite natural assumptions, we obtain an upper bound on the discretisation error – or a worst-case estimate for the rate of convergence – in terms of a difference in supremum norm between the value functions corresponding to the original control problem and the approximating control problems, respectively.

The approximation of the original control problem is carried out in two steps. The first step consists in constructing a sequence of control problems whose coefficients are piece-wise constant in both the time and segment variable. The admissible strategies are the same as those of the original problem. We obtain a rate of convergence for the controlled state processes, which is uniform in the strategies, thanks to the fact that the modulus of continuity of Itô diffusions with bounded coefficients has finite moments of all orders. This result can be found in Słomiński (2001), cf. Appendix A.2 below. The convergence rate for the controlled processes carries over to the approximation of the corresponding value functions.

The second discretisation step consists in approximating the original strategies by control processes which are piece-wise constant on a sub-grid of the time grid introduced in the first step. A main ingredient in the derivation of an error bound is the Principle of Dynamic Programming (PDP) or, as it is also known, Bellman's Principle of Optimality. The validity of the PDP for the "non-Markovian" dynamics at hand was proved in Larssen (2002), cf. Appendix A.1 below. A version of the PDP for controlled diffusions with time delay is also proved in Gihman and Skorohod (1979: Ch. 3); there are differences, though, in the formulation of the control problem.

We apply the PDP to obtain a global error bound from an estimate of the local truncation error. The fact that the value functions of the approximating problems from the first step are Lipschitz continuous under the supremum norm guarantees stability of the method. This way of error localisation and, in particular, the use of the PDP are adapted from Falcone and Ferretti (1994) and Falcone and Rosace (1996), who study deterministic optimal control problems with and without delay. Their proof technique is not confined to such simple approximation schemes as we adopt here; it extends the usual convergence analysis of finite difference methods for initial-boundary value problems, cf. Section 5.3 in Atkinson and Han (2001), for example.

To estimate the local truncation error we only need an error bound for the approximation by piece-wise constant strategies of finite-dimensional control problems with "constant coefficients"; that is, the cost rate and the coefficients of the state equation are functions of the control variable only. Such a result is provided by a stochastic mean value theorem due to Krylov (2001), which we cite in Appendix A.3. When the space of control actions is finite and the diffusion coefficient is not directly controlled, it is quite elementary to derive an analogous result with an error bound of higher order, namely of order $h^{1/2}$ instead of $h^{1/4}$, where h is the length of the time step in the discretisation.

In a last step, we put together the two error estimates to obtain a bound on the total approximation error. The error bound is of order nearly $h^{1/12}$ with h the length of the time step, see Theorem 4 in Section 5. To the best of our knowledge, this is the first result on the speed of convergence of a time-discretisation scheme for controlled systems with delay. We do not expect our estimate to be optimal, and understand the result as a benchmark on the way towards sharp error bounds. Moreover, the scheme's special structure can be exploited so that the computational requirements are lower than what might be expected by looking at the order of the error bound.

About ten years ago, N. V. Krylov was the first to obtain rates of convergence for finite difference schemes approximating finite-dimensional stochastic control problems with controlled and possibly degenerate diffusion matrix, see Krylov (1999, 2000) and the references therein. The error bound obtained there in the special case of a time-discretisation scheme with coefficients which are Lipschitz continuous in space and $\frac{1}{2}$-Hölder continuous in time is of order $h^{1/6}$ with h the length of the time step. Notice that in Krylov (1999) the order of convergence is given as $h^{1/3}$, where the time step has length h^2. When the space too is discretised, the ratio between time and space step is like h^2 against h or, equivalently, h vs. $\sqrt{h}$, which explains why the order of convergence is expressed in two different ways.

Using purely analytic techniques from the theory of viscosity solutions, Barles and Jakobsen (2005, to appear) obtain error bounds for a broad class of finite difference schemes for the approximation of partial differential equations involving operators of Hamilton-Jacobi-Bellman type. In the case of a simple time-discretisation scheme, the estimate for the speed of convergence they find is of order $h^{1/10}$ in the length h of the time step.

In the finite-dimensional setting, our two-step time-discretisation procedure allows to get from the case of "constant coefficients" to the case of general coefficients, even though it yields a worse rate of convergence in comparison with the results cited above, namely $\frac{1}{12}$ instead of $\frac{1}{6}$ and $\frac{1}{10}$, respectively. This is the price we pay for separating the approximation of the dynamics from that of the strategies. On the other hand, it is this separation that enables us to reduce the problem of strategy approximation to an elementary form. Observe that certain techniques like mollification of the value function employed in the works cited above are not available, because the space of initial values is not locally compact.

Our procedure also allows to estimate the error incurred when using strategies which are nearly optimal for the approximating problems with the dynamics of the original problem. This would be the way to apply the approximation scheme in many practically relevant situations. However, this method of nearly optimally controlling the original system is viable only if the available information includes perfect samples of the underlying noise process. The question is more complicated when information is restricted to samples of the state process.

The study of stochastic systems with delay has a long tradition, see Mohammed (1984) and the references therein. An overview of numerical methods for uncontrolled SDDEs is given in Buckwar (2000). The simplest approximation procedure is the Euler-Maruyama scheme. The work by Mao (2003) provides the rate of convergence for this scheme provided the SDDE has globally Lipschitz continuous coefficients and generalised distributed delays; Proposition 3 in Section 3 provides a partial generalisation of Mao's results and uses arguments similar to those in Calzolari et al. (to appear). The most common first order scheme is due to Milstein; see Hu et al. (2004) for the rate of convergence of this scheme applied to SDDEs with point delay.

There is a well-established method for approximating finite-dimensional stochastic control problems in continuous time, the so-called Markov chain method, which was introduced by H. J. Kushner about thirty years ago, see Kushner and Dupuis (2001) and the references therein. The idea is to construct a sequence of approximating control problems the dynamics of which are "locally consistent" with the dynamics of the original problem. If the cost functionals are reasonable approximations of the original performance criterion, then under very broad conditions local consistency of a scheme is sufficient to guarantee convergence of the corresponding value functions. The techniques used in Kushner and Dupuis (2001) and related work for the proofs of convergence are based on weak convergence of measures; they can be extended to cover control problems with delay, see Kushner (2005a) and also Fischer and Reiß (2007). In Kushner (2005b) numerical approximation of stochastic control problems with reflection, where generalised distributed delays may occur in the state and the control variable of the drift coefficient as well as in the reflection term, is treated using the Markov chain method. The starting point for the discretisation procedure in that work is not a controlled SDDE, but a representation of the system dynamics as a stochastic partial differential equation without delay. As the convergence proofs, which work for many different control problems and approximation schemes, are based on (local) consistency only, they usually do not yield any bound on the discretisation error.

Stochastic control problems with delay in the system dynamics have been the object of other recent studies; see Elsanosi et al. (2000) for certain explicitly available solutions and Øksendal and Sulem (2001) for the derivation of a maximum principle. If the dynamics of the control problem with delay exhibit a special structure, then the value function actually lives on a finite-dimensional space and the original problem can be reduced to a classical stochastic control problem without delay; see Larssen and Risebro (2003) and Bauer and Rieder (2005). We have already mentioned Larssen (2002) in connection with the PDP for delay systems. As in the finite-dimensional setting, the PDP can be invoked to derive a Hamilton-Jacobi-Bellman (HJB) equation for the value function. Such an HJB equation is not guaranteed to admit classical (i. e. Fréchet-differentiable) solutions, and the concept of viscosity solutions has to be introduced. The HJB equation can then be used as a starting point for constructing finite difference approximation schemes; see Chang et al. (2006) for this approach.

The rest of this paper is organised as follows. In Section 2, we introduce in detail the control problems we wish to approximate. In Section 3, we describe the first approximation step, that is, the time-discretisation of the dynamics. Section 4 is devoted to the second approximation step, the time-discretisation of the control processes. In Section 5, we derive the worst-case estimate of the total discretisation error. We also discuss the question of constructing nearly optimal strategies and address issues related to the numerical solution of the approximating control problems.

2 The original control problem

The dynamics of the control problems we want to approximate are described by a controlled d-dimensional stochastic delay (or functional) differential equation driven by a Wiener process. Both the drift and the diffusion coefficient may depend on the solution's history a certain amount of time into the past. The *delay length* gives a bound on the maximal time the system is allowed to look back into the past; it will be a finite deterministic time, say $r > 0$. For simplicity, we restrict attention to control problems with finite and deterministic time horizon. The performance of the admissible control processes or strategies will be measured in terms of a cost functional of evolutional type.

Typically, the solution process of an SDDE does not enjoy the Markov property, while the segment process associated with that solution does. For an $\mathbb{R}^d$-valued function $X = X(.)$ living on the time interval $[-r,\infty)$, the *segment* at time $t \in [0,\infty)$ is the function

$$X_t : [-r,0] \to \mathbb{R}^d, \qquad X_t(s) := X(t+s), \quad s \in [-r,0].$$

If X is a continuous function, then the segment X_t at time t is a continuous function defined on $[-r,0]$. Accordingly, if $(X(t))_{t\geq -r}$ is an $\mathbb{R}^d$-valued stochastic process with continuous trajectories, then the associated segment process $(X_t)_{t\geq 0}$ is a stochastic process taking its values in $\mathcal{C} := \mathbf{C}([-r,0],\mathbb{R}^d)$, the space of all $\mathbb{R}^d$-valued continuous functions on the interval $[-r,0]$. The space $\mathcal{C}$ comes equipped with the supremum norm, written $\|.\|$, induced by the standard norm on $\mathbb{R}^d$.

Let (Γ,ρ) be a complete and separable metric space, the set of *control actions*. We first state our control problem in the weak Wiener formulation, cf. Larssen (2002) and Yong and Zhou (1999: pp. 176-177). This is to justify our use of the Principle of Dynamic Programming. In subsequent sections we will only need the strong formulation.

Definition 1. A *Wiener basis* of dimension d_1 is a triple $((\Omega,\mathrm{P},\mathcal{F}),(\mathcal{F}_t),W)$ such that

(i) $(\Omega,\mathcal{F},\mathrm{P})$ is a complete probability space carrying a standard d_1-dimensional Wiener process W,

(ii) $(\mathcal{F}_t)$ is the completion by the P-null sets of $\mathcal{F}$ of the filtration induced by W.

A *Wiener control basis* is a quadruple $((\Omega, \mathrm{P}, \mathcal{F}), (\mathcal{F}_t), W, u)$ such that $((\Omega, \mathrm{P}, \mathcal{F}), (\mathcal{F}_t), W)$ is a Wiener basis and $u\colon [0, \infty) \times \Omega \to \Gamma$ is progressively measurable with respect to $(\mathcal{F}_t)$. The $(\mathcal{F}_t)$-progressively measurable process u is called a *control process*. Write $\mathcal{U}_W$ for the set of all Wiener control bases.

By abuse of notation, we will often hide the stochastic basis involved in the definition of a Wiener control basis; thus, we may write $(W, u) \in \mathcal{U}_W$ meaning that W is the Wiener process and u the control process of a Wiener control basis.

Let b, σ be Borel measurable functions defined on $[0, \infty) \times \mathcal{C} \times \Gamma$ and taking values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times d_1}$, respectively. The functions b, σ are the *coefficients* of the controlled SDDE that describes the dynamics of the control problem. The SDDE is of the form

$$(1) \qquad dX(t) \;=\; b\big(t_0{+}t, X_t, u(t)\big)dt \;+\; \sigma\big(t_0{+}t, X_t, u(t)\big)dW(t), \quad t > 0,$$

where $t_0 \geq 0$ is a deterministic initial time and $((\Omega, \mathrm{P}, \mathcal{F}), (\mathcal{F}_t), W, u)$ a Wiener control basis. The assumptions on the coefficients stated below will allow b, σ to depend on the segment variable in different ways. Let $\phi \in \mathcal{C}$ be a generic segment function. The coefficients b, σ may depend on ϕ through bounded Lipschitz functions of, for example,

$$\begin{aligned}
&\phi(-r_1), \ldots, \phi(-r_n), && (\textit{point delay}),\\
&\int_{-r}^{0} v_1(s, \phi(s)) w_1(s) ds, \ldots, \int_{-r}^{0} v_n(s, \phi(s)) w_n(s) ds && (\textit{distributed delay}),\\
&\int_{-r}^{0} \tilde{v}_1(s, \phi(s)) d\mu_1(s), \ldots, \int_{-r}^{0} \tilde{v}_n(s, \phi(s)) d\mu_n(s), && (\textit{generalised distributed delay}),
\end{aligned}$$

where $n \in \mathbb{N}$, $r_1, \ldots, r_n \in [0, r]$, $w_1, \ldots, w_n$ are Lebesgue integrable, $\mu_1, \ldots, \mu_n$ are finite Borel measures on $[0, r]$, v_i, $\tilde{v}_i$ are Lipschitz continuous in the second variable uniformly in the first, $v_i(.,0)w_i(.)$ is Lebesgue integrable and $\tilde{v}_i(.,0)$ is μ_i-integrable, $i \in \{1, \ldots, n\}$. Notice that the *generalised distributed delay* comprises the *point delay* as well as the Lebesgue absolutely continuous *distributed delay*. Let us call *functional delay* any type of delay that cannot be written in integral form. An example of a functional delay, which is also covered by the regularity assumptions stated below, is the dependence on the segment variable ϕ through bounded Lipschitz functions of

$$\sup_{s,t \in [-r,0]} \bar{v}_1(s, t, \phi(s), \phi(t)), \ldots, \sup_{s,t \in [-r,0]} \bar{v}_n(s, t, \phi(s), \phi(t)),$$

where $\bar{v}_i$ is a measurable function which is Lipschitz continuous in the last two variables uniformly in the first two variables and $\bar{v}_i(.,.,0,0)$ is bounded, $i \in \{1, \ldots, n\}$.

As initial condition for Equation (1), in addition to the time t_0, we have to prescribe the values of $X(t)$ for all $t \in [-r, 0]$, not only for $t = 0$. Thus, a deterministic initial condition for Equation (1) is a pair (t_0, ϕ), where $t_0 \geq 0$ is the initial time and $\phi \in \mathcal{C}$ the initial segment. We understand Equation (1) in the sense of an Itô equation. An adapted process X with continuous paths defined on the stochastic basis $(\Omega, \mathrm{P}, \mathcal{F}, (\mathcal{F}_t))$ of (W, u) is a *solution* with initial condition (t_0, ϕ) if it satisfies, P-almost-surely,

$$(2) \quad X(t) = \begin{cases} \phi(0) + \int_0^t b\big(t_0{+}s, X_s, u(s)\big)ds \ + \ \int_0^t \sigma\big(t_0{+}s, X_s, u(s)\big)dW(s), & t > 0, \\ \phi(t), \quad t \in [-r, 0]. \end{cases}$$

Observe that the solution process X always starts at time zero; it depends on the initial time t_0 only through the coefficients b, σ. As far as the control problem is concerned, this formulation is equivalent to the usual one, where the process X starts at time t_0 with initial condition $X_{t_0} = \phi$ and t_0 does not appear in the time argument of the coefficients.

A solution X to Equation (2) under (W, u) with initial condition (t_0, ϕ) is *strongly unique* if it is indistinguishable from any other solution $\tilde{X}$ satisfying Equation (2) under (W, u) with the same initial condition. A solution X to Equation (2) under (W, u) with initial condition (t_0, ϕ) is *weakly unique* if (X, W, u) has the same distribution as $(\tilde{X}, \tilde{W}, \tilde{u})$ whenever $(\tilde{W}, \tilde{u})$ has the same distribution as (W, u) and $\tilde{X}$ is a solution to Equation (2) under Wiener control basis $(\tilde{W}, \tilde{u})$ with initial condition (t_0, ϕ). Here, the space of Borel measurable functions $[0, \infty) \to \Gamma$ is equipped with the topology of convergence locally in Lebesgue measure.

Definition 2. A Wiener control basis $(W, u) \in \mathcal{U}_W$ is called *admissible* or an *admissible strategy* if, for each deterministic initial condition, Equation (2) has a strongly unique solution under (W, u) which is also weakly unique. Write $\mathcal{U}_{ad}$ for the set of admissible control bases.

Denote by $T > 0$ the finite deterministic time horizon. Let f, g be Borel measurable real-valued functions with f having domain $[0, \infty) \times \mathcal{C} \times \Gamma$ and g having domain $\mathcal{C}$. They will be referred to as the *cost rate* and the *terminal cost*, respectively. We introduce a *cost functional* J defined on $[0, T] \times \mathcal{C} \times \mathcal{U}_{ad}^J$ by setting

$$(3) \qquad J(t_0, \phi, (W, u)) := \ \mathbf{E}\left(\int_0^{T-t_0} f\big(t_0{+}s, X_s, u(s)\big)ds \ + \ g(X_{T-t_0})\right),$$

where X is the solution to Equation (2) under $(W, u) \in \mathcal{U}_{ad}^J$ with initial condition (t_0, ϕ) and $\mathcal{U}_{ad}^J \subseteq \mathcal{U}_{ad}$ is the set of all admissible Wiener control bases such that the expectation in (3) is well defined for all deterministic initial conditions.

The *value function* corresponding to Equation (2) and cost functional (3) is the function $V \colon [0, T] \times \mathcal{C} \to [-\infty, \infty)$ given by

$$(4) \qquad V(t_0, \phi) := \ \inf\big\{J(t_0, \phi, (W, u)) \mid (W, u) \in \mathcal{U}_{ad}^J\big\}.$$

It is this function that we wish to approximate.

Let us specify the hypotheses we make about the regularity of the coefficients b, σ, the cost rate f and the terminal cost g.

(A1) Measurability: the functions $b\colon [0,\infty)\times\mathcal{C}\times\Gamma\to\mathbb{R}^d$, $\sigma\colon [0,\infty)\times\mathcal{C}\times\Gamma\to\mathbb{R}^{d\times d_1}$, $f\colon [0,\infty)\times\mathcal{C}\times\Gamma\to\mathbb{R}$, $g\colon\mathcal{C}\to\mathbb{R}$ are jointly Borel measurable.

(A2) Boundedness: $|b|$, $|\sigma|$, $|f|$, $|g|$ are bounded by some constant $K>0$.

(A3) Uniform Lipschitz and Hölder condition: there is a constant $L>0$ such that for all $\phi,\tilde{\phi}\in\mathcal{C}$, $t,s\geq 0$, all $\gamma\in\Gamma$

$$\begin{aligned}|b(t,\phi,\gamma)-b(s,\tilde{\phi},\gamma)|\,\vee\,|\sigma(t,\phi,\gamma)-\sigma(s,\tilde{\phi},\gamma)| &\leq L\big(\|\phi-\tilde{\phi}\|+\sqrt{|t-s|}\big)\\ |f(t,\phi,\gamma)-f(s,\tilde{\phi},\gamma)|\,\vee\,|g(\phi)-g(\tilde{\phi})| &\leq L\big(\|\phi-\tilde{\phi}\|+\sqrt{|t-s|}\big).\end{aligned}$$

(A4) Continuity in the control: $b(t,\phi,.)$, $\sigma(t,\phi,.)$, $f(t,\phi,.)$ are continuous functions on Γ for any $t\geq 0$, $\phi\in\mathcal{C}$.

Here and in the sequel, $|.|$ denotes the Euclidean norm of appropriate dimension and $x\vee y$ denotes the maximum of x and y. The above measurability, boundedness and Lipschitz continuity assumptions on the coefficients b, σ guarantee the existence of a strongly unique solution $X=X^{t_0,\phi,u}$ to Equation (2) for every initial condition $(t_0,\phi)\in[0,T]\times\mathcal{C}$ and $(W,u)\in\mathcal{U}_W$ any Wiener control basis; see, for example, Theorem 2.1 and Remark 1.1(2) in Chapter 2 of Mohammed (1984). Moreover, weak uniqueness of solutions holds for all deterministic initial conditions. This is a consequence of a theorem due to Yamada and Watanabe, see Larssen (2002) for the necessary generalisation to SDDEs.

Consequently, under Assumptions (A1) – (A3), we have $\mathcal{U}_{ad}=\mathcal{U}_W$. Moreover, since f and g are assumed to be measurable and bounded, the expectation in (3) is always well defined, whence it holds that $\mathcal{U}_{ad}^J=\mathcal{U}_{ad}=\mathcal{U}_W$. Assumption (A4) will not be needed before Section 4.

The fact that weak uniqueness holds allows us to discard the weak formulation and consider our control problem in the strong Wiener formulation. Thus, we may work with a fixed Wiener basis. Under Assumptions (A1) – (A3), the admissible strategies will be precisely the natural strategies, that is, those that are representable as functionals of the driving Wiener process. From now on, let $((\Omega,\mathrm{P},\mathcal{F}),(\mathcal{F}_t),W)$ be a fixed d_1-dimensional Wiener basis. Denote by $\mathcal{U}$ the set of control processes defined on this stochastic basis.

The dynamics of our control problem are still given by Equation (2). Due to Assumptions (A1) – (A3), all control processes are admissible in the sense that Equation (2) has a (strongly) unique solution under any $u\in\mathcal{U}$ for every deterministic initial condition. In the definition of the cost functional, the Wiener process and the probability measure do not vary any more. The corresponding value function

$$[0,T]\times\mathcal{C}\ni(t_0,\phi)\to\inf\big\{J(t_0,\phi,u))\mid u\in\mathcal{U}\big\}$$

is identical to the function V determined by (4). By abuse of notation, we write $J(t_0,\phi,u)$ for $J(t_0,\phi,(W,u))$. We next state some important properties of the value function.

Proposition 1. *Assume (A1) – (A3). Then the value function V is bounded and Lipschitz continuous in the segment variable uniformly in the time variable. More precisely, there is $L_V > 0$ such that for all $t_0 \in [0,T]$, $\phi, \tilde{\phi} \in \mathcal{C}$,*

$$|V(t_0,\phi)| \;\leq\; K(T{+}1), \qquad |V(t_0,\phi) - V(t_0,\tilde{\phi})| \;\leq\; L_V\, \|\phi - \tilde{\phi}\|.$$

The constant L_V need not be greater than $3L(T+1)\exp(3T(T+4d_1)L^2)$. Moreover, V satisfies Bellman's Principle of Dynamic Programming, that is, for all $t \in [0, T{-}t_0]$,

$$V(t_0,\phi) \;=\; \inf_{u\in\mathcal{U}} \mathbf{E}\left(\int_0^t f\big(t_0{+}s, X^u_s, u(s)\big) ds \;+\; V\big(t_0{+}t, X^u_t\big) \right),$$

where X^u is the solution to Equation (2) *under control process u with initial condition (t_0,ϕ).*

Proof. For the boundedness and Lipschitz continuity of V see Proposition 7, for the Bellman Principle see Theorem 6 in Appendix A.1, where we set $\tilde{r} := r$, $\tilde{b} := b$ and so on. Notice that the Hölder continuity in time of the coefficients b, σ, f as stipulated in Assumption (A3) is not needed in the proofs. □

The value function V has some regularity in the time variable, too. It is Hölder continuous in time with parameter α for any $\alpha \leq \frac{1}{2}$ provided the initial segment is at least α-Hölder continuous. Notice that the coefficients b, σ, f need not be Hölder continuous in time. Except for the role of the initial segment, statement and proof of Proposition 2 are analogous to the non-delay case, see Krylov (1980: p. 167), for example.

Proposition 2. *Assume (A1) – (A3). Let $\phi \in \mathcal{C}$. If ϕ is α-Hölder continuous with Hölder constant not greater than L_H, then the function $V(.,\phi)$ is Hölder continuous; that is, there is a constant $\tilde{L}_V > 0$ depending only on L_H, K, T and the dimensions such that for all $t_0, t_1 \in [0,T]$*

$$|V(t_0,\phi) - V(t_1,\phi)| \;\leq\; \tilde{L}_V \left(|t_1{-}t_0|^\alpha \vee \sqrt{|t_1{-}t_0|} \right).$$

Proof. Let $\phi \in \mathcal{C}$ be α-Hölder continuous with Hölder constant not greater than L_H. Without loss of generality, we suppose that $t_1 = t_0 + h$ for some $h > 0$. We may also suppose $h \leq \frac{1}{2}$, because we can choose $\tilde{L}_V$ greater than $4K(T{+}1)$ so that the asserted inequality certainly holds for $|t_0{-}t_1| > \frac{1}{2}$. By Bellman's Principle as stated in Proposition 1, we see that

$$\begin{aligned}
&|V(t_0,\phi) - V(t_1,\phi)| \;=\; |V(t_0,\phi) - V(t_0{+}h,\phi)| \\
=\;& \Big|\inf_{u\in\mathcal{U}} \mathbf{E}\left(\int_0^h f\big(t_0{+}s, X^u_s, u(s)\big) ds \;+\; V(t_0{+}h, X^u_h) \right) \;-\; V(t_0{+}h,\phi)\Big| \\
\leq\;& \sup_{u\in\mathcal{U}} \mathbf{E}\left(\int_0^h \big|f\big(t_0{+}s, X^u_s, u(s)\big)\big| ds \right) \;+\; \sup_{u\in\mathcal{U}} \mathbf{E}\left(\big|V(t_0{+}h, X^u_h) - V(t_0{+}h,\phi)\big| \right) \\
\leq\;& K\,h \;+\; \sup_{u\in\mathcal{U}} L_V\, \mathbf{E}\left(\|X^u_h - \phi\| \right),
\end{aligned}$$

where K is the constant from Assumption (A2) and L_V the Lipschitz constant for V in the segment variable according to Proposition 1. We notice that $\phi = X^u_0$ for all $u \in \mathcal{U}$ since X^u is the solution to Equation (2) under control u with initial condition (t_0, ϕ). By Assumption (A2), Hölder's inequality, Doob's maximal inequality and Itô's isometry, for arbitrary $u \in \mathcal{U}$ it holds that

$$
\begin{aligned}
& \mathbf{E}\left(\|X^u_h - \phi\|\right) \\
\leq \;& \sup_{t\in[-r,-h]} |\phi(t+h) - \phi(t)| \;+\; \sup_{t\in[-h,0]} |\phi(0) - \phi(t)| \;+\; \mathbf{E}\left(\int_0^h \left|b\big(t_0+s, X^u_s, u(s)\big)\right| ds\right) \\
& + \mathbf{E}\left(\sup_{t\in[0,h]} \left|\int_0^t \sigma\big(t_0+s, X^u_s, u(s)\big) dW(s)\right|^2\right)^{\frac{1}{2}} \\
\leq \;& 2L_H\, h^{\alpha} \;+\; K\, h \;+\; 4K\, d_1 \sqrt{h}.
\end{aligned}
$$

Putting everything together yields the assertion. □

From the proof of Proposition 2 we see that the time regularity of the value function V is independent of the time regularity of the coefficients b, σ, f; it is always $\frac{1}{2}$-Hölder provided the initial segment is at least that regular.

3 First approximation step: piece-wise constant segments

The aim of this section is to define a sequence of approximating control problems where the coefficients of the dynamics, the cost rate, and the terminal cost are piece-wise constant functions of the time and segment variable, while the dependence on the strategies remains the same as in the original problem. We will obtain a bound for the approximation error which is uniform over all initial segments of a given Hölder continuity.

Let $N \in \mathbb{N}$. In order to construct the N-th approximating control problem, set $h_N := \frac{r}{N}$, and define $\lfloor . \rfloor_N$ by $\lfloor t \rfloor_N := h_N \lfloor \frac{t}{h_N} \rfloor$, where $\lfloor . \rfloor$ is the usual Gauss bracket, that is, $\lfloor t \rfloor$ is the integer part of the real number t. Set $T_N := \lfloor T \rfloor_N$ and $I_N := \{k\, h_N \mid k \in \mathbb{N}_0\} \cap [0, T_N]$. As T is the time horizon for the original control problem, T_N will be the time horizon for the N-th approximating problem. The set I_N is the time grid of discretisation degree N. Denote by Lin_N the operator $\mathcal{C} \to \mathcal{C}$ which maps a function in $\mathcal{C}$ to its piece-wise linear interpolation on the grid $\{k\, h_N \mid k \in \mathbb{Z}\} \cap [-r, 0]$.

We want to express the dynamics and the cost functional of the approximating problems in the same form as those of the original problem, so that the Principle of Dynamic Programming as stated in Appendix A.1 can be readily applied; see Propositions 5 and 6 in Section 4. To this end, the segment space has to be enlarged according to the discretisation degree N. Denote by $\mathcal{C}_N$ the space $\mathbf{C}([-r-h_N, 0], \mathbb{R}^d)$ of $\mathbb{R}^d$-valued continuous functions living on the interval $[-r-h_N, 0]$. For a continuous function or a continuous process Z defined on the time interval $[-r-h_N, \infty)$, let $\Pi_N(Z)(t)$ denote the segment of Z at time $t \geq 0$ of length $r+h_N$, that is, $\Pi_N(Z)(t)$ is the function $[-r-h_N, 0] \ni s \mapsto Z(t+s)$.

Given $t_0 \geq 0$, $\psi \in \mathcal{C}_N$ and $u \in \mathcal{U}$, we define the Euler-Maruyama approximation $Z = Z^{N,t_0,\psi,u}$ of degree N of the solution X to Equation (2) under control process u with initial condition (t_0,ψ) as the solution to

$$Z(t) = \begin{cases} \psi(0) & + \int_0^t b_N\big(t_0+s, \Pi_N(Z)(s), u(s)\big)ds \\ & + \int_0^t \sigma_N\big(t_0+s, \Pi_N(Z)(s), u(s)\big)dW(s), \quad t > 0, \\ \psi(t), & t \in [-r-h_N, 0], \end{cases} \tag{5}$$

where the coefficients b_N, σ_N are given by

$$\begin{aligned} b_N(t,\psi,\gamma) &:= b\left(\lfloor t\rfloor_N, \mathrm{Lin}_N\big([-r,0] \ni s \mapsto \psi(s+\lfloor t\rfloor_N - t)\big), \gamma\right), \\ \sigma_N(t,\psi,\gamma) &:= \sigma\left(\lfloor t\rfloor_N, \mathrm{Lin}_N\big([-r,0] \ni s \mapsto \psi(s+\lfloor t\rfloor_N - t)\big), \gamma\right), \quad t \geq 0,\ \psi \in \mathcal{C}_N,\ \gamma \in \Gamma. \end{aligned}$$

Thus, $b_N(t,\psi,\gamma)$ and $\sigma_N(t,\psi,\gamma)$ are calculated by evaluating the corresponding coefficients b and σ at $(\lfloor t\rfloor_N, \hat{\phi}, \gamma)$, where $\hat{\phi}$ is the segment in $\mathcal{C}$ which arises from the piece-wise linear interpolation with mesh size $\frac{r}{N}$ of the restriction of ψ to the interval $[\lfloor t\rfloor_N - t - r, \lfloor t\rfloor_N - t]$. Notice that the control action γ remains unchanged.

Assumptions (A1) – (A3) guarantee that, given any control process $u \in \mathcal{U}$, Equation (5) has a unique solution for each initial condition $(t_0,\psi) \in [0,\infty) \times \mathcal{C}_N$. Thus, the process $Z = Z^{N,t_0,\psi,u}$ of discretisation degree N is well defined. Notice that the approximating coefficients b_N, σ_N are still Lipschitz continuous in the segment variable uniformly in the time and control variables, although they are only piece-wise continuous in time.

Define the cost functional $J_N \colon [0,T_N] \times \mathcal{C}_N \times \mathcal{U} \to \mathbb{R}$ of discretisation degree N by

$$J_N(t_0,\psi,u) := \mathbf{E}\left(\int_0^{T_N - t_0} f_N\big(t_0+s, \Pi_N(Z)(s), u(s)\big)ds \;+\; g_N\big(\Pi_N(Z)(T_N - t_0)\big)\right), \tag{6}$$

where f_N, g_N are given by

$$\begin{aligned} f_N(t,\psi,\gamma) &:= f\left(\lfloor t\rfloor_N, \mathrm{Lin}_N\big([-r,0] \ni s \mapsto \psi(s+\lfloor t\rfloor_N - t)\big), \gamma\right), \\ g_N(\psi) &:= g\left(\mathrm{Lin}_N\big(\psi_{|[-r,0]}\big)\right), \qquad\qquad t \geq 0,\ \psi \in \mathcal{C}_N,\ \gamma \in \Gamma. \end{aligned}$$

As b_N, σ_N above, f_N, g_N are Lipschitz continuous in the segment variable (uniformly in time and control) under the supremum norm on $\mathcal{C}_N$. The value function V_N corresponding to (5) and (6) is the function $[0,T_N] \times \mathcal{C}_N \to \mathbb{R}$ determined by

$$V_N(t_0,\psi) := \inf\big\{J_N(t_0,\psi,u) \mid u \in \mathcal{U}\big\}. \tag{7}$$

If $t_0 \in I_N$, then $\lfloor t_0+s\rfloor_N = t_0 + \lfloor s\rfloor_N$ for all $s \geq 0$. Thus, the solution Z to Equation (5) under control process $u \in \mathcal{U}$ with initial condition $(t_0,\psi) \in I_N \times \mathcal{C}_N$ satisfies

$$\begin{aligned} Z(t) = \ & \psi(0) + \int_0^t b\big(t_0+\lfloor s\rfloor_N, \mathrm{Lin}_N\big(Z_{\lfloor s\rfloor_N}\big), u(s)\big)ds \\ & + \int_0^t \sigma\big(t_0+\lfloor s\rfloor_N, \mathrm{Lin}_N\big(Z_{\lfloor s\rfloor_N}\big), u(s)\big)dW(s) \quad \text{for all } t \geq 0. \end{aligned} \tag{8}$$

Moreover, $(Z(t))_{t\geq 0}$ depends on the initial segment ψ only through the restriction of ψ to the interval $[-r,0]$. In analogy, whenever $t_0 \in I_N$, the cost functional J_N takes on the form

$$(9)\quad J_N(t_0,\psi,u) \;=\; \mathbf{E}\left(\int_0^{T_N-t_0} f\big(t_0+\lfloor s\rfloor_N, \mathrm{Lin}_N\big(Z_{\lfloor s\rfloor_N}\big), u(s)\big)ds \;+\; g\big(\mathrm{Lin}_N\big(Z_{T_N-t_0}\big)\big)\right).$$

Hence, if $t_0 \in I_N$, then $J_N(t_0,\psi,u) = J_N(t_0,\psi_{|[-r,0]},u)$ for all $\psi \in \mathcal{C}_N$, $u \in \mathcal{U}$; that is, $J_N(t_0,.,.)$ coincides with its projection onto $\mathcal{C}\times\mathcal{U}$. Consequently, if $t_0 \in I_N$, then $V_N(t_0,\psi) = V_N(t_0,\psi_{|[-r,0]})$ for all $\psi \in \mathcal{C}_N$; that is, $V_N(t_0,.)$ can be interpreted as a function with domain $\mathcal{C}$ instead of $\mathcal{C}_N$. If $t_0 \in I_N$, by abuse of notation, we will write $V_N(t_0,.)$ also for this function. Notice that, as a consequence of Equations (8) and (9), in this case we have $V_N(t_0,\phi) = V_N(t_0,\mathrm{Lin}_N(\phi))$ for all $\phi \in \mathcal{C}$.

By Proposition 2, we know that the original value function V is Hölder continuous in time provided the initial segment is Hölder continuous. It is therefore enough to compare V and V_N on the grid $I_N \times \mathcal{C}$. This is the content of the next two statements. Again, the order of the error will be uniform only over those initial segments which are α-Hölder continuous for some $\alpha > 0$; the constant in the error bound also depends on the Hölder constant of the initial segment. We start with comparing solutions to Equations (2) and (5) for initial times in I_N.

Proposition 3. *Assume (A1)–(A3). Let $\phi \in \mathcal{C}$ be Hölder continuous with parameter $\alpha > 0$ and Hölder constant not greater than L_H. Then there is a constant C depending only on α, L_H, L, K, T and the dimensions such that for all $N \in \mathbb{N}$ with $N \geq 2r$, all $t_0 \in I_N$, $u \in \mathcal{U}$ it holds that*

$$\mathbf{E}\left(\sup_{t\in[-r,T]} |X(t) - Z^N(t)|\right) \;\leq\; C\left(h_N^{\alpha} \vee \sqrt{h_N \ln\big(\tfrac{1}{h_N}\big)}\right),$$

where X is the solution to Equation (2) under control process u with initial condition (t_0,ϕ) and Z^N is the solution to Equation (5) of discretisation degree N under u with initial condition (t_0,ψ) with $\psi \in \mathcal{C}_N$ being such that $\psi_{|[-r,0]} = \phi$.

Proof. Notice that $h_N \leq \frac{1}{2}$ since $N \geq 2r$, and observe that $Z := Z^N$ as defined in the assertion satisfies Equation (8), as the initial time t_0 lies on the grid I_N. Moreover, Z depends on the initial segment ψ only through $\psi_{|[-r,0]} = \phi$. Using Hölder's inequality, Doob's maximal inequality, Itô's isometry, Assumption (A3), and Fubini's theorem we find that

$$\begin{aligned}
&\mathbf{E}\left(\sup_{t\in[-r,T]} |X(t)-Z(t)|^2\right) \;=\; \mathbf{E}\left(\sup_{t\in[0,T]} |X(t)-Z(t)|^2\right)\\
\leq\;& 2T\,\mathbf{E}\left(\int_0^T \left|b\big(t_0+s, X_s, u(s)\big) - b\big(t_0+\lfloor s\rfloor_N, \mathrm{Lin}_N(Z_{\lfloor s\rfloor_N}), u(s)\big)\right|^2 ds\right)\\
&+\; 8d_1\,\mathbf{E}\left(\int_0^T \left|\sigma\big(t_0+s, X_s, u(s)\big) - \sigma\big(t_0+\lfloor s\rfloor_N, \mathrm{Lin}_N(Z_{\lfloor s\rfloor_N}), u(s)\big)\right|^2 ds\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq\ 4T\,\mathbf{E}\left(\int_0^T \left|b\big(t_0+\lfloor s\rfloor_N, X_s, u(s)\big) - b\big(t_0+\lfloor s\rfloor_N, \mathrm{Lin}_N(Z_{\lfloor s\rfloor_N}), u(s)\big)\right|^2 ds\right)\\
&\quad +\ 16d_1\,\mathbf{E}\left(\int_0^T \left|\sigma\big(t_0+\lfloor s\rfloor_N, X_s, u(s)\big) - \sigma\big(t_0+\lfloor s\rfloor_N, \mathrm{Lin}_N(Z_{\lfloor s\rfloor_N}), u(s)\big)\right|^2 ds\right)\\
&\quad +\ 4T(T+4d_1)L^2 h_N\\
&\leq\ 4(T+4d_1)L^2\left(T\,h_N + \int_0^T \mathbf{E}\left(\|X_s - \mathrm{Lin}_N(Z_{\lfloor s\rfloor_N})\|^2\right) ds\right)\\
&\leq\ 4(T+4d_1)L^2\left(T\,h_N +\ 3\int_0^T \left(\mathbf{E}\left(\|X_s - X_{\lfloor s\rfloor_N}\|^2\right) + \mathbf{E}\left(\|Z_{\lfloor s\rfloor_N} - \mathrm{Lin}_N(Z_{\lfloor s\rfloor_N})\|^2\right)\right) ds\right)\\
&\quad +\ 12(T+4d_1)L^2\int_0^T \mathbf{E}\left(\|X_{\lfloor s\rfloor_N} - Z_{\lfloor s\rfloor_N}\|^2\right) ds\\
&\leq\ 4T(T+4d_1)L^2\left(h_N +\ 18L_H^2\,h_N^{2\alpha}\ +\ 18C_{2,T}\,h_N\ln\big(\tfrac{1}{h_N}\big)\right)\\
&\quad +\ 12(T+4d_1)L^2\int_0^T \mathbf{E}\left(\sup_{t\in[-r,s]}|X(t)-Z(t)|^2\right) ds.
\end{aligned}
$$

Applying Gronwall's lemma, we obtain the assertion. In the last step of the above estimate Lemma 1 from Appendix A.2 and the Hölder continuity of ϕ have both been used twice. Firstly, to get for all $s\in[0,T]$

$$
\begin{aligned}
&\mathbf{E}\left(\|X_s - X_{\lfloor s\rfloor_N}\|^2\right)\\
\leq\ &2\,\mathbf{E}\left(\sup_{t,\tilde{t}\in[-r,0],|t-\tilde{t}|\leq h_N}|\phi(t)-\phi(\tilde{t})|^2\right)\ +\ 2\,\mathbf{E}\left(\sup_{t,\tilde{t}\in[0,T],|t-\tilde{t}|\leq h_N}|X(t)-X(\tilde{t})|^2\right)\\
\leq\ &2L_H^2\,h_N^{2\alpha}\ +\ 2C_{2,T}\,h_N\ln\big(\tfrac{1}{h_N}\big).
\end{aligned}
$$

Secondly, to obtain

$$
\begin{aligned}
&\mathbf{E}\left(\left\|Z_{\lfloor s\rfloor_N} - \mathrm{Lin}_N\big(Z_{\lfloor s\rfloor_N}\big)\right\|^2\right)\ =\ \mathbf{E}\left(\sup_{t\in[\lfloor s\rfloor_N-r,\lfloor s\rfloor_N]}\left|Z(t) - \mathrm{Lin}_N\big(Z_{\lfloor s\rfloor_N}\big)(t)\right|^2\right)\\
\leq\ &2\,\mathbf{E}\left(\sup_{t\in[-r,0)}\left|\phi(t)-\phi\big(\lfloor t\rfloor_N\big)\right|^2 + \left|\phi(t)-\phi\big(\lfloor t\rfloor_N+h_N\big)\right|^2\right)\\
&\quad +\ 2\,\mathbf{E}\left(\sup_{t\in[0,s)}\left|Z(t)-Z\big(\lfloor t\rfloor_N\big)\right|^2 + \left|Z(t)-Z\big(\lfloor t\rfloor_N+h_N\big)\right|^2\right)\\
\leq\ &4L_H^2\,h_N^{2\alpha}\ +\ 4\,\mathbf{E}\left(\sup_{t,\tilde{t}\in[0,s],|t-\tilde{t}|\leq h_N}|Z(t)-Z(\tilde{t})|^2\right)\\
\leq\ &4L_H^2\,h_N^{2\alpha}\ +\ 4C_{2,T}\,h_N\ln\big(\tfrac{1}{h_N}\big)\quad \text{for all } s\in[0,T].
\end{aligned}
$$

□

The order of the approximation error obtained in Proposition 3 for the underlying dynamics carries over to the approximation of the corresponding value functions. This works thanks to the Lipschitz continuity of the cost rate and terminal cost in the segment variable, the bound on the moments of the modulus of continuity from Lemma 1 in Appendix A.2, and the fact that the error bound in Proposition 3 is uniform over all strategies.

Theorem 1. *Assume (A1)–(A3). Let $\phi \in \mathcal{C}$ be Hölder continuous with parameter $\alpha > 0$ and Hölder constant not greater than L_H. Then there is a constant $\tilde{C}$ depending only on α, L_H, L, K, T and the dimensions such that for all $N \in \mathbb{N}$ with $N \geq 2r$, all $t_0 \in I_N$ it holds that*

$$\left|V(t_0,\phi) - V_N(t_0,\phi)\right| \;\leq\; \sup_{u\in\mathcal{U}} \left|J(t_0,\phi,u) - J_N(t_0,\psi,u)\right| \;\leq\; \tilde{C}\left(h_N^{\alpha} \vee \sqrt{h_N \ln(\tfrac{1}{h_N})}\right),$$

where $\psi \in \mathcal{C}_N$ is such that $\psi_{|[-r,0]} = \phi$.

Proof. To verify the first inequality, we distinguish the cases $V(t_0,\phi) > V_N(t_0,\phi)$ and $V(t_0,\phi) < V_N(t_0,\phi)$. First suppose that $V(t_0,\phi) > V_N(t_0,\phi)$. Then for each $\varepsilon \in (0,1]$ we find a strategy $u^{\varepsilon} \in \mathcal{U}$ such that $V_N(t_0,\phi) \geq J_N(t_0,\phi,u^{\varepsilon}) - \varepsilon$. Since $V(t_0,\phi) \leq J(t_0,\phi,u)$ for all $u \in \mathcal{U}$ by definition, it follows that

$$\begin{aligned} &|V(t_0,\phi) - V_N(t_0,\phi)| \;=\; V(t_0,\phi) - V_N(t_0,\phi) \;\leq\; J(t_0,\phi,u^{\varepsilon}) - J_N(t_0,\phi,u^{\varepsilon}) \;+\; \varepsilon \\ \leq\;\; &\sup_{u\in\mathcal{U}} |J(t_0,\phi,u) - J_N(t_0,\psi,u)| \;+\; \varepsilon. \end{aligned}$$

Sending ε to zero, we obtain the asserted inequality provided that $V(t_0,\phi) > V_N(t_0,\phi)$. If, on the other hand, $V(t_0,\phi) < V_N(t_0,\phi)$, then we choose a sequence of minimising strategies $u^{\varepsilon} \in \mathcal{U}$ such that $V(t_0,\phi) \geq J(t_0,\phi,u^{\varepsilon}) - \varepsilon$, notice that $|V(t_0,\phi) - V_N(t_0,\phi)| = V_N(t_0,\phi) - V(t_0,\phi)$ and obtain the asserted inequality as in the first case.

Now, let $u \in \mathcal{U}$ be any control process. Let X be the solution to Equation (2) under u with initial condition (t_0,ϕ) and $Z = Z^N$ be the solution to Equation (5) under u with initial condition (t_0,ψ). Using Assumption (A2) and the hypothesis that $t_0 \in I_N$, we get

$$\begin{aligned} |J(t_0,\phi,u) - J_N(t_0,\psi,u)| \;&\leq\; K\,|T - T_N| \;+\; \mathbf{E}\left(\left|g\big(\mathrm{Lin}_N(Z_{T_N - t_0})\big) - g\big(X_{T-t_0}\big)\right|\right) \\ &+\; \mathbf{E}\left(\int_0^{T_N - t_0} \left|f\big(t_0 + \lfloor s \rfloor_N, \mathrm{Lin}_N(Z_{\lfloor s \rfloor_N}), u(s)\big) - f\big(t_0 + s, X_s, u(s)\big)\right| ds\right). \end{aligned}$$

Recall that $|T - T_N| = T - \lfloor T \rfloor_N \leq h_N$. Hence, $K\,|T - T_N| \leq K\,h_N$. Now, using Assumption (A3), we see that

$$\begin{aligned} &\mathbf{E}\left(\left|g\big(\mathrm{Lin}_N(Z_{T_N - t_0})\big) - g\big(X_{T-t_0}\big)\right|\right) \\ \leq\;\; &L\Big(\mathbf{E}\big(\|Z_{T_N - t_0} - X_{T_N - t_0}\|\big) + \mathbf{E}\big(\|\,\mathrm{Lin}_N(Z_{T_N - t_0}) - Z_{T_N - t_0}\|\big) + \mathbf{E}\big(\|X_{T_N - t_0} - X_{T - t_0}\|\big)\Big) \\ \leq\;\; &L\left(C\left(h_N^{\alpha} \vee \sqrt{h_N \ln(\tfrac{1}{h_N})}\right) \;+\; 3L_H\, h_N^{\alpha} + 3C_{1,T}\sqrt{h_N \ln(\tfrac{1}{h_N})}\right), \end{aligned}$$

where C is a constant as in Proposition 3 and $C_{1,T}$ is a constant as in Lemma 1 in Appendix A.2. Notice that $(X(t))_{t\geq 0}$ as well as $(Z(t))_{t\geq 0}$ are Itô diffusions with coefficients bounded by the constant K from Assumption (A2). In the same way, also using the Hölder continuity of f in time and recalling that $|s - \lfloor s \rfloor_N| \leq h_N$ for all $s \geq 0$, we see that

$$\begin{aligned} &\mathbf{E}\left(\int_0^{T_N - t_0} \left| f\big(t_0 + \lfloor s \rfloor_N, \mathrm{Lin}_N\big(Z_{\lfloor s \rfloor_N}\big), u(s)\big) - f\big(t_0 + s, X_s, u(s)\big)\right| ds\right) \\ \leq\ & L\,(T_N - t_0)\left(\sqrt{h_N} \ +\ 3C_{1,T}\sqrt{h_N \ln(\tfrac{1}{h_N})} \ +\ (C + 3L_H)\left(h_N^{\alpha} \vee \sqrt{h_N \ln(\tfrac{1}{h_N})}\right)\right). \end{aligned}$$

Putting the three estimates together, we obtain the assertion. □

In virtue of Theorem 1, we can replace the original control problem of Section 2 with the sequence of approximating control problems defined above. The error between the problem of degree N and the original problem in terms of the difference between the corresponding value functions V and V_N is not greater than a multiple of $(\frac{r}{N})^{\alpha}$ for α-Hölder continuous initial segments if $\alpha \in (0, \frac{1}{2})$, where the proportionality factor is affine in the Hölder constant; it is less than a multiple of $\sqrt{\frac{\ln(N)}{N}}$ if $\alpha \geq \frac{1}{2}$.

From the proofs of Proposition 3 and Theorem 1 it is clear that the coefficients b, σ, f of the original problem, instead of being $\frac{1}{2}$-Hölder continuous as postulated by Assumption (A3), need only satisfy a bound of the form $\sqrt{|t-s| \ln(\frac{1}{|t-s|})}$, $t, s \in [0, T]$ with $|t-s|$ small, for the error estimates to hold.

Although we obtain an error bound for the approximation of V by the sequence of value functions $(V_N)_{N \in \mathbb{N}}$ only for Hölder continuous initial segments, the proofs of Proposition 3 and Theorem 1 also show that pointwise convergence of the value functions holds true for all initial segments $\phi \in \mathcal{C}$. Recall that a function $\phi : [-r, 0] \to \mathbb{R}^d$ is continuous if and only if $\sup_{t,s \in [-r,0], |t-s| \leq h} |\phi(t) - \phi(s)|$ tends to zero as $h \searrow 0$. Let us record the result for the value functions.

Corollary 1. *Assume (A1) – (A3). Then for all $(t_0, \phi) \in [0, T] \times \mathcal{C}$,*

$$\left| V(t_0, \phi) - V_N(\lfloor t_0 \rfloor_N, \phi) \right| \ \overset{N \to \infty}{\longrightarrow}\ 0.$$

Similarly to the value function of the original problem, also the function $V_N(t_0, .)$ is Lipschitz continuous in the segment variable uniformly in $t_0 \in I_N$ with Lipschitz constant not depending on the discretisation degree N. Since $t_0 \in I_N$, we may interpret $V_N(t_0, .)$ as a function defined on $\mathcal{C}$.

Proposition 4. *Assume (A1) – (A3). Let V_N be the value function of discretisation degree N. Then $|V_N|$ is bounded by $K(T+1)$. Moreover, if $t_0 \in I_N$, then $V_N(t_0, .)$ as a function of $\mathcal{C}$ satisfies the following Lipschitz condition:*

$$|V_N(t_0, \phi) - V_N(t_0, \tilde{\phi})| \ \leq\ 3L(T+1)\exp\big(3T(T + 4d_1)L^2\big)\, \|\phi - \tilde{\phi}\| \quad \textit{for all } \phi, \tilde{\phi} \in \mathcal{C}.$$

Proof. The assertion is again a consequence of Proposition 7 in Appendix A.1. To see this, set $\tilde{r} := r + h_N$, $\tilde{T} := T_N$, $\tilde{b} := b_N$, $\tilde{\sigma} := \sigma_N$, $\tilde{f} := f_N$, and $\tilde{g} := g_N$. Equation (5) then describes the same dynamics as Equation (12), $\tilde{J}$ is the same functional as J_N, whence $V_N = \tilde{V}$. The hypotheses of Appendix A.1 are satisfied. Finally, recall that $T_N \leq T$ and that, since $t_0 \in I_N$, $V_N(t_0, \psi)$ depends on $\psi \in \mathcal{C}_N$ only through $\psi_{|[-r,0]}$. □

4 Second approximation step: piece-wise constant strategies

In Section 3, we have discretised the time as well as the segment space in time. The resulting control problem of discretisation degree $N \in \mathbb{N}$ has dynamics described by Equation (5), cost functional J_N defined by (6) and value function V_N given by (7). Here, we will also approximate the control processes $u \in \mathcal{U}$, which up to now have been those of the original problem, by introducing further control problems defined over sets of piece-wise constant strategies. To this end, for $n \in \mathbb{N}$, set

$$(10) \quad \mathcal{U}_n := \big\{ u \in \mathcal{U} \mid u(t) \text{ is } \sigma(W(k\tfrac{r}{n}), k \in \mathbb{N}_0)\text{-measurable and } u(t) = u(\lfloor t \rfloor_n),\ t \geq 0 \big\}.$$

Recall that $\lfloor t \rfloor_n = \frac{r}{n} \lfloor \frac{n}{r} t \rfloor$. Hence, $\mathcal{U}_n$ is the set of all Γ-valued $(\mathcal{F}_t)$-progressively measurable processes which are right-continuous and piece-wise constant in time relative to the grid $\{k\frac{r}{n} \mid k \in \mathbb{N}_0\}$ and, in addition, are $\sigma(W(k\frac{r}{n}), k \in \mathbb{N}_0)$-measurable. In particular, if $u \in \mathcal{U}_n$ and $t \geq 0$, then the random variable $u(t)$ can be represented as

$$u(t)(\omega) \ = \ \theta\big(\lfloor \tfrac{n}{r} t \rfloor, W(0)(\omega), \ldots, W(\lfloor \tfrac{n}{r} t \rfloor)(\omega)\big), \quad \omega \in \Omega,$$

where θ is some Γ-valued Borel measurable function depending on u and n. For the purpose of approximating the control problem of degree N we will use strategies in $\mathcal{U}_{N \cdot M}$ with $M \in \mathbb{N}$. Let us write $\mathcal{U}_{N,M}$ for $\mathcal{U}_{N \cdot M}$.

With the same dynamics and the same performance criterion as before, for each $N \in \mathbb{N}$, we introduce a family of value functions $V_{N,M}$, $M \in \mathbb{N}$, defined on $[0, T_N] \times \mathcal{C}_N$ by setting

$$(11) \qquad V_{N,M}(t_0, \psi) := \ \inf\big\{ J_N(t_0, \psi, u) \mid u \in \mathcal{U}_{N,M} \big\}.$$

We will refer to $V_{N,M}$ as the value function of degree (N, M). By construction, it holds that $V_N(t_0, \psi) \leq V_{N,M}(t_0, \psi)$ for all $(t_0, \psi) \in [0, T_N] \times \mathcal{C}_N$. Hence, in estimating the approximation error, we only need an upper bound for $V_{N,M} - V_N$.

As with V_N, if the initial time t_0 lies on the grid I_N, then $V_{N,M}(t_0, \psi)$ depends on ψ only through its restriction $\psi_{|[-r,0]} \in \mathcal{C}$ to the interval $[-r, 0]$. We write $V_{N,M}(t_0, .)$ for this function, too. The dynamics and costs, in this case, can again be represented by Equations (8) and (9), respectively. And again, if $t_0 \in I_N$, we have $V_{N,M}(t_0, \phi) = V_{N,M}(t_0, \mathrm{Lin}_N(\phi))$ for all $\phi \in \mathcal{C}$.

Propositions 5 and 6 state Bellman's Principle of Dynamic Programming for the value functions V_N and $V_{N,M}$, respectively. The special case when the initial time as well as the time step lie on the grid I_N is given separately, as it is this representation which will be used in the approximation result; see the proof of Theorem 2.

Proposition 5. *Assume (A1) – (A3). Let $t_0 \in [0, T_N]$, $\psi \in \mathcal{C}_N$. Then for $t \in [0, T_N - t_0]$,*

$$V_N(t_0, \psi) \ = \ \inf_{u \in \mathcal{U}} \mathbf{E} \left(\int_0^t f_N\big(t_0 + s, \Pi_N(Z^u)(s), u(s)\big) ds \ + \ V_N\big(t_0 + t, \Pi_N(Z^u)(t)\big) \right),$$

where Z^u is the solution to Equation (5) *of degree N under control process u and with initial condition (t_0, ψ). If $t_0 \in I_N$ and $t \in I_N \cap [0, T_N - t_0]$, then*

$$V_N(t_0, \phi) \;=\; \inf_{u \in \mathcal{U}} \mathbf{E}\left(\int_0^t f\big(t_0 + \lfloor s \rfloor_N, \mathrm{Lin}_N(Z^u_{\lfloor s \rfloor_N}), u(s)\big) ds \;+\; V_N\big(t_0 + t, \mathrm{Lin}_N(Z^u_t)\big) \right),$$

where $V_N(t_0, .)$, $V_N(t_0 + t, .)$ are defined as functionals on $\mathcal{C}$, and ϕ is the restriction of ψ to the interval $[-r, 0]$.

Proof. Apply Theorem 6 in Appendix A.1. To this end, let $\tilde{\mathcal{U}}$ be the set of strategies $\mathcal{U}$ and set $\tilde{r} := r + h_N$, $\tilde{T} := T_N$, $\tilde{b} := b_N$, $\tilde{\sigma} := \sigma_N$, $\tilde{f} := f_N$, and $\tilde{g} := g_N$. Observe that Equation (5) describes the same dynamics as Equation (12), that $\tilde{J} = J_N$, whence $V_N = \tilde{V}$, and verify that the hypotheses of Appendix A.1 are satisfied. $\square$

Proposition 6. *Assume (A1) – (A3). Let $t_0 \in [0, T_N]$, $\psi \in \mathcal{C}_N$. Then for $t \in I_{N \cdot M} \cap [0, T_N - t_0]$*

$$V_{N,M}(t_0, \psi) \;=\; \inf_{u \in \mathcal{U}_{N,M}} \mathbf{E}\left(\int_0^t f_N\big(t_0 + s, \Pi_N(Z^u)(s), u(s)\big) ds \;+\; V_{N,M}\big(t_0 + t, \Pi_N(Z^u)(t)\big) \right),$$

where Z^u is the solution to Equation (5) *of degree N under control process u and with initial condition (t_0, ψ). If $t_0 \in I_N$ and $t \in I_N \cap [0, T_N - t_0]$, then*

$$V_{N,M}(t_0, \phi) \;=\; \inf_{u \in \mathcal{U}_{N,M}} \mathbf{E}\left(\int_0^t f\big(t_0 + \lfloor s \rfloor_N, \mathrm{Lin}_N(Z^u_{\lfloor s \rfloor_N}), u(s)\big) ds \;+\; V_{N,M}\big(t_0 + t, \mathrm{Lin}_N(Z^u_t)\big) \right),$$

where $V_{N,M}(t_0, .)$, $V_{N,M}(t_0 + t, .)$ are defined as functionals on $\mathcal{C}$, and ϕ is the restriction of ψ to the interval $[-r, 0]$.

Proof. Apply Theorem 6 of Appendix A.1 as in the proof of Proposition 5, except for the fact that we choose $\mathcal{U}_{N,M} = \mathcal{U}_{N \cdot M}$ instead of $\mathcal{U}$ as the set of strategies $\tilde{\mathcal{U}}$. Notice that, by hypothesis, the intermediate time t lies on the grid $I_{N \cdot M}$. $\square$

The next result gives a bound on the order of the global approximation error between the value functions of degree N and (N, M) provided that the local approximation error is of order greater than one in the discretisation step.

Theorem 2. *Assume (A1) – (A3). Let $N, M \in \mathbb{N}$. Suppose that for some constants $\hat{K}, \delta > 0$ the following holds: for any $t_0 \in I_N$, $\phi \in \mathcal{C}$, $u \in \mathcal{U}$ there is $\bar{u} \in \mathcal{U}_{N,M}$ such that*

$$(*) \qquad \begin{aligned} & \mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), \bar{u}(s)\big) ds + V_N(t_0 + h_N, \bar{Z}_{h_N}) \right) \\ \leq\; & \mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), u(s)\big) ds + V_N(t_0 + h_N, Z_{h_N}) \right) \;+\; \hat{K}\, h_N^{1+\delta}, \end{aligned}$$

where Z is the solution to Equation (5) *of degree N under control process u, $\bar{Z}$ the solution to Equation* (5) *of degree N under $\bar{u}$, both with initial condition (t_0, ψ) for some $\psi \in \mathcal{C}_N$ such that $\psi_{|[-r,0]} = \phi$. Then*

$$\big| V_{N,M}(t_0, \phi) - V_N(t_0, \phi) \big| \;\leq\; T\, \hat{K}\, h_N^{\delta} \quad \textit{for all } t_0 \in I_N,\ \phi \in \mathcal{C}.$$

Proof. Let $N, M \in \mathbb{N}$. Recall that $V_{N,M} \geq V_N$ by construction. It is therefore enough to prove the upper bound for $V_{N,M} - V_N$. Suppose Condition $(*)$ is fulfilled for N, M and some constants $\hat{K}, \delta > 0$. Observe that $V_N(T_N, .) = g(\mathrm{Lin}_N(.)) = V_{N,M}(T_N, .)$.

Let $t_0 \in I_N \setminus \{T_N\}$. Let $\phi \in \mathcal{C}$, and choose any $\psi \in \mathcal{C}_N$ such that $\psi_{|[-r,0]} = \phi$. Given $\varepsilon > 0$, in virtue of Proposition 5, we find a control process $u \in \mathcal{U}$ such that

$$V_N(t_0, \phi) \;\geq\; \mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), u(s)\big) ds \;+\; V_N\big(t_0 + h_N, \mathrm{Lin}_N(Z_{h_N})\big) \right) \;-\; \varepsilon,$$

where Z is the solution to Equation (5) of degree N under control process u with initial condition (t_0, ψ). For this u, choose $\bar{u} \in \mathcal{U}_{N,M}$ according to $(*)$, and let $\bar{Z}$ be the solution to Equation (5) of degree N under control process $\bar{u}$ with the same initial condition as for Z. Then, using the above inequality and Proposition 6, we see that

$$\begin{aligned}
& V_{N,M}(t_0, \phi) - V_N(t_0, \phi) \\
\leq\; & V_{N,M}(t_0, \phi) \;-\; \mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), u(s)\big) ds \;+\; V_N\big(t_0 + h_N, \mathrm{Lin}_N(Z_{h_N})\big) \right) \;+\; \varepsilon \\
\leq\; & \mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), \bar{u}(s)\big) ds \;+\; V_{N,M}\big(t_0 + h_N, \mathrm{Lin}_N(\bar{Z}_{h_N})\big) \right) \;+\; \varepsilon \\
& -\; \mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), u(s)\big) ds \;+\; V_N\big(t_0 + h_N, \mathrm{Lin}_N(Z_{h_N})\big) \right) \\
=\; & \mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), \bar{u}(s)\big) ds \;+\; V_N\big(t_0 + h_N, \mathrm{Lin}_N(\bar{Z}_{h_N})\big) \right) \\
& -\; \mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), u(s)\big) ds \;+\; V_N\big(t_0 + h_N, \mathrm{Lin}_N(Z_{h_N})\big) \right) \\
& +\; \mathbf{E}\Big(V_{N,M}\big(t_0 + h_N, \mathrm{Lin}_N(\bar{Z}_{h_N})\big) - V_N\big(t_0 + h_N, \mathrm{Lin}_N(\bar{Z}_{h_N})\big) \Big) \;+\; \varepsilon \\
\leq\; & \hat{K}\, h_N^{1+\delta} \;+\; \sup_{\tilde{\phi} \in \mathcal{C}} \big\{ V_{N,M}(t_0 + h_N, \tilde{\phi}) - V_N(t_0 + h_N, \tilde{\phi}) \big\} \;+\; \varepsilon,
\end{aligned}$$

where in the last line Condition $(*)$ has been exploited. Since $\varepsilon > 0$ was arbitrary and neither the first nor the last line of the above inequalities depend on u or $\bar{u}$, it follows that for all $t_0 \in I_N \setminus \{T_N\}$,

$$\sup_{\phi \in \mathcal{C}} \big\{ V_{N,M}(t_0, \phi) - V_N(t_0, \phi) \big\} \quad\leq\quad \hat{K}\, h_N^{1+\delta} \;+\; \sup_{\phi \in \mathcal{C}} \big\{ V_{N,M}(t_0 + h_N, \phi) - V_N(t_0 + h_N, \phi) \big\}.$$

Recalling the equality $V_{N,M}(T_N, .) = V_N(T_N, .)$, we conclude that for all $t_0 \in I_N$,

$$\sup_{\phi \in \mathcal{C}} \big\{ V_{N,M}(t_0, \phi) - V_N(t_0, \phi) \big\} \quad\leq\quad \frac{1}{h_N}\,(T_N - t_0)\, \hat{K}\, h_N^{1+\delta} \quad\leq\quad T\, \hat{K}\, h_N^{\delta},$$

which yields the assertion. □

Statement and proof of Theorem 2 should be compared to Theorem 7 in Falcone and Rosace (1996). We note, though, that the deterministic analogue of Condition $(*)$ in

Theorem 2 is weaker than the corresponding conditions (37) and (38) in Falcone and Rosace (1996). In particular, it is not necessary to require that any controlled process Z can be approximated with local error of order $h^{1+\delta}$ by some process $\bar{Z}$ using only control processes which are piece-wise constant in time on a grid of width h. In the stochastic case, such a requirement would in general be too strong to be satisfiable.

In order to be able to apply Theorem 2, we must check whether and how Condition $(*)$ can be satisfied. Given a grid of width $\frac{r}{N}$ for the discretisation in time and segment space, we would expect the condition to be fulfilled provided we choose the sub-grid for the piece-wise constant controls fine enough; that is, the time discretisation of the control processes should be of degree M with M sufficiently big in comparison to N. Indeed, if we choose M of any order greater than three in N, then Condition $(*)$ holds. This is the content of Theorem 3. The theorem, in turn, relies on a kind of mean value theorem, due to Krylov, which we cite as Theorem 7 in Appendix A.3.

Theorem 3. *Assume (A1) – (A4). Let $\beta > 3$. Then there is a number $\hat{K} > 0$ depending only on K, r, L, T, the dimensions and β such that Condition $(*)$ in Theorem 2 is satisfied with constants $\hat{K}$ and $\delta := \frac{\beta-3}{4}$ for all $N, M \in \mathbb{N}$ such that $N \geq r$ and $M \geq N^{\beta}$.*

Proof. Let $N, M \in \mathbb{N}$ be such that $N \geq r$ and $M \geq N^{\beta}$. Let $t_0 \in I_N$, $\phi \in \mathcal{C}$. Define the following functions:

$$
\begin{aligned}
&\tilde{b}\colon \Gamma \to \mathbb{R}^d, && \tilde{b}(\gamma) := b\big(t_0, \mathrm{Lin}_N(\phi), \gamma\big),\\
&\tilde{\sigma}\colon \Gamma \to \mathbb{R}^{d\times d_1}, && \tilde{\sigma}(\gamma) := \sigma\big(t_0, \mathrm{Lin}_N(\phi), \gamma\big),\\
&\tilde{f}\colon \Gamma \to \mathbb{R}, && \tilde{f}(\gamma) := f\big(t_0, \mathrm{Lin}_N(\phi), \gamma\big),\\
&\tilde{g}\colon \mathbb{R}^d \to \mathbb{R}^d, && \tilde{g}(x) := V_N\big(t_0{+}h_N, \mathrm{Lin}_N(S(\phi,x))\big),
\end{aligned}
$$

where $S(\phi, x)$ is the function in $\mathcal{C}$ given by

$$
S(\phi,x)\colon [-r,0] \ni s \mapsto \begin{cases} \phi(s{+}h_N) & \text{if } s \in [-r,-h_N],\\ \phi(0) + \frac{s+h_N}{h_N}\,x & \text{if } s \in (-h_N, 0]. \end{cases}
$$

As a consequence of Assumption (A4), $\tilde{b}$, $\tilde{\sigma}$, $\tilde{f}$ as just defined are continuous functions on (Γ, ρ). By Assumption (A2), $|\tilde{b}|$, $|\tilde{\sigma}|$, $|\tilde{f}|$ are all bounded by K. As a consequence of Proposition 4, the function $\tilde{g}$ is Lipschitz continuous and for the Lipschitz constant we have

$$
\sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|\tilde{g}(x) - \tilde{g}(y)|}{|x-y|} \;\leq\; 3L(T{+}1)\exp\big(3T(T{+}4d_1)L^2\big).
$$

Let $u \in \mathcal{U}$, and let Z^u be the solution to Equation (5) of degree N under control process u with initial condition (t_0, ψ) for some $\psi \in \mathcal{C}_N$ such that $\psi_{|[-r,0]} = \phi$. As Z also satisfies Equation (8), we see that

$$
Z^u(t) - \phi(0) \;=\; \int_0^t \tilde{b}\big(u(s)\big)ds \;+\; \int_0^t \tilde{\sigma}\big(u(s)\big)dW(s) \quad \text{for all } t \in [0, h_N].
$$

By Theorem 7 in Appendix A.3, we find $\bar{u} \in \mathcal{U}_{N,M}$ such that

$$\begin{aligned} &\mathbf{E}\left(\int_0^{h_N} \tilde{f}\big(\bar{u}(s)\big)ds + \tilde{g}\big(X^{\bar{u}}(h_N)\big)\right) \;-\; \mathbf{E}\left(\int_0^{h_N} \tilde{f}\big(u(s)\big)ds + \tilde{g}\big(Z^{u}(h_N) - \phi(0)\big)\right) \\ \leq\;\; &\bar{C}(1+h_N)\left(\frac{r}{N\cdot M}\right)^{\frac{1}{4}}\left(\left(\frac{r}{N\cdot M}\right)^{\frac{1}{4}} \sup_{\gamma\in\Gamma}|\tilde{f}(\gamma)| \;+\; \sup_{x,y\in\mathbb{R}^d, x\neq y}\frac{|\tilde{g}(x)-\tilde{g}(y)|}{|x-y|}\right), \end{aligned}$$

where $X^{\bar{u}}$ satisfies

$$X^{\bar{u}}(t) \;=\; \int_0^t \tilde{b}\big(\bar{u}(s)\big)ds \;+\; \int_0^t \tilde{\sigma}\big(\bar{u}(s)\big)dW(s) \quad \text{for all } t \geq 0.$$

Notice that the constant $\bar{C}$ above only depends on K and the dimensions d and d_1. Let $Z^{\bar{u}}$ be the solution to Equation (5) of degree N under control process $\bar{u}$ with initial condition (t_0, ψ), where $\psi_{|[-r,0]} = \phi$ as above. Then, by construction, $Z^{\bar{u}}(t) - \phi(0) = X^{\bar{u}}(t)$ for all $t \in [0, h_N]$. Set

$$\hat{K} := 2\bar{C}\, r^{-\frac{\beta}{4}}\left(K + 3L(T+1)\exp\big(3T(T+4d_1)L^2\big)\right).$$

Since $M \geq N^{\beta}$ by hypothesis, $\frac{1+\beta}{4} = 1+\delta > 1$ and $h_N = \frac{r}{N}$, we have

$$r^{\frac{1}{4}}(N\cdot M)^{-\frac{1}{4}} \;\leq\; r^{\frac{1}{4}}\cdot N^{-\frac{1+\beta}{4}} \;=\; r^{-\frac{\beta}{4}}\cdot h_N^{1+\delta}.$$

Recalling the definition of the coefficients $\tilde{b}$, $\tilde{\sigma}$, $\tilde{f}$, $\tilde{g}$, we have thus found a piece-wise constant strategy $\bar{u} \in \mathcal{U}_{N,M}$ such that

$$\begin{aligned} &\mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), \bar{u}(s)\big)ds + V_N\big(t_0+h_N, Z^{\bar{u}}_{h_N}\big)\right) \\ \leq\;\; &\mathbf{E}\left(\int_0^{h_N} f\big(t_0, \mathrm{Lin}_N(\phi), u(s)\big)ds + V_N\big(t_0+h_N, Z^{u}_{h_N}\big)\right) \;+\; \hat{K}\, h_N^{1+\delta}, \end{aligned}$$

where Z^u, $Z^{\bar{u}}$ are the solutions corresponding to u and $\bar{u}$, respectively, as above. □

We note that the constant $\hat{K}$ appearing in Theorem 3 and its proof depends on β only through the factor $r^{-\frac{\beta}{4}}$. Moreover, $\hat{K}$ also depends on the delay length r only through the factor $r^{-\frac{\beta}{4}}$. Theorem 2 and Theorem 3, together with the above observation, yield the following bound on the difference between the value functions of degree N and degree (N, M), respectively.

Corollary 2. *Assume (A1) – (A4). Then there is a positive constant $\bar{K}$ depending only on K, L, T and the dimensions such that for all $\beta > 3$, all $N \in \mathbb{N}$ with $N \geq r$, all $M \in \mathbb{N}$ with $M \geq N^{\beta}$, all $t_0 \in I_N$, all $\phi \in \mathcal{C}$ it holds that*

$$\big|V_{N,M}(t_0,\phi) - V_N(t_0,\phi)\big| \;\leq\; \bar{K}\, r^{-\frac{\beta}{4}}\left(\frac{r}{N}\right)^{\frac{\beta-3}{4}}.$$

In particular, with $M = \lceil N^{\beta}\rceil$, where $\lceil x\rceil$ is the least integer not smaller than x, the upper bound on the discretisation error can be rewritten as

$$\big|V_{N,\lceil N^{\beta}\rceil}(t_0,\phi) - V_N(t_0,\phi)\big| \;\leq\; \bar{K}\, r^{-\frac{\beta}{1+\beta}}\left(\frac{r}{N^{1+\beta}}\right)^{\frac{\beta-3}{4(1+\beta)}}.$$

From Corollary 2 we see that, in terms of the total number of time steps $N\lceil N^{\beta}\rceil$, we can achieve any rate of convergence smaller than $\frac{1}{4}$ by choosing the sub-discretisation order β sufficiently large.

5 Overall discretisation error

Here, we put together the error bounds from Sections 3 and 4 in order to obtain an overall estimate for the rate of convergence, that is, a bound on the discretisation error incurred in passing from the original value function to the value function of degree (N,M). In addition, we address the question of whether and in which sense nearly optimal strategies for the discrete problems can be used as nearly optimal strategies for the original system. Finally, we briefly discuss the question of how to numerically compute the value functions of degree (N,M).

Let us return to the overall discretisation error. As in Corollary 2, we express the error bound in terms of the total number of discretisation steps or, taking into account the presence of the delay length r, in terms of the length of the time step.

Theorem 4. *Assume (A1) – (A4). Let $\alpha > 0$, $L_H > 0$. Then there is a constant $\bar{C}$ depending only on α, L_H, L, K, T and the dimensions such that for all $\beta > 3$, all $N \in \mathbb{N}$ with $N \geq 2r$, all $t_0 \in I_N$, all α-Hölder continuous $\phi \in \mathcal{C}$ with Hölder constant not greater than L_H, it holds that, with $h = \frac{r}{N^{1+\beta}}$,*

$$\left|V(t_0,\phi) - V_{N,\lceil N^{\beta}\rceil}(t_0,\phi)\right| \;\leq\; \bar{C}\left(r^{\frac{\alpha\cdot\beta}{1+\beta}}\, h^{\frac{\alpha}{1+\beta}} \,\vee\, r^{\frac{\beta}{2(1+\beta)}}\sqrt{\ln\left(\tfrac{1}{h}\right)}\, h^{\frac{1}{2(1+\beta)}} \;+\; r^{-\frac{\beta}{1+\beta}}\, h^{\frac{\beta-3}{4(1+\beta)}}\right).$$

In particular, with $\beta = 5$ and $h = \frac{r}{N^6}$, it holds that

$$\left|V(t_0,\phi) - V_{N,N^5}(t_0,\phi)\right| \;\leq\; \bar{C}\left(r^{\frac{5\alpha}{6}} h^{\frac{2\alpha-1}{12}} \,\vee\, r^{\frac{5}{12}}\sqrt{\ln\left(\tfrac{1}{h}\right)} \;+\; r^{-\frac{5}{6}}\right) h^{\frac{1}{12}}.$$

Proof. Clearly, $|V - V_{N,\lceil N^{\beta}\rceil}| \leq |V - V_N| + |V_N - V_{N,\lceil N^{\beta}\rceil}|$. The assertion now follows from Corollary 2 and Theorem 1, where $\ln(\frac{1}{h_N}) = \ln(\frac{N}{r})$ is bounded by $\ln(\frac{N^{1+\beta}}{r}) = \ln(\frac{1}{h})$. $\square$

The choice $\beta = 5$ in Theorem 4 yields the same rate for both summands in the error estimate provided the initial segment is at least $\frac{1}{2}$-Hölder continuous, because $\frac{1}{2} = \frac{\beta-3}{4}$ implies $\beta = 5$. Thus, the best overall error bound we obtain without additional assumptions is of order $h^{1/12}$ up to neglecting the logarithmic term.

The rate $\frac{1}{12}$ is a worst-case estimate. Moreover, it is easy to obtain better error bounds in special situations. Suppose, for instance, that the space of control actions Γ is finite and the diffusion coefficient σ is not directly controlled, that is, $\sigma(t,\phi,\gamma)$ does not vary with $\gamma \in \Gamma$. In this case, in order to find a bound on the local approximation error, it is enough to have a "mean value theorem", analogous to Theorem 7 in Appendix A.3, for processes of the form $X^u(t) := \int_0^t \tilde{b}\big(u(s)\big)ds + \tilde{\sigma}\, W(t)$, where u is a strategy from $\mathcal{U}$ or $\mathcal{U}_n$. The "mean value" error is seen to be of order $(\frac{r}{n})^{1/2}$, instead of $(\frac{r}{n})^{1/4}$ as in Theorem 7. Thus, when

Γ is finite and σ not directly controlled, we have a bound on the overall error of order $h^{1/6}$ times a logarithmic factor.

Recall that $V_{N,M} \geq V_N$ for all $N, M \in \mathbb{N}$ by construction. If, instead of the two-sided error bound of Theorem 4, we were merely interested in obtaining an upper bound for V, we would simply compute $V_{N,M}$ with $M = 1$. Theorem 1 implies that we would incur an error of order nearly $\frac{1}{2}$; that is, we would have

$$V \quad \leq \quad V_{N,1} \;+\; \text{constant} \times \sqrt{\frac{\ln(N)}{N}} \quad \text{for all } N \in \mathbb{N},\; N \geq 2r,$$

where the initial segments are supposed to be at least $\frac{1}{2}$-Hölder continuous. This direction, however, is the less informative one, since we do not expect the minimal costs for the discretised system to be lower than the minimal costs for the original system.

Up to this point, we have been concerned with convergence of value functions only. A natural question to ask is the following: Suppose we have found a strategy $\bar{u} \in \mathcal{U}_{N,M}$ which is ε-optimal for the control problem of degree (N, M) under initial condition (t_0, ϕ). Will this same strategy $\bar{u}$ also be nearly optimal for the original control problem?

The hypothesis that $\bar{u}$ be ε-optimal for the problem of degree (N, M) under initial condition (t_0, ϕ) means that $J_N(t_0, \phi, \bar{u}) - V_{N,M}(t_0, \phi) \leq \varepsilon$. Recall that the cost functional for the problem of degree (N, M) is identical to the one for the problem of degree N, namely J_N, and that, by construction, $J_N \geq V_{N,M} \geq V_N$ over the set of strategies $\mathcal{U}_{N,M}$. The strategy $\bar{u}$ is nearly optimal for the original control if there is $\tilde{\varepsilon}$ which must be small for ε small and N, M big enough such that $J(t_0, \phi, \bar{u}) - V(t_0, \phi) \leq \tilde{\varepsilon}$. Recall that $\mathcal{U}_{N,M} \subset \mathcal{U}$, whence $J(t_0, \phi, \bar{u})$ is well-defined. The next theorem states that nearly optimal strategies for the approximating problems are indeed nearly optimal for the original problem, too.

Theorem 5. *Assume (A1) – (A4). Let $\alpha > 0$, $L_H > 0$. Then there is a constant $\bar{C}_r$ depending only on α, L_H, L, K, T, the dimensions and the delay length r such that for all $\beta > 3$, all $N, M \in \mathbb{N}$ with $N \geq 2r$ and $M \geq N^{\beta}$, all $t_0 \in I_N$, all α-Hölder continuous $\phi \in \mathcal{C}$ with Hölder constant not greater than L_H the following holds:*

If $\bar{u} \in \mathcal{U}_{N,M}$ is such that $J_N(t_0, \phi, \bar{u}) - V_{N,M}(t_0, \phi) \leq \varepsilon$, then, with $h = \frac{r}{N^{1+\beta}}$,

$$J(t_0, \phi, \bar{u}) - V(t_0, \phi) \;\leq\; \bar{C}_r \left(h^{\frac{\alpha}{1+\beta}} \vee \sqrt{\ln\left(\tfrac{1}{h}\right)}\, h^{\frac{1}{2(1+\beta)}} \;+\; h^{\frac{\beta-3}{4(1+\beta)}} \right) \;+\; \varepsilon.$$

Proof. Let $\bar{u} \in \mathcal{U}_{N,M}$ be such that $J_N(t_0, \phi, \bar{u}) - V_{N,M}(t_0, \phi) \leq \varepsilon$. Then

$$\begin{aligned}
& J(t_0, \phi, \bar{u}) - V(t_0, \phi) \\
\leq\; & J(t_0, \phi, \bar{u}) - J_N(t_0, \phi, \bar{u}) \;+\; J_N(t_0, \phi, \bar{u}) - V_{N,M}(t_0, \phi) \;+\; V_{N,M}(t_0, \phi) - V(t_0, \phi) \\
\leq\; & \sup_{u \in \mathcal{U}} \left| J(t_0, \phi, u) - J_N(t_0, \phi, u) \right| \;+\; \varepsilon \;+\; V_{N,M}(t_0, \phi) - V(t_0, \phi).
\end{aligned}$$

The assertion is now a consequence of Theorem 1 and Theorem 4. $\square$

Let us suppose we have found a strategy $\bar{u}$ for the problem of degree (N, M) with fixed initial condition $(t_0, \phi) \in I_N \times \mathcal{C}$ which is ε-optimal or optimal and a *feedback control.* The latter means here that $\bar{u}$ can be written in the form

$$\bar{u}(t)(\omega) \;=\; \bar{u}_0\big(\lfloor t\rfloor_{N\cdot M}, \Pi_N(Z^u)(\lfloor t\rfloor_{N\cdot M})(\omega)\big) \quad \text{for all } \omega \in \Omega,\ t \geq 0,$$

where Z^u is the solution to Equation (8) under control $\bar{u}$ and initial condition (t_0, ϕ) and $\bar{u}_0$ is some measurable Γ-valued function defined on $[0, \infty) \times \mathcal{C}_N$ or, because of the discretisation, on $\{k\frac{r}{N\cdot M} \mid k \in \mathbb{N}_0\} \times \mathbb{R}^{d(N\cdot M+M+1)}$. We would like to use $\bar{u}_0$ as a feedback control for the original system. It is not clear whether this is possible unless one assumes some regularity like Lipschitz continuity of $\bar{u}_0$ in its segment variable. The problem is that we have to replace solutions to Equation (8) with solutions to Equation (2).

Something can be said, though. Recall the definition of $\mathcal{U}_{N,M}$ at the beginning of Section 4. Strategies in $\mathcal{U}_{N,M}$ are not only piece-wise constant, they are also adapted to the filtration generated by $W(k\frac{r}{N\cdot M})$, $k \in \mathbb{N}_0$. Thus, if $\bar{u} \in \mathcal{U}_{N,M}$ is a feedback control, then it can be re-written as

$$\bar{u}(t)(\omega) \;=\; \bar{u}_1\big(\lfloor t\rfloor_{N\cdot M}, W(\lfloor t\rfloor_{N\cdot M} - k\tfrac{r}{N\cdot M})(\omega),\, k = 0, \ldots, (N+1)M\big), \quad \omega \in \Omega,\ t \geq 0,$$

where $\bar{u}_1$ is some measurable Γ-valued function depending on the initial condition (t_0, ϕ) and defined on $\{k\frac{r}{N\cdot M} \mid k \in \mathbb{N}_0\} \times \mathbb{R}^{d_{N,M}}$ with $d_{N,M} := d(N\cdot M+M+1)$. The above equality has to be read keeping in mind the convention that $W(t) = 0$ if $t < 0$. The function $\bar{u}_1$ can be used as a feedback control for the original problem as it directly depends on the underlying noise process, which is the same for the control problem of degree (N, M) and the original problem. By Theorem 5 we know that $\bar{u}_1$ induces a nearly optimal strategy for the original control problem provided $\bar{u}$ was nearly optimal for the discretised problem.

We now turn to the question of how to compute the discretised value functions. The value function of degree (N, M) is the value function of a finite-dimensional optimal control problem in discrete time. One time step corresponds to a step of length $\frac{r}{N\cdot M}$ in continuous time. The noise component of the control problem of degree (N, M) is given by a finite sequence of independent Gaussian random variables with mean zero and variance $\frac{r}{N\cdot M}$, because the time horizon is finite and the strategies in $\mathcal{U}_{N,M}$ are not only piece-wise constant, but also adapted to the filtration generated by $W(k\frac{r}{N\cdot M})$, $k \in \mathbb{N}_0$.

By construction of the approximation to the dynamics in Section 3, the segment space for the problem of degree (N, M) is the subspace of $\mathcal{C}_N$ consisting of all functions which are piece-wise linear relative to the grid $\{k\frac{r}{N\cdot M} \mid k \in \mathbb{Z}\} \cap [-r-\frac{r}{N}, 0]$. The segment space of degree (N, M), therefore, is finite-dimensional and isomorphic to $\mathbb{R}^{d_{N,M}}$ with $d_{N,M} = d(N\cdot M+M+1)$. The functions of interest are actually those whose nodes are multiples of $\frac{r}{N}$ units of time apart, but in each step of the evolution the segment functions (and their nodes) get shifted in time by $\frac{r}{N\cdot M}$ units.

Theoretically, the Principle of Dynamic Programming as expressed in Proposition 6 could be applied to compute the value function $V_{N,M}$. Practically, however, it is not possible to use an algorithm based on directly applying one-step Dynamic Programming. This difficulty arises, because the state space of the controlled discrete-time Markov chains we are dealing with is $\mathbb{R}^{d_{N,M}}$ and the (semi-)discrete value function $V_{N,M}$ is defined on $I_{N\cdot M} \times \mathbb{R}^{d_{N,M}}$ or, in the fully discrete case, on a $d_{N,M}$-dimensional grid. In view of Theorem 4, the dimension $d_{N,M}$ is expected to be very large so that storing the values of $V_{N,M}$ for all initial conditions – as required by the Dynamic Programming method – becomes impossible.

The situation, however, is not as desperate as it might seem. Recall that $V_{N,M}$ is an approximation of the value function V_N constructed in Section 3, which in turn approximates V, the value function of the original problem. For any initial time $t_0 \in I_N$, the function $V_N(t_0,.)$ is defined on $\mathbb{R}^{d_N}$ with $d_N := d(N+1)$. An approximation to V_N can be computed by backward iteration starting from time T_N and proceeding in time steps of length $\frac{r}{N}$. To compute $V_N(t_0 - \frac{r}{N}, \phi)$ when $V(t_0,.)$ is available and $t_0 \in I_N$, an "inner" backward iteration can be performed over the grid $\{t_0 - \frac{r}{N} + k\frac{r}{N\cdot M} \mid k = 0, \ldots, M\}$. The computational complexity of one time step of length $\frac{r}{N\cdot M}$ according to the PDP is not greater than in the case of dynamics and costs without delay, except for the more costly evaluation of the value function at the first step of the backward iteration.

A different approach is linked to the observation that Monte Carlo simulation of trajectories of the state process as given by Equation (2) or, in the discretised version, by Equation (8) with piece-wise constant controls is feasible even for extremely fine time grids. In this respect, a method recently introduced by Rogers (2006) for computing value functions of high-dimensional discrete-time Markovian optimal control problems may be useful. The method is based on path-wise optimisation and Monte Carlo simulation of trajectories of a reference Markov chain. It uses minimisation over functions which can be interpreted as candidates for the value function. If those candidates can be chosen from a computationally "nice" class, then the value function can be computed at any given point without the need to store its values for the entire state space. Unlike schemes directly employing the PDP, Rogers's method does not yield an approximation of the value function over the entire state space, but only its value at a given initial point.

A Appendix

A.1 Bellman's Principle of Optimality

Let $((\Omega, \mathcal{F}, \mathrm{P}), (\mathcal{F}_t), W)$ be a Wiener basis of dimension d_1. Let $\mathcal{U}$ be the associated set of control processes. For $n \in \mathbb{N}$, define the set $\mathcal{U}_n \subset \mathcal{U}$ of piece-wise constant strategies according to (10) at the beginning of Section 4. Let $\tilde{\mathcal{U}}$ be either $\mathcal{U}$ or $\mathcal{U}_n$ for some $n \in \mathbb{N}$.

Let $\tilde{r} > 0$ and set $\tilde{\mathcal{C}} := \mathbf{C}([-\tilde{r}, 0], \mathbb{R}^d)$. If Y is an $\mathbb{R}^d$-valued process, then the notation Y_t in this subsection denotes the segment of length $\tilde{r}$. We introduce the following hypotheses:

(H1) Measurability: $\tilde{b}\colon [0,\infty) \times \tilde{\mathcal{C}} \times \Gamma \to \mathbb{R}^d$, $\tilde{\sigma}\colon [0,\infty) \times \tilde{\mathcal{C}} \to \mathbb{R}^{d\times d_1}$, $\tilde{f}\colon [0,\infty) \times \tilde{\mathcal{C}} \times \Gamma \to \mathbb{R}$, $\tilde{g}\colon \tilde{\mathcal{C}} \to \mathbb{R}$ are Borel measurable functions.

(H2) Boundedness: $|\tilde{b}|$, $|\tilde{\sigma}|$, $|\tilde{f}|$, $|\tilde{g}|$ are bounded by some positive constant K.

(H3) Uniform Lipschitz condition: there is a constant $L > 0$ such that for all $\phi, \psi \in \tilde{\mathcal{C}}$, all $t \geq 0$, all $\gamma \in \Gamma$

$$\begin{aligned} |\tilde{b}(t,\phi,\gamma) - \tilde{b}(t,\psi,\gamma)| \vee |\tilde{\sigma}(t,\phi,\gamma) - \tilde{\sigma}(t,\psi,\gamma)| &\leq L\,\|\phi - \psi\|, \\ |\tilde{f}(t,\phi,\gamma) - \tilde{f}(t,\psi,\gamma)| \vee |\tilde{g}(\phi) - \tilde{g}(\psi)| &\leq L\,\|\phi - \psi\|. \end{aligned}$$

Let $\tilde{T} > 0$. Define the cost functional $\tilde{J}$ on $[0,\tilde{T}] \times \tilde{\mathcal{C}} \times \mathcal{U}$ by

$$\tilde{J}(t_0,\psi,u) := \mathbf{E}\left(\int_0^{\tilde{T}-t_0} \tilde{f}\big(t_0{+}s, Y_s, u(s)\big) ds \;+\; \tilde{g}\big(Y_{\tilde{T}-t_0}\big) \right),$$

where Y is the solution to the controlled SDDE

$$(12) \quad Y(t) = \begin{cases} \psi(0) + \int_0^t \tilde{b}\big(t_0{+}s, Y_s, u(s)\big) ds \;+\; \int_0^t \tilde{\sigma}\big(t_0{+}s, Y_s, u(s)\big) dW(s), & t > 0, \\ \psi(t), \quad t \in [-\tilde{r}, 0]. \end{cases}$$

Define the associated value function $\tilde{V} \colon [0,\tilde{T}] \times \tilde{\mathcal{C}} \to \mathbb{R}$ by

$$\tilde{V}(t_0,\psi) := \inf\big\{ \tilde{J}(t_0,\psi,u) \mid u \in \tilde{\mathcal{U}} \big\}.$$

Observe that $\tilde{V}$ thus defined gives the minimal costs over the set $\mathcal{U}$ of all control processes or just over a set of strategies which are piece-wise constant relative to the grid $\{k\frac{r}{n} \mid k \in \mathbb{N}_0\}$ for some $n \in \mathbb{N}$, depending on the choice of $\tilde{\mathcal{U}}$. The following property of $\tilde{V}$ is useful.

Proposition 7. *Assume (H1) – (H3). Let $\tilde{V}$ be the value function defined above. Then $\tilde{V}$ is bounded and Lipschitz continuous in the segment variable uniformly in the time variable. More precisely, $|\tilde{V}|$ is bounded by $K(\tilde{T}{+}1)$ and for all $t_0 \in [0,\tilde{T}]$, all $\phi, \psi \in \tilde{\mathcal{C}}$,*

$$|\tilde{V}(t_0,\phi) - \tilde{V}(t_0,\psi)| \;\leq\; 2\sqrt{2}\, L(\tilde{T}{+}1) \exp\big(3\tilde{T}(\tilde{T}+ 4d_1)L^2\big)\, \|\phi - \psi\|.$$

Proof. Boundedness of $\tilde{V}$ is an immediate consequence of its definition and Hypothesis (H2). Let $t_0 \in [0,\tilde{T}]$, let $\phi, \psi \in \tilde{\mathcal{C}}$. Recall the inclusion $\tilde{\mathcal{U}} \subset \mathcal{U}$ and observe that, in virtue of the definition of $\tilde{V}$, we have

$$|\tilde{V}(t_0,\phi) - \tilde{V}(t_0,\psi)| \quad \leq \quad \sup_{u \in \mathcal{U}} |\tilde{J}(t_0,\phi,u) - \tilde{J}(t_0,\psi,u)|.$$

By Hypothesis (H3), for all $u \in \mathcal{U}$ we get

$$\begin{aligned} &|\tilde{J}(t_0,\phi,u) - \tilde{J}(t_0,\psi,u)| \\ \leq\;& \mathbf{E}\left(\int_0^{\tilde{T}-t_0} \big|\tilde{f}\big(t_0{+}s, X^u_s, u(s)\big) - \tilde{f}\big(t_0{+}s, Y^u_s, u(s)\big)\big| ds \;+\; \big|\tilde{g}\big(X^u_{\tilde{T}-t_0}\big) - \tilde{g}\big(Y^u_{\tilde{T}-t_0}\big)\big| \right) \\ \leq\;& L(1 + \tilde{T}{-}t_0)\; \mathbf{E}\left(\sup_{t \in [-\tilde{r},\tilde{T}]} |X^u(t) - Y^u(t)|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where X^u, Y^u are the solutions to Equation (12) under control process u with initial conditions (t_0, ϕ) and (t_0, ψ), respectively. Now, for every $T \in [0, \tilde{T}]$,

$$\mathbf{E}\left(\sup_{t\in[-\tilde{r},T]} |X^u(t) - Y^u(t)|^2\right) \quad \leq \quad 2\,\mathbf{E}\left(\sup_{t\in[0,T]} |X^u(t) - Y^u(t)|^2\right) \;+\; 2\|\phi - \psi\|^2,$$

while Hölder's inequality, Doob's maximal inequality, Itô's isometry, Fubini's theorem and Hypothesis (H3) together yield

$$\begin{aligned}
&\mathbf{E}\left(\sup_{t\in[0,T]} |X^u(t) - Y^u(t)|^2\right) \\
\leq\;& 3|\phi(0) - \psi(0)|^2 \;+\; 3T\,\mathbf{E}\left(\int_0^T \left|\tilde{b}\big(t_0{+}s, X^u_s, u(s)\big) - \tilde{b}\big(t_0{+}s, Y^u_s, u(s)\big)\right|^2 ds\right) \\
&+\; 3d_1 \sum_{i=1}^{d}\sum_{j=1}^{d_1} \mathbf{E}\left(\sup_{t\in[0,T]}\left(\int_0^t \big(\tilde{\sigma}_{ij}\big(t_0{+}s, X^u_s, u(s)\big) - \tilde{\sigma}_{ij}\big(t_0{+}s, Y^u_s, u(s)\big)\big)\, dW^j(s)\right)^2\right) \\
\leq\;& 3|\phi(0) - \psi(0)|^2 \;+\; 3T\,L^2 \int_0^T \mathbf{E}\left(|X^u_s - Y^u_s|^2\right) ds \\
&+\; 12d_1\,\mathbf{E}\left(\int_0^T \sum_{i=1}^{d}\sum_{j=1}^{d_1} \big(\tilde{\sigma}_{ij}\big(t_0{+}s, X^u_s, u(s)\big) - \tilde{\sigma}_{ij}\big(t_0{+}s, Y^u_s, u(s)\big)\big)^2\, ds\right) \\
\leq\;& 3|\phi(0) - \psi(0)|^2 \;+\; 3(T{+}\,4d_1)L^2 \int_0^T \mathbf{E}\Big(\sup_{t\in[-\tilde{r},s]} |X^u(t) - Y^u(t)|^2\Big) ds.
\end{aligned}$$

Since $|\phi(0){-}\psi(0)| \leq \|\phi - \psi\|$, Gronwall's lemma gives

$$\mathbf{E}\left(\sup_{t\in[-\tilde{r},\tilde{T}]} |X^u(t) - Y^u(t)|^2\right) \quad \leq \quad 8\|\phi - \psi\|^2 \exp\big(6\tilde{T}(\tilde{T}{+}\,4d_1)L^2\big).$$

Putting the estimates together, we obtain the assertion. □

Recall that the the value function $\tilde{V}$ has been defined over the set of strategies $\tilde{\mathcal{U}}$. If $\tilde{\mathcal{U}} = \mathcal{U}$ set $\tilde{I} := [0, \infty)$, else if $\tilde{\mathcal{U}} = \mathcal{U}_n$ set $\tilde{I} := \{k\frac{r}{n} \,|\, k \in \mathbb{N}_0\}$. The following version of Bellman's Principle of Optimality or Principle of Dynamic Programming holds.

Theorem 6 (PDP). *Assume (H1) – (H3). Then for all* $t_0 \in [0, \tilde{T}]$, *all* $t \in \tilde{I} \cap [0, \tilde{T}{-}t_0]$, *all* $\psi \in \tilde{\mathcal{C}}$,

$$\tilde{V}(t_0, \psi) \;=\; \inf_{u\in\mathcal{U}} \mathbf{E}\left(\int_0^t \tilde{f}\big(t_0{+}s, Y^u_s, u(s)\big) ds \;+\; \tilde{V}(t_0{+}t, Y^u_t)\right),$$

where Y^u *is the solution to Equation* (12) *under control process* u *with initial condition* (t_0, ψ).

Theorem 6 is proved in the same way as Theorem 4.2 in Larssen (2002), also see the proof of Theorem 4.3.3 in Yong and Zhou (1999: p. 180). We merely point out the differences in the problem formulation and the hypotheses. Here, all coefficients, those of the dynamics and those of the cost functional, are bounded, while Larssen (2002) also allows for sub-linear growth. Since Equation (12) has unique solutions, boundedness of the coefficients guarantees that the cost functional $\tilde{J}$ as well as the value function $\tilde{V}$ are well defined. Notice that we express dependence on the initial time in a different, but equivalent way in comparison with Larssen (2002). Notice further that in Theorem 6 only deterministic times appear.

We have stated the control problem and given Bellman's principle in the strong Wiener formulation, cf. Section 2. Although the weak Wiener formulation is essential for the proof, the resulting value functions are the same for both versions. This is due to the fact that weak uniqueness holds for Equation (12). Also the infimum in the dynamic programming equation can be taken over all Wiener control bases or just over all control processes associated with a fixed Wiener basis.

There are two respects in which our hypotheses are more general than those of Theorem 4.2 in Larssen (2002). The first is that we do not require the integrand $\tilde{f}$ of the cost functional to be uniformly continuous in its three variables. But this assumption is not needed for the dynamic programming equation, while it is important for versions of the Hamilton-Jacobi-Bellman partial differential equation. The second is that we allow the optimisation problem to be formulated for certain subclasses of admissible strategies, namely the subclasses $\mathcal{U}_n$ of piece-wise constant strategies. The set of allowed intermediate times must be chosen accordingly.

A.2 Moments of the modulus of continuity of Itô diffusions

A typical trajectory of standard Brownian motion is Hölder continuous of any order less than one half. If such a trajectory is evaluated at two different time points $t_1, t_2 \in [0, T]$ with $|t_1 – t_2| \leq h$ small, then the difference between the values at t_1 and t_2 is not greater than a multiple of $\sqrt{h \ln(\frac{1}{h})}$, where the proportionality factor depends on the trajectory and the time horizon T, but not on the choice of the time points t_1, t_2. This is a consequence of Lévy's exact modulus of continuity for Brownian motion. The modulus of continuity of a stochastic process is a random element. Lemma 1 below shows that the modulus of continuity of Brownian motion and, more generally, that of any Itô diffusion with bounded coefficients have finite moments of any order. The result can be found in Słomiński (2001), cf. Lemma A.4 there.

Lemma 1 (Słomiński)**.** *Let W be a d_1-dimensional Wiener process living on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $Y = (Y^{(1)}, \ldots, Y^{(d)})^{\mathsf{T}}$ be an Itô diffusion of the form*

$$Y(t) \;=\; y_0 + \int_0^t \tilde{b}(s)ds \;+\; \int_0^t \tilde{\sigma}(s)dW(s), \quad t \geq 0,$$

where $y_0 \in \mathbb{R}^d$ and $\tilde{b}$, $\tilde{\sigma}$ are $(\mathcal{F}_t)$-adapted processes with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d_1}$, respectively. If $|\tilde{b}|$, $|\tilde{\sigma}|$ are bounded by some positive constant K, then it holds that for every $p > 0$, every

$T > 0$ *there is a constant* $C_{p,T}$ *depending only on* K*, the dimensions,* p *and* T *such that*

$$\mathbf{E}\left(\sup_{t,s\in[0,T],|t-s|\leq h}\left|Y(t)-Y(s)\right|^{p}\right) \;\leq\; C_{p,T}\left(h\ln(\tfrac{1}{h})\right)^{\frac{p}{2}} \quad \textit{for all } h\in(0,\tfrac{1}{2}].$$

It is enough to prove Lemma 1 for the special case of one-dimensional Brownian motion. The full statement is then derived by a component-wise estimate and a time-change argument, cf. Theorem 3.4.6 in Karatzas and Shreve (1991: p. 174), for example. One way of proving the assertion for Brownian motion is to follow the derivation of Lévy's exact modulus of continuity as suggested in Exercise 2.4.8 of Stroock and Varadhan (1979). The main ingredient there is an inequality due to Garsia, Rodemich, and Rumsey, see Theorem 2.1.3 in Stroock and Varadhan (1979: p. 47) and Garsia et al. (1970).

A.3 A stochastic mean value theorem due to Krylov

The theorem we cite here is a reduced version, adapted to our notation, of Theorem 2.7 in Krylov (2001). It provides an estimate of the error in approximating constant-coefficient controlled Itô diffusions by diffusions with piece-wise constant strategies. The error is measured in terms of cost-functional-like expectations with Lipschitz (or Hölder) coefficients; see Section 1 in Krylov (2001) for a discussion of various error criteria.

Let $((\Omega,\mathcal{F},\mathrm{P}),(\mathcal{F}_t),W)$ be a Wiener basis of dimension d_1 in the sense of Definition 1. As above, let (Γ,ρ) be a complete and separable metric space, and denote by $\mathcal{U}$ the set of all $(\mathcal{F}_t)$-progressively measurable processes $[0,\infty)\times\Omega\to\Gamma$. For $n\in\mathbb{N}$, let $\mathcal{U}_n$ be the subset of $\mathcal{U}$ given by (10). Thus, if $\bar{u}\in\mathcal{U}_n$, then $\bar{u}$ is right-continuous and piece-wise constant in time relative to the grid $\{k\frac{r}{n}\,|\,k\in\mathbb{N}_0\}$ and $\bar{u}(t)$ is measurable with respect to the σ-algebra generated by $W(k\frac{r}{n})$, $k=0,\ldots,\lfloor t\frac{n}{r}\rfloor$. We have incorporated the delay length r in the partition in order to be coherent with the notation of Section 4.

Let $\tilde{b}\colon\Gamma\to\mathbb{R}^d$, $\tilde{\sigma}\colon\Gamma\to\mathbb{R}^{d\times d_1}$ be continuous functions with $|\tilde{b}|$, $|\tilde{\sigma}|$ bounded by K. For $u\in\mathcal{U}$ denote by X^u the process

$$X^u(t) := \int_0^t \tilde{b}\big(u(s)\big)ds \;+\; \int_0^t \tilde{\sigma}\big(u(s)\big)dW(s), \quad t\geq 0.$$

The next result provides an error estimate for the approximation of a process X^u, where $u\in\mathcal{U}$, by processes X^{u_n}, $n\in\mathbb{N}$, where $u_n\in\mathcal{U}_n$, in terms of suitable cost functionals.

Theorem 7 (Krylov)**.** *Let* $\bar{T}>0$*. There is a constant* $\bar{C}>0$ *depending only on* K *and the dimensions such that the following holds: For any* $n\in\mathbb{N}$ *such that* $n\geq r$*, any bounded continuous function* $\tilde{f}\colon\Gamma\to\mathbb{R}$*, any bounded Lipschitz continuous function* $\tilde{g}\colon\mathbb{R}^d\to\mathbb{R}$*, any* $u\in\mathcal{U}$ *there exists* $u_n\in\mathcal{U}_n$ *such that*

$$\begin{aligned}&\mathbf{E}\left(\int_0^{\bar{T}}\tilde{f}\big(u_n(s)\big)ds+\tilde{g}\big(X^{u_n}(\bar{T})\big)\right) \;-\; \mathbf{E}\left(\int_0^{\bar{T}}\tilde{f}\big(u(s)\big)ds+\tilde{g}\big(X^{u}(\bar{T})\big)\right)\\ &\leq\; \bar{C}(1+\bar{T})\left(\frac{r}{n}\right)^{\frac{1}{4}}\left(\left(\frac{r}{n}\right)^{\frac{1}{4}}\sup_{\gamma\in\Gamma}|\tilde{f}(\gamma)| \;+\; \sup_{x,y\in\mathbb{R}^d,x\neq y}\frac{|\tilde{g}(x)-\tilde{g}(y)|}{|x-y|}\right).\end{aligned}$$

Note that in Theorem 7 the difference between the two expectations may be inverted, since we can take $-\tilde{f}$ in place of $\tilde{f}$ and $-\tilde{g}$ in place of $\tilde{g}$.

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